JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 11, Number 3, Fall 1999

STABILITY OF APPROXIMATION METHODS ON LOCALLY NON-EQUIDISTANT MESHES FOR SINGULAR INTEGRAL EQUATIONS

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ABSTRACT. A number of numerical methods for singular integral operators with continuous coefficients on curves with corners are studied. Necessary and sufficient conditions for stability of the methods under consideration are given. These conditions are formulated in terms of invertibility of some model operators. It is shown that the model operators belong to a well-known operator algebra, and their Fredholm properties are investigated. It is proved that the indices of the operators mentioned are equal to zero.

Let Γ be a simple closed counter-clockwise oriented curve in the complex plane **C**. Up to now a variety of approximation methods have been proposed and investigated, see [**1, 2, 4, 7, 12**–**23**] for solving singular integral equations of Cauchy type

$$(Ax)(\tau) := a(\tau)x(\tau) + \frac{b(\tau)}{\pi i}\int_\Gamma \frac{x(t)dt}{t-\tau} + \int_\Gamma k(t,\tau)x(t)\,dt = f(\tau), \quad t \in \Gamma. \tag{0.1}$$

As a rule, the meshes for the underlying numerical schemes were supposed to be regular. But if the solution of the original equation has a singularity at some point $t_0 \in \Gamma$ then sometimes one uses meshes which are concentrated at the point t_0. The usual way to do this is to use a continuously differentiable transformation. The original equation is transformed into a more complicated equation but one which has the advantage that it can be solved by using regular meshes. Some general discussion of this topic can be found in [**12**, Section 6.3], see also [**4, 7, 8, 13, 14, 16, 17, 23**]. But there exist sequences of meshes which cannot be described by such transformations. They appear, for instance, in adaptive algorithms and require a separate investigation.

Received by the editors in revised form on December 8, 1998.
The work of the first author was supported by the Priority Research Program, "Boundary Element Methods" of the German Research Foundation (Deutsche Forschungsgemeinschaft).

To be more precise, we consider the following characteristic of partitions. Let $\{\mathcal{P}_n\}_{n\in\mathbf{N}}$ be the sequence of the meshes which are employed to construct a numerical method, and let t^* be an arbitrary fixed point of Γ. By $\{t_k^{(n)}\}$, $t_k^{(n)} \in \Gamma$, we denote the ordered set of all points defining the mesh $\mathcal{P}_n$ and by $t_{k_0}^{(n)}$ the nearest point of the mesh $\mathcal{P}_n$ lying on the left side with respect to the point t^*. Let $r_n(t^*)$, $n \in \mathbf{N}$, refer to the quotient

$$r_n(t^*) = \frac{|t_{k_{0+2}}^{(n)} - t_{k_{0+1}}^{(n)}|}{|t_{k_0}^{(n)} - t_{k_{0-1}}^{(n)}|}.$$

If t^* coincides with some point of $\mathcal{P}_n$, then, for definiteness, we put $t_{k_0}^{(n)} := t^*$, and $r_n(t^*)$ is defined as

$$r_n(t^*) = \frac{|t_{k_{0+1}}^{(n)} - t_{k_0}^{(n)}|}{|t_{k_0}^{(n)} - t_{k_{0-1}}^{(n)}|}.$$

Notice that all meshes which are obtained by continuously differentiable transformations turns out to be locally equidistant. That is,

$$\lim_{n\to\infty} r_n(t^*) = 1 \quad \forall\, t^* \in \Gamma. \tag{0.2}$$

Here a more general approach to nonregular partitions is proposed. Namely, we consider meshes which appear by using noncontinuously differentiable functions. However, it is still supposed that the partition sequence preserves some regularity in the sense that

$$\lim_{n\to\infty} r_n(t^*) = p(t^*) \quad \forall\, t^* \in \Gamma, \tag{0.3}$$

and only for a finite number of points, say $t_1, \dots, t_m$, the limit is not equal to one:

$$p(t^*) = \begin{cases} 1 & \text{if } t^* \notin \{t_1, t_2, \dots, t_m\} \\ p_k & \text{if } t^* = t_k,\ k = 1, 2, \dots, m, \end{cases} \tag{0.4}$$

and $p_k \in (0, +\infty)$, $p_k \neq 1$ for all $k = 1, 2, \dots, m$.

In the sequel, partitions with the properties (0.3)–(0.4) are called locally non-equidistant.

The problems of the stability of methods based on locally non-equidistant meshes as well as effects arising at the break points of the partitions were almost not considered earlier. We know only one reference, namely [**12**, Sections 5.2.4–5.26, Section 5.5.1], where such questions are mentioned for some spline Galerkin methods. The aim of this paper is to make the underyling ideas more explicit and to extend them to other numerical methods. Thereby, we explain how these considerations can be used to investigate the stability of adaptive algorithms.

In the present paper we start with analyzing the situation when a singular integral equation is considered on the angle $\Gamma = \Gamma_\omega$, $0 < \omega < 2\pi$, the underlying meshes are equidistant on each of the arcs of Γ_ω, and the coefficients a and b of the initial equation are supposed to be constant. Thus, limit (0.3) can be not equal to one only at the point 0. Under the above assumptions the stability of a quadrature method is studied. It should be noted that this problem is crucial for all the further considerations because it represents a so-called local model for the general problem, and because the stability conditions of the corresponding quadrature method for the equation (0.1) can be formulated by means of the characteristics of some operators, say A_τ, $\tau \in \Gamma$, which are immediately connected with the local models mentioned, see Section 3. In particular, we show that all the operators A_τ belong to the smallest closed subalgebra of the Banach algebra $\mathcal{L}(l^2_\rho)$ of all linear operators acting in the space l^2_ρ, which contains all Toeplitz operators with piecewise continuous generating functions. In addition, we compute the symbols of these operators and study their Fredholmness. An interesting fact to be noted here also is that the Fredholm properties of A_τ, $\tau \in \Gamma$, do not depend on the change of the partition parameters.

Note that the proofs of these results essentially use methods which were proposed by Prößdorf and Rathsfeld in [**14, 15**]. Alternatively, the considerations can also be based on the approach suggested in [**12**], see Remark 3. However, we prefer here the methods of Prößdorf and Rathsfeld because they seem to be more elementary in the case at hand.

Section 2 is devoted to analogous problems for ε-collocation and qualocation methods for singular integral equations.

The results of Sections 1–2 are then used to settle the general situation in Section 3. This will be made by help of local method.

Finally, by use of the quadrature method it will be demonstrated in Section 4 how the stability analysis can be carried out for some adaptive methods.

It should also be noted that a special collocation method as well as some associated Galerkin method based on non-equidistant partitions were considered in [**1**]. But the meshes used in [**1**] do not have the property (0.3)–(0.4). Therefore, the underlying stability analysis requires some special assumptions concerning the initial singular integral equations and is quite different from the methods presented in this paper.

1. Quadrature methods for non-equidistant partitions. Approximation operators and their symbols. Let Γ_ω, $0 < \omega < 2\pi$, be an angle,

$$\Gamma_\omega := \mathbf{R}^+ \cup \mathbf{R}^+ e^{i\omega},$$

where $\mathbf{R}^+ e^{i\omega}$ is directed to 0 and $\mathbf{R}^+$ is directed away from 0. Given $\rho \in (-1/2, 1/2)$, let $L^2_\rho := L^2_\rho(\Gamma_\omega)$ denote the Lebesgue space of all measurable functions x on the angle Γ_ω with the norm

$$\|x\| := \left(\int_{\Gamma_\omega} |x(t)|^2 |t|^{2\rho}\, dt \right)^{1/2} < +\infty.$$

In this section our main aim is the study of quadrature methods for the singular integral equation

$$(Ax)(\tau) := (aI + bS_{\Gamma\omega})x(\tau) = f(\tau), \quad \tau \in \Gamma_\omega, \tag{1.1}$$

where the element x is in L^2_ρ, the operator S_{Γ_ω} is given by

$$(S_{\Gamma_\omega} x)(\tau) = \frac{1}{\pi i} \int_{\Gamma_\omega} \frac{x(t)dt}{t - \tau}, \quad x \in L^2_\rho,$$

and the coefficients a and b are supposed to be constant. It will be seen later on that equation (1.1) and the corresponding quadrature method constitute a so-called local model, which is of great importance in studying the general situation. Namely, the stability conditions of numerical methods for singular integral equations with continuous coefficients on piecewise Lyapunov contours will be formulated in terms of the mentioned local models. Let $\mathbf{R}^+$ and $\mathbf{R}^+e^{i\omega}$, $0<\omega<2\pi$, be the arcs of the angle Γ_ω. In all that follows we assume that the arcs $\mathbf{R}^+$ and $\mathbf{R}^+e^{i\omega}$ are equipped with equidistant meshes $\mathcal{P}_n$ and $\mathcal{P}_n^{(1)}$, $n\in\mathbf{N}$, respectively. Moreover, by p_n and $p_n^{(1)}$, we denote the corresponding partition sizes.

Suppose that
$$\frac{p_n}{p_n^{(1)}} = p \quad \forall\, n\in\mathbf{N}.$$
Without loss of generality we can assume that $p_n^{(1)} = 1$. Then the parameter p, more precisely, the pair $(1,p)$, can be viewed as the pressure coefficient of the mesh at the point $t=0$. The topic of this section is to study the behavior of some numerical methods which are constructed by means of such meshes. According to the choice of the pressure coefficient p, we will give a parametrization of the angle Γ_ω in the following way

$$t := \gamma(s) = \begin{cases} ps & \text{if } s\ge 0\\ -se^{i\omega} & \text{if } s<0. \end{cases} \tag{1.2}$$

Let $n\in\mathbf{N}$. We choose real numbers ε,δ, $0<\varepsilon\neq\delta<1$, and define points $t_k^{(n)}$ and $\tau_k^{(n)}$, $k\in\mathbf{Z}$, as follows

$$t_k^{(n)} := \begin{cases} p(k+\delta)/n & \text{if } k\ge 0\\ -((k+\delta)/n)e^{i\omega} & \text{if } k<0; \end{cases} \tag{1.3}$$

$$\tau_k^{(n)} := \begin{cases} p(k+\varepsilon)/n & \text{if } k\ge 0\\ -((k+\varepsilon)/n)e^{i\omega} & \text{if } k<0. \end{cases} \tag{1.4}$$

In the following we make the assumption that the function f occurring in (1.1) belongs to the class$R_\rho^2(\Gamma_\omega)$ of Riemann integrable functions on Γ_ω. This class consists of all functions which are Riemann integrable on each finite subarc of Γ_ω and for which the norm

$$\|f\|_{R_\rho^2(\Gamma_\omega)} = \|f\|_{L_\rho^2(\Gamma_\omega)} + \left(\sum_{k=0}^{\infty}\sup_{t\in[k,k+1)}|f(t)|^2\right)^{1/2} + \left(\sum_{k=0}^{\infty}\sup_{t\in e^{i\omega}[k,k+1)}|f(t)|^2\right)^{1/2}$$

is finite.

To determine approximate values $\xi_j^{(n)}$, $j \in \mathbf{Z}$, of the exact solution $x = x(t)$ at the points $t_j^{(n)}$, $j \in \mathbf{Z}$, we will solve the following system of linear algebraic equations

$$\tag{1.5} \begin{cases} (a - ib\cot(\pi(\varepsilon-\delta)))\xi_k^{(n)} \\ +\dfrac{b}{\pi i}\sum_{j\in\mathbf{Z}} \dfrac{\Delta t_j^{(n)}}{t_j^{(n)} - \tau_k^{(n)}}\xi_j^{(n)} = f(\tau_k^{(n)}), \\ k \in \mathbf{Z} \end{cases}$$

where

$$\tag{1.6} \Delta t_j^{(n)} := \begin{cases} (p/n) & \text{if } j \geq 0 \\ -(e^{i\omega}/n) & \text{if } j < 0. \end{cases}$$

If $\varphi_k^{(n)}(t)$, $k \in \mathbf{Z}$, denote the functions

$$\tag{1.7} \varphi_k^{(n)}(t) := \begin{cases} \begin{cases} 1 & \text{if } t \in p[(k/n), ((k+1)/n)) \\ 0 & \text{otherwise} \end{cases} & \text{if } k \geq 0 \\ \begin{cases} 1 & \text{if } t \in -e^{i\omega}[((k+1)/n), (k/n)) \\ 0 & \text{otherwise} \end{cases} & \text{if } k < 0 \end{cases}$$

then the approximate solution of (1.1) is given by

$$\tag{1.8} x_n(t) = \sum_{k\in\mathbf{Z}} \xi_k^{(n)}\varphi_k^{(n)}(t).$$

Let L_n stand for the orthogonal projection onto the subspace of $L^2_\rho(\Gamma_\omega)$ generated by all the elements of (1.8). Then the approximation method (1.5)–(1.8) can be rewritten as an operator equation

$$\tag{1.9} \breve{A}_n x_n = f_n, \quad x_n, f_n \in \operatorname{im} L_n,$$

with some operator $\breve{A}_n : \operatorname{im} L_n \to \operatorname{im} L_n$. In this section we will deal with investigating the properties of the operators $\breve{A}_n$. However, before it will be done we are going to illustrate in more detail how the approximate method (1.5) arises.

Rewriting the equation (1.1) in the form

$$
\begin{aligned}
(1.10)\quad & ax(\gamma(s)) + \frac{b}{\pi i}\int_{-\infty}^{+\infty} \frac{x(\gamma(\sigma))}{\sigma - s}\,d\sigma \\
&\quad + \frac{b}{\pi i}\int_{-\infty}^{+\infty}\left(\frac{\gamma'(\sigma)}{\gamma(\sigma)-\gamma(s)} - \frac{1}{\sigma - s}\right)x(\gamma(\sigma))\,ds = f(\gamma(s)),
\end{aligned}
$$

we see that the integral

$$
\int_{-\infty}^{+\infty}\left(\frac{\gamma'(\sigma)}{\gamma(\sigma)-\gamma(s)} - \frac{1}{\sigma - s}\right)x(\gamma(\sigma))\,d\sigma,
$$

has only fixed singularities. We approximate the integral $\int_{-\infty}^{+\infty} x(\sigma)\,d\sigma$ by the quadrature rule

$$
\int_{-\infty}^{+\infty} x(\sigma)\,d\sigma \sim \sum_{j\in\mathbf{Z}} x(\sigma_j^{(n)})\frac{1}{n}, \qquad \sigma_j^{(n)} = \frac{j+\delta}{n}, \quad j \in \mathbf{Z}.
$$

Applying this formula to the values of the regularized singular integral at the points $s_k^{(n)} = (k+\varepsilon)/n$, see [**12, 14, 17**], we observe that

$$
\begin{aligned}
(1.11)\quad \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{x(\gamma(\sigma))d\sigma}{\sigma - s_k^{(n)}} &= \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{x(\gamma(\sigma)) - x(\gamma(s_k^{(n)}))}{\sigma - s_k^{(n)}}\,d\sigma \\
&\sim \frac{1}{\pi i}\sum_{j\in\mathbf{Z}}\frac{x(\gamma(\sigma_j^{(n)}))}{\sigma_j^{(n)} - s_k^{(n)}}\cdot\frac{1}{n} - x(\gamma(s_k^{(n)}))i\cot(\pi(\varepsilon-\delta)).
\end{aligned}
$$

Now one can evaluate both sides of the equation (1.10) at the points $s_k^{(n)}$, $k \in \mathbf{Z}$. Approximating $(S_{\mathbf{R}}x)(s_k^{(n)})$, $k \in \mathbf{Z}$, by (1.11) and replacing $x(\gamma(s_k^{(n)}))$ by $x(\gamma(\sigma_k^{(n)}))$ we obtain the following system of linear algebraic equations

$$
\begin{cases}
(a - ib\cot(\pi(\varepsilon-\delta)))x(\sigma_k^{(n)}) \\
\quad + \dfrac{b}{\pi i}\sum_{j\in\mathbf{Z}}\dfrac{x(\gamma(\sigma_j^{(n)}))}{\sigma_j^{(n)} - s_k^{(n)}}\dfrac{1}{n} \\
\quad + \dfrac{b}{\pi i}\sum_{j\in\mathbf{Z}}\left(\dfrac{\gamma'(\sigma_j^{(n)})}{\gamma(\sigma_j^{(n)}) - \gamma(s_k^{(n)})} - \dfrac{1}{\sigma_j^{(n)} - s_k^{(n)}}\right) \\
\qquad\qquad \cdot x(\gamma(\sigma_j^{(n)}))\dfrac{1}{n} = f(\gamma(s_k^{(n)})), \\
\qquad k \in \mathbf{Z}.
\end{cases}
$$

Denoting by $\xi_j^{(n)}$, $j \in \mathbf{Z}$, the terms $x(\gamma(\sigma_j^{(n)}))$ we see that the latter system is nothing else than (1.5).

Remark 1. In case $\varepsilon = \delta$ the points $s_j^{(n)}$ and $\sigma_j^{(n)}$ coincide. Therefore, we approximate the values of the singular integral $(S_{\mathbf{R}}x)(s_k^{(n)})$ by

$$\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{x(\gamma(\sigma))d\sigma}{\sigma - s_k^{(n)}} \sim \frac{1}{\pi i}\sum_{\substack{j\in\mathbf{Z}\\ j\neq k}}\frac{x(\gamma(s_j^{(n)}))}{s_j^{(n)} - s_k^{(n)}}\cdot\frac{1}{n} + \frac{1}{n}\frac{1}{\pi i}x'(\gamma(s_k^{(n)}))$$

and neglect the small terms $x'(\gamma(s_k^{(n)}))/(i\pi n)$, $k \in \mathbf{Z}$. This leads to the system

$$\begin{cases} a\xi_k^{(n)} + \dfrac{b}{\pi i}\sum_{\substack{j\in\mathbf{Z}\\ j\neq k}}\dfrac{\Delta t_j^{(n)}}{t_j^{(n)} - t_k^{(n)}}\xi_j^{(n)} = f(t_k^{(n)}) \\ k \in \mathbf{Z}, \end{cases} \tag{1.12}$$

where $t_k^{(n)}$, $k \in \mathbf{Z}$, are defined by (1.3).

Let $\tilde{l}^2_\nu$, $\nu \in \mathbf{R}$, stand for the Banach space of all sequences $\{\xi_k\}_{k\in\mathbf{Z}}$ of numbers $\xi_k \in \mathbf{C}$, $k \in \mathbf{Z}$, endowed with the norm

$$\|\{\xi_k\}_{k\in\mathbf{Z}}\|_{\tilde{l}^2_\nu} := \left(\sum_{k\in\mathbf{Z}}|\xi_k|^2(|k|+1)^{2\nu}\right)^{1/2}.$$

In the case $\nu = 0$, we simply write $\tilde{l}^2$.

Analogously to [**3**], one can show that there exists a number $m > 0$ satisfying

$$\begin{aligned}\frac{1}{m}\left\|\sum_{k\in\mathbf{Z}}\xi_k^{(n)}\varphi_k^{(n)}\right\|_{L^2(\Gamma_{\omega,\rho})} &\leq n^{-1/2-\rho}\|\{\xi_k^{(n)}\}_{k\in\mathbf{Z}}\|_{\tilde{l}^2_{-\rho}} \\ &\leq m\left\|\sum_{k\in\mathbf{Z}}\xi_k^{(n)}\varphi_k^{(n)}\right\|_{L^2(\Gamma_{\omega,\rho})},\end{aligned} \tag{1.13}$$

where $\varphi_k^{(n)}$, $k \in \mathbf{Z}$, are defined by (1.7).

Clearly we can now identify the system (1.5), and also (1.12), with an operator equation of the form

$$A_n \xi_n = \eta_n, \tag{1.14}$$

where $\xi_n = \{\xi_k^{(n)}\}_{k\in\mathbf{Z}}$, $\eta_k = \{f(\tau_k^{(n)})\}_{k\in\mathbf{Z}}$ and $A_n \in \mathcal{L}(\tilde{l}^2_{-\rho})$. Now, using (1.13), it is easily seen that the operator sequence $\{A_n\}_{n\in\mathbf{N}}$ is stable if and only if the same holds for the sequence of the approximate operators $\{\breve{A}_n\}_{n\in\mathbf{N}}$ of (1.9) corresponding to the method (1.5).

Proposition 1. *The sequence $\{A_n\}$ is stable if and only if the operator $A_1 \in \mathcal{L}(\tilde{l}^2_{-\rho})$ is invertible.*

The proof immediately follows from the fact that the matrices A_n are independent of n.

It should be noted that the question of invertibility of the operator A_1 is a serious problem. Up to now there are no effective methods to solve such problems in general. However, a first step towards the solution of the invertibility problem is to find out whether A_1 belongs to some special class of operators and to investigate its Fredholm properties. To do this, we firstly identify the Hilbert space $\tilde{l}^2_{-\rho}$ with the direct sum $l^2_{-\rho} \dot{+} l^2_{-\rho}$ of the Hilbert spaces of one-sided sequences and, secondly, we consider the operators on $\tilde{l}^2_{-\rho}$ as 2×2 matrices the elements of which are operators acting on $l^2_{-\rho}$. Let $\mathcal{T}_{-\rho}$ stand for the smallest closed subalgebra of the algebra $\mathcal{L}(l^2_{-\rho})$ which contains all Toeplitz operators $T(a)$ with piecewise continuous generating functions a. Then $\mathcal{T}^{2\times 2}_{-\rho}$ is a subalgebra of $\mathcal{L}(\tilde{l}^2_{-\rho}) \cong \mathcal{L}(l^2_{-\rho} \dot{+} l^2_{-\rho}) \cong \mathcal{L}(l^2_{-\rho})^{2\times 2}$ consisting of all 2×2 matrices with entries in $\mathcal{T}_{-\rho}$. Notice that a symbol calculus is available for the algebra $\mathcal{T}^{2\times 2}_{-\rho}$, see [**10**]. By means of this, the Fredholm properties of our operator can be studied completely.

Proposition 2. *Let $\rho \in (-1/2, 1/2)$. Then the operator $A_1 = A_1^{\rho,p,\omega}$ defined in* (1.5) *and* (1.14) *belongs to the algebra $\mathcal{T}^{2\times 2}_{-\rho}$ and its symbol* $\operatorname{smb} A_1(t,\mu)$*, $\mu \in [0,1]$, $t \in \mathbf{T}$, $\mathbf{T} := \{t \in \mathbf{C} : |t| = 1\}$, can be written in the following form*

$$\begin{aligned} (\operatorname{smb} A_1)(t,\mu) &\\ &= \begin{pmatrix} a + bf^{(\varepsilon-\delta)}(t) & 0 \\ 0 & a - bf^{(\delta-\varepsilon)}(t) \end{pmatrix}, \quad t \neq 1,\ \mu \in [0,1], \end{aligned} \tag{1.15}$$

$$(\operatorname{smb} A_1)(1,\mu)$$

$$= \begin{pmatrix} a - ib\cot(\pi\theta) & ib(e^{-i(\pi-\omega)\theta}p^{-\theta}/\sin(\pi\theta)) \\ -ib(e^{-i(\omega-\pi)\theta}p^{\theta}/\sin(\pi\theta)) & a + ib\cot(\pi\theta) \end{pmatrix}, \tag{1.16}$$

$$\mu \in [0,1],$$

where

$$f^{(\alpha)}(t) = f^{(\alpha)}(e^{i2\pi s}) = 2\frac{\sin(-\pi\alpha s)}{\sin(-\pi\alpha)}e^{-i\pi\alpha(s-1)} - 1, \quad s \in [0,1],$$

and

$$\theta = \theta(\mu) = \frac{1}{2} - \rho + \frac{i}{2\pi}\ln\frac{\mu}{1-\mu}.$$

Proof. First suppose that $\rho = 0$ and rewrite the operator A_1 as the matrix operator

$$A_1 = \begin{pmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{pmatrix}, \tag{1.17}$$

where $A_1^{ij} \in \mathcal{L}(l^2)$, $i,j = 1,2$. Let us show that each of the operators A_1^{ij}, $i,j = 1,2$, belongs to $\mathcal{T} := \mathcal{T}_0$. Indeed, the operators A_1^{ij}, $i,j = 1,2$, have the form

$$A_1^{11} = (a - ib\cot(\pi(\varepsilon-\delta)))I + \frac{b}{\pi i}\left(\frac{1}{-(k-j)-(\varepsilon-\delta)}\right)_{k,j=0}^{+\infty}, \tag{1.18}$$

$$A_1^{12} = -\frac{b}{\pi i}\left(\frac{1}{(j+1-\delta) - p(k+\varepsilon)e^{i(2\pi-\omega)}}\right)_{k,j=0}^{+\infty}, \tag{1.19}$$

$$A_1^{21} = \frac{b}{\pi i}\left(\frac{1}{(j+\delta) + p^{-1}(-k-1+\varepsilon)e^{i\omega}}\right)_{k,j=0}^{+\infty}, \tag{1.20}$$

$$A_1^{22} = (a + ib\cot(\pi(\delta-\varepsilon)))I + \frac{b}{\pi i}\left(\frac{1}{(k-j)+(\delta-\varepsilon)}\right)_{k,j=0}^{+\infty}. \tag{1.21}$$

It is well known [**14, 17**] that A_1^{11} and A_1^{22} are Toeplitz operators. More precisely,

$$A_1^{11} = T(a + bf^{(\varepsilon-\delta)}), \qquad A_1^{22} = T(a - bf^{(\delta-\varepsilon)}). \tag{1.22}$$

Now let us show that $A_1^{21} \in \mathcal{T}$. To this end we set $y := p^{-1}x$ in the well-known formula [**9**]

$$\frac{1}{\pi i}\frac{1}{1-ye^{i\omega}} = \frac{1}{2\pi i}\int_{\mathrm{Re}\,z=1/2} y^{-z}\Big\{-i\frac{e^{-i(\omega-\pi)z}}{\sin(\pi z)}\Big\}\,dz.$$

Then

$$\frac{1}{\pi i}\frac{1}{1-p^{-1}xe^{i\omega}} = \frac{1}{2\pi i}\int_{\mathrm{Re}\,z=1/4} (p^{-1}x)^{-z}\Big\{-i\frac{e^{-i(\omega-\pi)z}}{\sin(\pi z)}\Big\}\,dz. \tag{1.23}$$

From (1.23) and from the residue theorem, one can conclude that

$$\begin{aligned} &\frac{1}{\pi i}\frac{p^{-1}x}{1-p^{-1}xe^{i\omega}} \\ &\quad = \frac{1}{\pi i}e^{-i\omega} + \frac{e^{-i\omega}}{2\pi i}\int_{\mathrm{Re}\,z=1/4} (p^{-1}x)^{-z}\Big\{-i\frac{e^{-i(\omega-\pi)z}}{\sin(\pi z)}\Big\}\,dz. \end{aligned} \tag{1.24}$$

Combining (1.23) with (1.24) we obtain

$$\begin{aligned} &\frac{1-x}{1-p^{-1}xe^{i\omega}} \\ &\quad = -pe^{-i\omega} + \frac{1-pe^{-i\omega}}{2}\int_{\mathrm{Re}\,z=1/4} x^{-z}\Big\{-i\frac{e^{-i(\omega-\pi)z}p^{z}}{\sin(\pi z)}\Big\}\,dz. \end{aligned} \tag{1.25}$$

The latter relation enables us to present the operator A_1^{21} in the form

$$\begin{aligned} A_1^{21} &= \left(\frac{1-(k+1-\varepsilon)/(j+\delta)}{1-p^{-1}((k+1-\varepsilon)/(j+\delta))e^{i\omega}}\,\frac{1}{\pi i}\,\frac{1}{(j+\delta)-(k+1-\varepsilon)}\right)_{k,j=0}^{\infty} \\ &= -bpe^{-i\omega}\left(\frac{1}{(j+\delta)-(k+1-\varepsilon)}\right)_{k,j=0}^{\infty} \end{aligned}$$

$$
\begin{aligned}
&+ b\bigg(\frac{1-pe^{-i\omega}}{2}\int_{\operatorname{Re} z=1/4}\bigg\{-\,i\frac{e^{-i(\omega-\pi)z}p^z}{\sin(\pi z)}\bigg\}\\
&\qquad\qquad\qquad\cdot\frac{(k+1-\varepsilon)/(j+\delta)^{-z}}{(j+\delta)-(k+1-\varepsilon)}\,dz\bigg)_{k,j=0}^{+\infty}\\
&= K - bpe^{-i\omega}T(f^{(1-\varepsilon-\delta)}) - b\frac{1-pe^{-i\omega}}{2}\\
&\quad\cdot\int_{\operatorname{Re} z=1/4}\bigg\{-\,i\frac{e^{-i(\omega-\pi)z}p^z}{\sin(\pi z)}\bigg\}((k+1-\varepsilon)^{-z}\delta_{kj})_{k,j=0}^{+\infty}\\
&\qquad\qquad\qquad\cdot T(f^{(1-\varepsilon-\delta)})((j+\delta)^{z}\delta_{kj})_{k,j=0}^{+\infty}\,dz,
\end{aligned}
$$

where K is a compact operator.

Given a number $z\in\mathbf{C}$, $\operatorname{Re} z\in(-1,1)$, we define $\Lambda^z := ((k+1)^{-z}\delta_{kj})_{k,j=0}^{+\infty}$, and, similar to [**14, 17**], we denote the operator $\Lambda^{-z}T(f^{(1-\varepsilon-\delta)})\Lambda^z$ by $\mathcal{A}^z$. Now it follows immediately that there exists a compact operator K_1 such that

$$
\begin{aligned}
(1.26)\quad A_1^{21} &= b\frac{1-pe^{-i\omega}}{2}\int_{\operatorname{Re} z=1/4}\bigg\{-\,i\frac{e^{-i(\omega-\pi)z}p^z}{\sin(\pi z)}\bigg\}\mathcal{A}^z\,dz\\
&\qquad - bpe^{-i\omega}T(f^{(1-\varepsilon-\delta)}) + K_1.
\end{aligned}
$$

Since $\mathcal{A}^z\in\mathcal{T}$, [**14, 17**] and all compact operators belong to $\mathcal{T}$, the equality (1.26) gives $\mathcal{A}_1^{21}\in\mathcal{T}$ and

$$
\begin{aligned}
\operatorname{smb} A_1^{21} &= b\frac{1-pe^{-i\omega}}{2}\int_{\operatorname{Re} z=1/4}\bigg\{-\,i\frac{e^{-i(\omega-\pi)z}p^z}{\sin(\pi z)}\bigg\}\operatorname{smb}\mathcal{A}^z\,dz\\
&\qquad - bpe^{-i\omega}\operatorname{smb}\mathcal{A}^0.
\end{aligned}
$$

Extending the function $z\to\operatorname{smb}\mathcal{A}^z$ to a 1-periodic analytic function we can rewrite the latter expression as

$$
\begin{aligned}
\operatorname{smb} A_1^{21} = \frac{b}{2}\bigg\{&\int_{\operatorname{Re} z=1/4}\bigg\{-\,i\frac{e^{-i(\omega-\pi)z}p^z}{\sin(\pi z)}\bigg\}\operatorname{smb}\mathcal{A}^z\,dz\\
&-\int_{\operatorname{Re} z=5/4}\bigg\{-\,i\frac{e^{-i(\omega-\pi)z}p^z}{\sin(\pi z)}\bigg\}\operatorname{smb}\mathcal{A}^z\,dz\bigg\}\\
&- bpe^{-i\omega}\operatorname{smb}\mathcal{A}^0.
\end{aligned}
$$

Applying the residue theorem again and remembering that the function $z \to (\operatorname{smb} \mathcal{A}^z)(t,\mu)$ is constant in the strip $\operatorname{Re} z \in (1/4, 5/4)$, if $t \neq 1$ and has the simple pole at $z_0 = 1/2 + (i/2\pi)\ln \mu/(1-\mu)$ if $t = 1$, we get

$$(1.27)\quad (\operatorname{smb} A_1^{21})(t,\mu) = \begin{cases} 0 & \text{if } t \neq 1 \\ b(-i)\dfrac{e^{-i(\omega-\pi)(\frac{1}{2}+\frac{i}{2\pi}\ln\frac{\mu}{1-\mu})}p^{(\frac{1}{2}+\frac{i}{2\pi}\ln\frac{\mu}{1-\mu})}}{\sin(\pi(\frac{1}{2}+\frac{i}{2\pi}\ln\frac{\mu}{1-\mu}))} & \text{otherwise.} \end{cases}$$

The same arguments can also be applied to A_1^{12}. It turns out that A_1^{12} is in $\mathcal{T}$, too, and

$$(1.28)\quad (\operatorname{smb} A_1^{12})(t,\mu) = \begin{cases} 0 & \text{if } t \neq 1 \\ -b(-i)\dfrac{e^{-i(\pi-\omega)(\frac{1}{2}+\frac{i}{2\pi}\ln\frac{\mu}{1-\mu})}p^{-(\frac{1}{2}+\frac{i}{2\pi}\ln\frac{\mu}{1-\mu})}}{\sin(\pi(\frac{1}{2}+\frac{i}{2\pi}\ln\frac{\mu}{1-\mu}))} & \text{otherwise.} \end{cases}$$

Thus, for $\rho = 0$ the relations (1.15) and (1.16) follow from (1.27), (1.28), (1.22) and from corresponding results of [**10**] for Toeplitz operators. We now turn to the case $\rho \neq 0$. Let us introduce the operator $\Lambda_\sigma^\rho = ((k+\sigma)^\rho \delta_{kj})_{k,j=0}^{+\infty}$, where σ is a fixed real number. Since, for each $\sigma \in (0,1)$, the operators $\Lambda_\sigma^\rho : l^2 \to l^2_{-\rho}$ and $\Lambda_\sigma^{-\rho} : l^2_{-\rho} \to l^2$ are continuously invertible, the operator $A^{21} \in \mathcal{L}(l^2_{-\rho})$ is Fredholm if and only if the operator $B^{21} = \Lambda_\delta^{-\rho} A^{21} \Lambda_{1-\varepsilon}^\rho \in \mathcal{L}(l^2)$ is so. Let ψ be some positive number such that $\psi \pm \rho \in (-1/2, 1/2)$. Then immediately from (1.25) we obtain

$$\frac{1-x}{1-p^{-1}xe^{i\omega}}x^{-\rho} = -pe^{-i\omega}x^{-\rho} + \frac{1-pe^{-i\omega}}{2} \cdot \int_{\operatorname{Re} z=\psi} x^{-(z+\rho)}\left\{-\,i\frac{e^{-i(\omega-\pi)z}p^z}{\sin(\pi z)}\right\}dz.$$

Using this formula we can represent the operator B^{21},

$$B_1^{21} = b\left(\frac{1-(k+1-\varepsilon)/(j+\delta)}{1-p^{-1}((k+1-\varepsilon)/(j+\delta))e^{i\omega}}\,\frac{1}{\pi i}\,\frac{((k+1-\varepsilon)/(j+\delta))^{-\rho}}{(j+\delta)-(k+1-\varepsilon)}\right)_{k,j=0}^{\infty}$$

in the form

$$B_1^{21} = b\frac{1-pe^{-i\omega}}{2}\int_{\operatorname{Re} z=\psi}\left\{-i\frac{e^{-i(\omega-\pi)z}p^z}{\sin(\pi z)}\right\}\mathcal{A}^{z+\rho}\,dz \\ - bpe{-}i\omega\mathcal{A}^{\rho} + K_2$$

with K_2 being some compact operator. Further considerations literally repeat corresponding ones for the case $\rho = 0$. The only difference is that the function $z \to (\operatorname{smb}\mathcal{A}^{z+\rho})(1,\mu)$ has the simple pole $z_0 = 1/2 - \rho + (i/2\pi)\ln\mu/(1-\mu)$ in the strip $\operatorname{Re} z \in (\psi, \psi+1)$. Thus, the proof of our proposition is finished.

Corollary 1. *For all parameters $\rho \in (-1/2, 1/2)$, $\rho \in (0,\infty)$ and $\omega \in (0, 2\pi)$ the operators $A_1 = A_1^{\rho,p,\omega}$ are simultaneously Fredholm or not. When these operators are Fredholm their indices vanish.*

Proof. Since for $t \neq 1$ the symbol $\operatorname{smb} A_1^{\rho,p,\omega}(t,\mu)$ of the operator $A_1^{\rho,p,\omega}$ is independent of the parameters ρ, p and ω, it remains to consider $\operatorname{smb} A_1^{\rho,p,\omega}(1,\mu)$. Straightforward computations give us the following relation

$$\det\{\operatorname{smb} A_1^{\rho,p,\omega}(1,\mu)\} = a^2 - b^2.$$

This completes the proof of the first assertion. To show the second part of the claim we can suppose $\rho = 0$, $p = 1$ and $\omega = \pi$ because $\det\{\operatorname{smb} A_1^{\rho,p,\omega}(t,\mu)\}$ is independent of these parameters. But in this case the operator

$$A_1^{0,1,\pi} = aI + b(f_{k-j}^{\varepsilon-\delta})_{k,j\in\mathbf{Z}} \in \mathcal{L}(\tilde{l}^2)$$

is a convolution operator which is Fredholm if and only if

$$a + bf^{(\varepsilon-\delta)}(t) \neq 0 \quad \text{for any } t \in \mathbf{T}, \tag{1.29}$$

and its index always vanishes.

Corollary 2. *The operators $A_1^{\rho,p,\omega}$, $\rho \in (-1/2, 1/2)$, $p \in (0,\infty)$, $\omega \in (0, 2\pi)$ are Fredholm if and only if*

$$(a+b)(a-b)^{-1} \notin \Gamma^{\varepsilon,\delta}, \tag{1.30}$$

where

$$\Gamma^{\varepsilon,\delta} := \{-te^{-i\pi(\varepsilon-\delta)}, t \in [0,+\infty)\}.$$

In view of Corollary 1 we can again suppose $p = 1$, $\rho = 0$, $\omega = \pi$ and employ the equivalence of conditions (1.29) and (1.30), see [**14, 17**].

Thus the preceding corollaries show that partition change does not influence the Fredholm properties of the approximation operators. But such a step can lead to the change of their invertibility properties.

Remark 2. The approximation method (1.12) can be treated in the same manner. The results obtained are similar to those for method (1.5). More precisely, the symbol of the approximation operator $\hat{A}_1$ for method (1.12) has the form (1.15), (1.16). However, the function $f^{(\alpha)}(t)$ in (1.15) should be replaced by the function

$$\hat{f}(t) = f(e^{i2\pi s}) = 2s - 1, \quad 0 \le s < 1.$$

Remark 3. The proof of Proposition 2 can also be done by the methods of [**12**] because the operator $A \in \mathcal{L}(L^2(\Gamma_\omega, \rho))$ in (1.10) is isometrically isomorphic to the matrix operator

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{11} & A_{22} \end{pmatrix},$$

where $A_{jk} \in \mathcal{L}(L^2(\mathbf{R}^+, \rho))$, $j, k = 1, 2$,

$$A_{11} = aI + bS_{\mathbf{R}^+}, \qquad A_{22} = aI - bS_{\mathbf{R}^+}.$$

Moreover, A_{12} and A_{21} are Mellin convolution operators

$$(\mathcal{M}x)(s) = \int_{\mathbf{R}^+} k\left(\frac{s}{u}\right) x(u) \frac{du}{u}$$

with the kernels

$$k_{12}(t) = \frac{b}{\pi i} \frac{1}{1 - p^{-1}e^{i\omega}t}, \qquad k_{21}(t) = -\frac{b}{\pi i} \frac{1}{1 - pe^{i(2\pi-\omega)}t},$$

respectively.

2. Collocation and qualocation methods for non-equidistant partitions. Fix $n \in N$, and let $\varphi_k^{(n)} = \varphi_k^{(n)}(t)$, $k \in \mathbf{Z}$, be the functions introduced in (1.7). An approximate solution $x_n(t)$ of (1.1) is sought in the form (1.8) but here the coefficients $\xi_k^{(n)}$, $k \in \mathbf{Z}$, of (1.8) are determined by solving the following system

$$Ax_n(\tau_k^{(n)}) = f(\tau_k^{(n)}), \quad k \in \mathbf{Z}, \tag{2.1}$$

where the $\tau_k^{(n)}$, $k \in \mathbf{Z}$ are defined in (1.4). Introduce the interpolation operator $K_{n,\varepsilon}$ by

$$(K_{n,\varepsilon} f)(t) = \sum_{k \in \mathbf{Z}} f(\tau_k^{(n)}) \varphi_k^{(n)}(t).$$

Then one can write the system (2.1) in the form $\tilde{A}_{n,\varepsilon} x_n = f_n$, where

$$\tilde{A}_{n,\varepsilon} := K_{n,\varepsilon} A|_{Im\, L_n} \in \mathcal{L}(Im L_n), \quad f_n = K_n f,$$

and where L_n, $n \in \mathbf{N}$, denote the orthogonal projections onto the subspaces span $\{\varphi_k^{(n)}, k \in \mathbf{Z}\}$. It should also be noted that the matrices $\tilde{A}_{n,\varepsilon}$ are independent of n.

Proposition 3. *Let* $\rho \in (-1/2, 1/2)$. *Then the operator* $\tilde{A}_{1,\varepsilon} = \tilde{A}_{1,\varepsilon}^{\rho,p,\omega}$ *of* (2.1) *belongs to the algebra* $\mathcal{T}_{-\rho}^{2\times 2}$, *and its symbol* $\operatorname{smb} \tilde{A}_{1,\varepsilon}(t,\mu)$, $t \in \mathbf{T}$, $\mu \in [0,1]$, *can be written in the form*

$$\begin{aligned}(\operatorname{smb} \tilde{A}_{1,\varepsilon}^{\rho,p,\omega})(t,\mu) \\ = \begin{pmatrix} a + b\int_0^1 f^{(\varepsilon-\delta)}(t)\, d\delta & 0 \\ 0 & a - b\int_0^1 f^{(\delta-\varepsilon)}(t)\, d\delta \end{pmatrix}\end{aligned} \tag{2.2}$$

for $t \in \mathbf{T}\backslash\{1\}$, $\mu \in [0,1]$, *and*

$$(\operatorname{smb} \tilde{A}_{1,\varepsilon}^{\rho,p,\omega})(1,\mu) = (\operatorname{smb} A_1^{\rho,p,\omega})(1,\mu), \quad \mu \in [0,1]. \tag{2.3}$$

Proof. Let us consider the matrix representation for the operator $\tilde{A}_{1,\varepsilon} = \tilde{A}^{\rho,p,\omega}_{1,\varepsilon}$, which is

$$\tilde{A}_{1,\varepsilon} = aI + b\left(\frac{1}{\pi i}\int_{\Gamma_\omega} \frac{\varphi_j^{(1)}(t)dt}{t-\tau_k^{(1)}}\right)_{k,j\in\mathbf{Z}}. \tag{2.4}$$

Denote by t_k^δ, $k \in \mathbf{Z}$, $\delta \in (0,1)$, the points

$$t_k^\delta = \begin{cases} p(k+\delta) & \text{if } k \geq 0 \\ -(k+\delta)e^{i\omega} & \text{if } k < 0. \end{cases}$$

Then straightforward computations lead to

$$\frac{1}{\pi i}\int_{\Gamma_\omega} \frac{\varphi_j^{(1)}dt}{t-\tau_k^{(1)}} = \frac{1}{\pi i}\int_0^{1/2}\left(\frac{2\varepsilon\Delta t_j^{(1)}}{t_j^{2\varepsilon\delta}-\tau_k^{(1)}} - \frac{2(1-\varepsilon)\Delta t_j^{(1)}}{t_j^{1+2(\varepsilon-1)\delta}-\tau_k^{(1)}}\right)d\delta, \tag{2.5}$$

where

$$\Delta t_j^{(1)} = \begin{cases} p & \text{if } j \geq 0 \\ -e^{i\omega} & \text{if } j < 0. \end{cases} \tag{2.6}$$

Now setting

$$B_\varepsilon^\delta = (a - ib\cot(\pi(\varepsilon-\delta)))I + b\left(\frac{1}{\pi i}\frac{\Delta t_j^{(1)}}{t_j^\delta - \tau_k^{(1)}}\right)_{k,j\in\mathbf{Z}}$$

and using (2.4) and (2.5), we represent the initial operator $\tilde{A}^{\rho,p,\omega}_{1,\varepsilon}$ in the form

$$\tilde{A}^{\rho,p,\omega}_{1,\varepsilon} = \int_0^{1/2} (2\varepsilon B_\varepsilon^{2\varepsilon\delta} + 2(\varepsilon-1)B_\varepsilon^{1+2(\varepsilon-1)\delta})\, d\delta. \tag{2.7}$$

Let us consider the operator-function $R(\delta) = 2\varepsilon B_\varepsilon^{2\varepsilon\delta} + 2(\varepsilon-1)$ $B_\varepsilon^{1+2(\varepsilon-1)\delta}$, $\delta \in [0,1/2]$. By Proposition 2, the operators $B_\varepsilon^{2\delta\varepsilon}$ and $B_\varepsilon^{1+2(1-\varepsilon)\delta}$ belong to $\mathcal{T}^{2\times 2}_{-\rho}$. In addition, the function $R : [0,1/2] \to \mathcal{T}^{2\times 2}_{-\rho}$ is continuous, because

$$\frac{2\varepsilon}{t_j^{2\varepsilon\delta}-\tau_k^{(1)}} - \frac{2(1-\varepsilon)}{t_j^{1+2(1-\varepsilon)\delta}-\tau_k^{(1)}} = 0, \quad \text{if } k = j,$$

and
$$\left|\frac{1}{t_j^{2\varepsilon\delta_1}-\tau_k^{(1)}}-\frac{1}{t_j^{2\varepsilon\delta_2}-\tau_k^{(1)}}\right| \le \frac{c_1|\delta_1-\delta_2|}{|j-k|^2}, \quad \text{if } k \neq j,$$

with some constant c_1 dependent on p, ε and ω only. Therefore, the operator $\tilde{A}_{1,\varepsilon}^{\rho,p,\omega}$ also belongs to the algebra $\mathcal{T}_{-\rho}^{2\times 2}$ and its symbol $(\operatorname{smb}\tilde{A}_{1,\varepsilon}^{\rho,p,\omega})(t,\mu)$, $t \in \mathbf{T}$, $\mu \in [0,1]$, has the form

$$(\operatorname{smb}\tilde{A}_{1,\varepsilon}^{\rho,p,\omega})(t,\mu) = \int_0^{1/2} (2\varepsilon(\operatorname{smb} B_\varepsilon^{2\varepsilon\delta})(t,\mu) + 2(\varepsilon-1)(\operatorname{smb} B_\varepsilon^{1+2(\varepsilon-1)\delta})(t,\mu))\, d\delta. \tag{2.8}$$

Remembering (1.15) and (1.16), we obtain (2.2) and (2.3).

Corollary 3. *For all parameters $\rho \in (-1/2,1/2)$, $p \in (0,\infty)$ and $\omega \in (0,2\pi)$, the operators $\tilde{A}_{1,\varepsilon}^{\rho,p,\omega}$ are simultaneously Fredholm or not. When these operators are Fredholm their indices vanish.*

Now let Γ^ε be the curve

$$\Gamma^\varepsilon := \{(\psi_\varepsilon(s)-1)/\psi_\varepsilon(s), s \in [0,1]\},$$

where $\psi_\varepsilon(s)$ is

$$\psi_\varepsilon(s) = \int_0^1 e^{-\pi(\varepsilon-\delta)(s-1)} \frac{\sin(-\pi(\varepsilon-\delta)\mu)}{(-\pi(\varepsilon-\delta))}\, d\delta.$$

Corollary 4. *The operators $\tilde{A}_{1,\varepsilon}^{\rho,p,\omega}$ are Fredholm if and only if $(a+b)(a-b)^{-1} \in \Gamma^\varepsilon$.*

Note that the proofs of Corollaries 3 and 4 literally repeat the argumentation of the proofs of Corollaries 1 and 2.

Now we would like to apply the above results to the study of the qualocation method for the equation (1.1).

Let $0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{m-1}$ and $w_r \in (0,1)$, $r = 1,2,\ldots,m-1$, be real numbers such that $\sum_{r=1}^{m-1} w_r = 1$, and let Q_n designate the quadrature rule on Γ_ω:

$$Q_n(g) = \sum_{k\in\mathbf{Z}} \frac{\Delta t_k^{(n)}}{n} \sum_{r=0}^{m-1} w_r g(\tau_k^{n,\varepsilon_r}), \tag{2.9}$$

where

$$\tau_k^{n,\varepsilon_r} = \begin{cases} p(k+\varepsilon_r)/n & \text{if } k \geq 0 \\ -((k+\varepsilon_r)/n)e^{i\omega} & \text{if } k < 0, \end{cases}$$

and $\Delta t_k^{(n)}$ are defined by (1.6).

The approximate solution $x_n(t)$ of (1.1) is again sought in the form (1.8) but according to the qualocation method [**12, 21, 22**] the coefficients $\xi_k^{(n)}$, $k \in \mathbf{Z}$, are defined by solving the following operator equation

$$Q_n(\varphi_j^{(n)} A x_n) = Q_n(\varphi_j^{(n)} f), \quad j \in \mathbf{Z}.$$

The latter equation is equivalent to the following system of linear algebraic equations

$$\sum_{r=0}^{m-1} w_r A x_n(\tau_k^{n,\varepsilon_r}) = \sum_{r=0}^{m-1} w_r f(\tau_k^{n,\varepsilon_r}), \quad k \in \mathbf{Z}. \tag{2.10}$$

By $A_{n,Q}$ we denote the operator which corresponds to (2.10).

Proposition 4. *Let $\rho \in (-1/2, 1/2)$. Then*

(i) *the matrices of the operator $A_{n;Q}$, $n \in \mathbf{N}$, are independent of n;*

(ii) *the operator $A_Q = A_{1,Q}^{\rho,p,\omega}$ belongs to the algebra $\mathcal{T}_{-\rho}^{2\times 2}$, and its symbol* $\operatorname{smb} A_Q(t,\mu)$, $t \in \mathbf{T}$, $\mu \in [0,1]$, *can be written in the form*

$$\begin{aligned} &(\operatorname{smb} A_Q^{\rho,p,\omega})(t,\mu) \\ &= \begin{pmatrix} a + b\sum_{r=0}^{m-1} \omega_r \int_0^1 f^{(\varepsilon_r-\delta)}(t)\,d\delta & 0 \\ 0 & a - b\sum_{r=0}^{m-1} \omega_r \int_0^1 f^{(\delta-\varepsilon_r)}(t)\,d\delta \end{pmatrix} \end{aligned} \tag{2.11}$$

for $t \in \mathbf{T}\backslash\{1\}$, $\mu \in [0,1]$, and

$$\begin{aligned} (\operatorname{smb} A_Q^{\rho,p,\omega})(1,\mu) &= (\operatorname{smb} \tilde{A}_{1,\varepsilon}^{\rho,p,\omega})(1,\mu) \\ &= (\operatorname{smb} A_1^{\rho,p,\omega})(1,\mu), \mu \in [0,1]. \end{aligned} \tag{2.12}$$

(iii) *The operator* $A_Q^{\rho,p,\omega}$ *is Fredholm if and only if* $(a+b)(a-b)^{-1} \notin \Gamma^Q$, *where*

$$\Gamma^Q := \left\{ \left(\sum_{r=0}^{m-1} w_r \psi_{\varepsilon_r}(s) - 1 \right) \Big/ \sum_{r=0}^{m-1} w_r \psi_{\varepsilon_r}(s), s \in [0,1] \right\}$$

and ψ_{ε_r}, $r = 1, 2, \dots, m-1$, *are as above.*

The proof of Proposition 4 follows immediately from Proposition 3 because

$$A_Q = \sum_{r=0}^{m-1} w_r \tilde{A}_{1,\varepsilon_r}.$$

3. Approximation methods with non-equidistant partitions on curves with corners. Let Γ be a simple closed piecewise smooth curve in the complex plane $\mathbf{C}$, and let $\tilde{\gamma} : \mathbf{R} \to \Gamma$ be some 1-periodic continuous parametrization of Γ. By $\mathcal{C}$ we denote the set $\{\tilde{t}_0, \tilde{t}_1, \dots, \tilde{t}_{\tilde{m}_0}\}$ of all corner points of Γ, and let $\tilde{s}_j$, $j = 1, 2, \dots, \tilde{m}_0$, be real numbers such that $0 < \tilde{s}_1 < \tilde{s}_2 < \dots < \tilde{s}_{\tilde{m}_0} < 1$ and

$$\tilde{\gamma}(\tilde{s}_j) = \tilde{t}_j, \quad j = 1, 2, \dots, \tilde{m}_0.$$

From now on, we make the following assumptions concerning the function $\tilde{\gamma}$:

(i) $\tilde{\gamma}$ is twice continuously differentiable on each open interval $(\tilde{s}_j, \tilde{s}_{j+1})$, $j = 1, 2, \dots, \tilde{m}_0$, where $\tilde{s}_{\tilde{m}_0+1} := \tilde{s}_1 + 1$.

(ii) There exist finite one-sided limits $\tilde{\gamma}'(\tilde{s}_j \pm 0)$ and $\tilde{\gamma}''(\tilde{s}_j \pm 0)$ for all $j = 1, 2, \dots, \tilde{m}_0$.

In this section we restrict our attention to the study of a quadrature method for the equation (0.1), which is based on non-equidistant meshes on Γ. To this end, we choose some finite set $\mathcal{B}$ of points $t_1^*, t_2^*, \dots, t_{m_0^*}^*$ on Γ, which will be viewed as the break points of the mesh sequence. Now the sequence of meshes will be constructed in such a way that

$$\lim_{n \to \infty} r_n(t_k^*) = p_k^*, \quad k = 1, 2, \dots, m_0^*,$$

where p_k^*, $k = 1, 2, \dots, m_0^*$ are some fixed numbers and $p_k^* \neq 1$ for any $k = 1, 2, \dots, m_0^*$. Note that an algorithm of construction of the meshes with prescribed properties will be given in Section 4.

By $\mathcal{M}$ we denote the union of the sets $\mathcal{C}$ and $\mathcal{B}$, i.e.,

$$\mathcal{M} := \mathcal{C} \cup \mathcal{B},$$

$\mathcal{M} = \{t_1, t_2, \dots, t_{m_0}\}$ and let s_j, $j = 1, 2, \dots, m_0$, be the real numbers such that $0 < s_1 < s_2 < \cdots < s_{m_0} < 1$ and

$$\tilde{\gamma}(s_j) = t_j, \quad j = 1, 2, \dots, m_0.$$

Due to the assumptions (i) and (ii) we can choose a 1-periodic parametrization $\gamma : \mathbf{R} \to \Gamma$ of the curve Γ such that:

(i) $t_j = \gamma(j/m_0)$, $j = 1, 2, \dots, m_0$;

(ii) the parametrization γ is twice continuously differentiable on each interval $(j/m_0, (j+1)/m_0)$, $j = 1, 2, \dots, m_0$;

(iii) there exist finite one-sided limits $\gamma'(j/m_0 \pm 0)$ and $\gamma''(j/m_0 \pm 0)$ for any $j = 1, 2, \dots, m_0$;

(iv) $|\gamma'(j/m_0 + 0)| = p_j |\gamma'(j/m_0 - 0)|$, $j = 1, 2, \dots, m_0$, where

$$p_j = \begin{cases} p_k^* & \text{if } \gamma(j/m_0) = t_k^*,\ t_k^* \in \mathcal{B} \\ 1 & \text{if } \gamma(j/m_0) \notin \mathcal{B}. \end{cases}$$

Thus the point $t_0 = \gamma(s_0)$ of Γ belongs to the set $\mathcal{M}$ if and only if at least one of the following two conditions is fulfilled:

(i) $\arg \gamma'(s_0 + 0) \neq \arg \gamma'(s_0 - 0)$;

(ii) $|\gamma'(s_0 + 0)| \neq |\gamma'(s_0 - 0)|$.

Let us define the function $\nu := \nu(t)$ by the relation

$$\nu(t) := \prod_{j=1}^{m_0} |t - t_j|^{\rho_j}, \quad t \in \Gamma,$$

where $\rho_j \in (-1/2, 1/2)$, $j = 1, 2, \dots, m_0$. In the following the operator A defined by (0.1) will be considered as an operator acting in the space $L^2_\nu := L^2_\nu(\Gamma)$ of all Lebesgue measurable functions x such that

$$\|x\|_{L^2_\nu(\Gamma)} := \left(\int_\Gamma |x(t)|^2 \nu^2(t) |dt| \right)^{1/2}.$$

Let $n = lm_0$, $l \in \mathbf{N}$, and let $\varepsilon, \delta \in (0,1)$ be real numbers such that $\varepsilon \neq \delta$. For each $k = 1, 2, \dots, n-1$, we put

$$t_k^{(n)} = \gamma\left(\frac{k+\delta}{n}\right), \qquad \tau_k^{(n)} = \gamma\left(\frac{k+\varepsilon}{n}\right).$$

Then, due to the choice of the parameterization γ, the sequence of meshes $\{\mathcal{P}_n\}_{n\in N}$, where $\mathcal{P}_n := \{t_0^{(n)}, t_1^{(n)}, \dots, t_{n-1}^{(n)}\}$, has the property that

$$\lim_{n\to\infty} r_n(t) = \begin{cases} p_k^* & \text{if } t = t^* \in \mathcal{B} \\ 1 & \text{if } t \notin \mathcal{B}. \end{cases}$$

We will determine the approximate values $\xi_k^{(n)}$, $k = 0, 1, \dots, n-1$, for the exact solution of the equation (0.1) at the points $t_k^{(n)}$, $k = 0, 1, \dots, n-1$, by solving the following discrete system:

$$\text{(3.1)} \qquad \begin{cases} (a(\tau_k^{(n)}) - b(\tau_k^{(n)}) i \cot(\pi(\varepsilon - \delta)))\xi_k^{(n)} \\ \quad + \dfrac{1}{\pi i} \sum_{j=0}^{n-1} \dfrac{b(\tau_k^{(n)})\Delta t_j^{(n)}}{t_j^{(n)} - \tau_k^{(n)}} \xi_j^{(n)} \\ \quad + \sum_{j=0}^{n-1} k(\tau_k^{(n)}, t_j^{(n)}) \Delta t_j^{(n)} \xi_j^{(n)} = g(\tau_k^{(n)}), \\ \qquad\qquad k = 0, 1, \dots, n-1, \end{cases}$$

where $\Delta t_j^{(n)} = \gamma((j+1)/n) - \gamma(j/n)$. If $\chi_j^{(n)}$ stands for the characteristic function of the arc joining $\gamma(j/n)$ to $\gamma((j+1)/n)$, then the approximate solution of (0.1) is given by

$$\text{(3.2)} \qquad x_n(t) = \sum_{j=0}^{n-1} \xi_j^{(n)} \chi_j^{(n)}(t).$$

The stability conditions for the quadrature method (3.1)–(3.2) can be obtained with the help of localizing techniques for approximate methods, cf. [**12**]. Roughly speaking, it means that we assign a family of simpler problems to the original problem. Namely, with each point $\tau \in \Gamma$, we connect a quadrature method (local model) of the form (1.5). The parameters of this one reflect the local behavior of the original problem at the point τ. More precisely, let $\tau \in \Gamma$, and let $s_\tau \in [0,1)$

be chosen in such a way that $\tau = \gamma(s_\tau)$. By ω_τ, p_τ and ρ_τ we denote the following real numbers

$$\begin{aligned}
\omega_\tau &= \arg(-\gamma'(s_\tau - 0)/\gamma'(s_\tau + 0)), \quad \omega_\tau \in (0, 2\pi),\\
p_\tau &= |\gamma'(s_\tau + 0)/\gamma'(s_\tau - 0)|,\\
\rho_\tau &= \begin{cases} \rho_j & \text{if } \tau = t_j,\ j = 1, 2, \dots, m_0,\\ 0 & \text{if } \tau \notin \mathcal{M}. \end{cases}
\end{aligned}$$

Further, we consider the approximate method (1.5)–(1.8) with the underlying operator

$$A^\tau = a(\tau)I + b(\tau)S_{\Gamma_{\omega_\tau}},$$

which can be viewed as an operator acting in the space $L^2(\Gamma_{\omega_\tau}, \rho_\tau)$.

Let A_τ stand for the associated operator of the form (1.17)–(1.21), where a, b, p and ω are equal to $a(\tau), b(\tau), p_\tau$ and ω_τ, respectively, and let $R = R(\Gamma)$ be the set of all Riemann integrable functions on Γ. The remainder of this section will be devoted to the proof of the following result.

Theorem 1. *Let $a, b \in C(\Gamma)$ and $k \in C(\Gamma \times \Gamma)$. Then*

(i) *the approximate method* (3.1) *is stable if and only if the operator $A \in \mathcal{L}(L^2_\nu(\Gamma))$ and the corresponding operators $A_\tau \in \mathcal{L}(l^2_{-\rho_\tau})$, $\tau \in \mathcal{M}$, are invertible*;

(ii) *the systems* (3.1) *are uniquely solvable for n sufficiently large, and the approximate solutions* (3.2) *converge to the exact solution of the equation* (0.1) *in the norm of $L^2_\nu(\Gamma)$ under the assumption that the method* (3.1) *is stable and $f \in R(\Gamma)$.*

As was said before, the proof of this theorem can be done by using localization methods. For the convenience of the reader, we recall some results. However, from now on, all operators under consideration are supposed to act in spaces without weight. Note that the general situation can again be translated to this one, see, for example, Section 1.

Let $L_n \in \mathcal{L}(L^2(\Gamma))$ be the orthogonal projection onto the subspace (3.2). It is well known, see, e.g., [**12, 17**], that L_n converges strongly to the identity operator as $n \to \infty$. By $\mathcal{A}$ we refer to the set of all sequences $\{A_n\}$ of all continuous operators $A_n : \operatorname{im} L_n \to \operatorname{im} L_n$ such that there exists an operator $A \in \mathcal{L}(L^2(\Gamma))$ for which $A_n L_n \to A$ and $A_n^* L_n^* \to A^*$ strongly as $n \to \infty$. In addition, let $\mathcal{J}$ be the collection of all sequences having the form $\{L_n K L_n + G_n\}$ where K belongs to the set $\mathcal{K}(L^2)$ of all compact operators, and where $G_n : \operatorname{im} L_n \to \operatorname{im} L_n$ and $\|G_n\| \to 0$ as $n \to \infty$. Then it is easily seen that $\mathcal{A}$ becomes a Banach algebra if it is provided with the natural operations of a linear space and with the norm

$$\|\{A_n\}\| = \sup_n \|A_n L_n\|.$$

The set $\mathcal{J}$ is a closed ideal of $\mathcal{A}$. The next result gives us a powerful tool for proving stability of many numerical methods we know, cf. [**12**].

Theorem 2. *Let $\{A_n\} \in \mathcal{A}$ and $A_n L_n \to A$ strongly. The sequence $\{A_n\}$ is stable if and only if the operator A is invertible in $\mathcal{L}(L^2(\Gamma))$ and if the coset $\{A_n\}^0 := \{A_n\} + \mathcal{J}$ is invertible in the quotient-algebra $\mathcal{A}/\mathcal{J}$.*

From now on, we denote by $\{A_n\}$ the sequence of operators $A_n : \operatorname{im} L_n \to \operatorname{im} L_n$ corresponding to numerical method (3.1) and start with proving the sufficiency part of Theorem 1. As the sequence $\{A_n\} \in \mathcal{A}$ (what immediately follows from [**12, 14**]), then in view of Theorem 2 we have to show the invertibility of $\{A_n\}^0$ in $\mathcal{A}/\mathcal{J}$. It can be done by means of the local principle of Gohberg and Krupnik. For the convenience of the reader, we recall some facts concerning this subject, for details see [**11**].

Suppose that $\mathcal{U}$ is a Banach algebra with identity e. A subset $M_\tau \subset \mathcal{U}$ is said to be a localizing class if $0 \notin M_\tau$ and if, for any $u_1, u_2 \in M_\tau$ there exists an element $u \in M_\tau$ such that $u_l u = u u_l$, $l = 1, 2$. A system $\{M_\tau\}_{\tau \in T}$ is called covering if, for each collection $\{u_\tau\}_{\tau \in T}$, $u_\tau \in M_\tau$, there is a finite subset of elements u_τ the sum of which is invertible.

An element $x \in \mathcal{U}$ is said to be M_τ-invertible if there are elements $z_1, z_2 \in \mathcal{U}$ and $u_1, u_2 \in M_\tau$ such that

$$u_1 x z_1 = u_1, \qquad u_2 x z_2 = u_2,$$

and two elements $x, y \in \mathcal{U}$ are called M_τ-equivalent if

$$\inf_{u \in M_\tau} \|u(x-y)\| = \inf_{u \in M_\tau} \|(x-y)u\| = 0.$$

Theorem 3 [**11**]. *Let $\{M_\tau\}_{\tau\in T}$ be a covering system of localizing classes and let, for all $\tau \in T$, the element $x \in \mathcal{U}$ be M_τ-equivalent to $x_\tau \in \mathcal{U}$. If x commutes with all elements in $\cup_{\tau\in T} M_\tau$, then x is invertible in $\mathcal{U}$ if and only if the elements x_τ are M_τ-invertible for all $\tau \in \mathcal{U}$.*

Now we can return to the proof of Theorem 1. First of all, we have to find a covering system of localizing classes in $\mathcal{A}$. To do this, we consider the interpolation projection K_n^ε which sends each Riemann integrable function f on Γ into the function

$$K_n^\varepsilon f(t) = \sum_{j=0}^{n-1} f(\tau_j^{(n)})\chi_j^{(n)}(t)$$

where $\chi_j^{(n)}$, $j = 0, 1, \ldots, n-1$, are the characteristic functions of the arcs $[\gamma(j/n), \gamma((j+1)/n))$. Let $\tau \in \Gamma$, and let $\mathcal{L}_\tau$ be the set of all real-valued Lipschitz functions h on Γ taking values between 0 and 1 and such that $h(t) = 1$ in some neighborhood of τ. By h_n we denote the operator $K_n^\varepsilon f L_n$.

Lemma 1. *The set $\{M_\tau\}_{\tau\in\Gamma}$, $M_\tau := \{\{h_n\}^0, h \in L_\tau\}$ is a covering system of localizing classes in $\mathcal{A}/\mathcal{J}$, and the elements of $\cup_{\tau\in\Gamma} M_\tau$ commute with the coset $\{A_n\}^0$ in $\mathcal{A}/\mathcal{J}$.*

The proof of the first assertion of this lemma immediately follows from the definition of the operators $h_n, h \in L_\tau$. To prove the second assertion of Lemma 1 we can use the methods presented in [**15, 17**], see, for example, [**17**, p. 407] or, alternatively, quote the assertion (a) of Proposition 5.12 from [**12**].

Let $P_n \in \mathcal{L}(\tilde{l}^2)$ stand for the projection $P_n : (x_k)_{k\in\mathbf{Z}} \to (y_k)_{k\in\mathbf{Z}}$ with

$$y_k = \begin{cases} x_k & \text{if } -n/2 < k \le n/2 \\ 0 & \text{otherwise.} \end{cases}$$

For fixed $\tau \in \Gamma$ and $n \in \mathbf{Z}^+$ we introduce the operator $W_n^\tau \in \mathcal{L}(\operatorname{im} P_n, \operatorname{im} L_n)$ in the following way: first we define the number $j(\tau, n) \in \{0, 1, \ldots, n-1\}$ as that index for which $\tau \in (\gamma(j(\tau, n)/n),$

$\gamma((j(\tau,n)+1)/n)]$. Then we set

$$W_n^\tau(\delta_{jk})_{k\in\mathbf{Z}} = \chi^{(n)}_{j+j(\tau,n)}.$$

It is evident that $W_n^\tau : \operatorname{im} P_n \to \operatorname{im} L_n$ is a one-to-one mapping and applying (1.13) we conclude that

$$\|W_n^\tau\|\,\|W_{-n}^\tau\| < +\infty, \tag{3.3}$$

where W_{-n}^τ is the inverse operator to W_n^τ. Let us now consider the sequences $\{W_{-n}^\tau A_n W_n^\tau\}$ and $\{(W_{-n}^\tau A_n W_n^\tau)^*\}$. Calculating the strong limits of both of them we find that

$$s-\lim W_{-n}^\tau A_n W_n^\tau = A_\tau, \qquad s-\lim(W_{-n}^\tau A_n W_n^\tau)^* = A_\tau^*, \tag{3.4}$$

where A_τ is defined above.

Let $\mathbf{T}$ be the unit circle, $\mathbf{T} := \{t\in\mathbf{C} : |t|=1\}$. By $C_1(\mathbf{T})$ we denote the set of all functions f of $C(\mathbf{T})\backslash\{1\}$ which have finite limits as t tends to 1 ± 0. Given a function $f\in C_1(\mathbf{T})$ with the Fourier coefficient sequence $\{f_k\}$, we let $T^0(f)$ denote the operator

$$T^0(f) := (f_{j-k})_{j,k\in\mathbf{Z}}.$$

Then $\operatorname{alg}(T^0(f),P)$ refers to the smallest closed subalgebra of $\mathcal{L}(\tilde{l}^2)$ which contains all operators $T^0(f)$ with $f\in C_1(\mathbf{T})$ and the projection $P:\tilde{l}^2\to\tilde{l}^2$,

$$P(x_k) = \begin{cases} x_k & \text{if } k\ge 0\\ 0 & \text{if } k<0.\end{cases}$$

Now we are going to show that all the operators A_τ, $\tau\in\Gamma$, belong to $\operatorname{alg}^{2\times 2}(T^0(f),P)$. Actually, let G_i, $i=1,2$, stand for the operators

$$G_i = \left(\int_m^{m+1}\int_n^{n+1} k_i\left(\frac{t}{s}\right)\frac{ds}{s}\,dt\right)_{m,n=0}^{+\infty},$$

with the kernels k_1,k_2 being defined as follows

$$k_1(x) = \frac{b}{\pi i}\frac{1}{1-p^{-1}e^{i\omega}x}, \qquad k_2(x) = \frac{b}{\pi i}\frac{1}{1-pe^{i(2\pi-\omega)}x}.$$

Using Corollary 2.1 of [**12**] we represent the operators (1.19)–(1.20) in the form

$$A_1^{12} = G_1 + K_1, \qquad A_1^{21} = G_2 + K_2,$$

with some compact operators K_1, K_2. Since the algebra $\operatorname{alg}(T^0(f), P)$ contains the operators G_1, G_2, see [**12**, Proposition 2.11], as well as all compact operators we get that the operator

$$A_\tau = \begin{pmatrix} T(a + bf^{(\varepsilon-\delta)}) & G_1 + K_1 \\ G_2 + K_2 & T(a - bf^{(\delta-\varepsilon)}) \end{pmatrix}$$

is in $\operatorname{alg}^{2\times 2}(T^0(f), P)$.

Suppose now that the operator A_τ is invertible. Taking into account the inverse closedness of $\operatorname{alg}(T^0(f), P)$ in $\mathcal{L}(\tilde{l}^2)$ we obtain that A_τ^{-1} is in $\operatorname{alg}^{2\times 2}(T^0(f), P)$ as well. Therefore, both of the sequences $\{W_n^\tau A_\tau W_{-n}^\tau\}$, $\{W_n^\tau A_\tau^{-1} W_{-n}^\tau\} \in \mathcal{A}$, [**14, 15**]. In addition, one can write

$$W_n^\tau A_\tau W_{-n}^\tau \cdot W_n^\tau A_\tau^{-1} W_{-n}^\tau = L_n + W_n^\tau A_\tau (I - W_{-n}^\tau W_n^\tau) A_\tau^{-1} W_{-n}^\tau. \tag{3.5}$$

Let us consider the term $W_n^\tau A_\tau (I - W_{-n}^\tau W_n^\tau) A_\tau^{-1} W_{-n}^\tau$. It follows from Lemma 11.10 of [**17**] that, for each $\varepsilon > 0$, there exists a function $h^\tau \in L_\tau$ such that

$$\|(K_n h^\tau L_n) W_n^\tau A_\tau (I - W_{-n}^\tau W_n^\tau) A_\tau^{-1} W_{-n}^\tau\| < \varepsilon. \tag{3.6}$$

Hence, the operator

$$L_n + (K_n h^\tau L_n) W_n^\tau A_\tau (I - W_{-n}^\tau W_n^\tau) A_\tau^{-1} W_{-n}^\tau$$

is invertible if h^τ is chosen in a suitable way. A little thought shows that $\{B_n\} = \{(L_n + (K_n h^\tau L_n) W_n^\tau A_\tau (I - W_{-n}^\tau W_n^\tau) A_\tau^{-1} W_{-n}^\tau)^{-1}\}$ is in $\mathcal{A}$, too. Multiplying (3.5) by $(K_n h^\tau L_n)$ from the left side and by B_n from the right, we obtain that $\{W_n^\tau A_\tau W_{-n}^\tau\}$ is M_τ-invertible from the right. Analogous considerations are used to prove M_τ-invertibility of $\{W_n^\tau A_\tau W_{-n}^\tau\}$ from the right. Thus, if all operators A_τ, $\tau \in \Gamma$, are invertible, then all cosets $\{W_n^\tau A_\tau W_{-n}^\tau\}^0$ are M_τ-invertible in $\mathcal{A}/\mathcal{J}$.

We are left to show that $\{A_n\}^0$ and $\{W_n^\tau A_\tau W_{-n}^\tau\}^0$ are M_τ-equivalent for each $\tau \in \Gamma$, i.e., we have to prove that the following term

$$r(h, n) = \|h_n (A_n - W_n^\tau A_\tau W_{-n}^\tau)\|_{\mathcal{A}/\mathcal{J}} \tag{3.7}$$

can be made sufficiently small by appropriate choice of the function $h \in L_\tau$. Evidently, we can prove this only for operators A generating the initial algebra. Let, for instance, $A = S_\Gamma$. If $\tau \notin \mathcal{M}$, then one may refer to the corresponding results of [**12, 14, 17**]. Let $\tau \in \mathcal{M}$. Without loss of generality, we can suppose that $\tau = \gamma(0)$. Now choose a neighborhood U_τ of τ such that U_τ contains only one point of $\mathcal{M}$, namely, the point τ, and define a mapping $\psi_\tau : U_\tau \to \Gamma_\tau$ by

$$\psi_\tau(\gamma(s)) = \begin{cases} p_\tau s & \text{if } s \geq 0 \\ -e^{i\omega_\tau} s & \text{if } s < 0, \end{cases}$$

with $s \in \mathbf{R}$ satisfying condition $\gamma(s) \in U_\tau$. We also let $V_\tau = \psi_\tau(U_\tau)$. Then the operator $\psi_\tau S_{U_\tau} \psi_\tau^{-1} \chi_{V_\tau} I - SV_\tau$ is compact, see, e.g., [**12**]. Due to this property the term $r(h,n)$ from (3.7) can be made smaller than any prescribed $\varepsilon > 0$ if h is chosen in such a way that $\operatorname{supp} h \subset U_\tau$ and U_τ is sufficiently small, and using Theorem 3 completes the proof of Theorem 1. Note that S_{U_τ} and S_{V_τ} are Cauchy singular integral operators over U_τ and V_τ, respectively.

Thus, the cosets $\{A_n\}^0$ and $\{W_n^\tau A_\tau W_{-n}^\tau\}^0$ are M_τ-equivalent and applying Theorem 3 finishes the proof of the sufficiency part of Theorem 1. The necessity part immediately follows from the existence of the strong limits (3.4).

Remark 4. Both the collocation method and the qualocation one for equation (0.1) which are based on locally non-equidistant meshes of Γ can be handled in the same way.

4. Stability of the adaptive quadrature method. In this section it is shown that the methods which were developed for investigating numerical methods on locally non-equidistant meshes can also be used to study the stability problem of so-called adaptive algorithms for singular integral equations. Let us shortly recall the matter of the problem.

We consider the sequence of the meshes $\{\mathcal{P}_n\}_{n\in\mathbf{N}}$ which are employed to construct a numerical method. The meshes $\mathcal{P}_n$, $n \in \mathbf{N}$, are defined by the collections of the points $\{t_k^{(n)}\}$. To get more precise information about the solution of the initial problem in a neighborhood of some point t^* the meshes $\mathcal{P}_n$, $n \in \mathbf{N}$, are modified in the following way.

Firstly, an arc $l_0(t^*)$ of Γ is chosen such that $t^* \in l_0(t^*)$ and, secondly, to the points $t_k^{(n)} \in l_0(t^*) \cap \mathcal{P}_n$ are added new points of $l_0(t^*)$. It leads to a reduction of the mesh size on $l_0(t^*)$. On the other part $\Gamma \backslash l_0(t^*)$ of Γ the points of the meshes $\mathcal{P}_n$ are retained. As a result, a new sequence of the meshes $\{\mathcal{P}'_n\}$ appears. This one is used now to construct a modified approximation method. If we get insufficient information by using the meshes sequence $\{\mathcal{P}'_n\}$ we choose a new arc $l_1(t^*) \subset l_0(t^*)$ and the mesh on $l_1(t^*)$ is refined again. This process can be repeated many times. Thus, we get a sequence of arcs

$$l_m(t^*) \subset l_{m-1}(t^*) \subset \cdots \subset l_0(t^*).$$

These arcs have the property that, for each $k = 1, 2, \ldots, m$, the density of the mesh on $l_k(t^*)$ is greater than on $l_{k-1}(t^*) \backslash l_k(t^*)$. In connection with this, such arcs $l_k(t^*)$, $k = 1, 2, \ldots, m$, are said to be the condensation arcs of the meshes. It is well known that such methods often reduce the computing costs which are needed to get a given accuracy. We show that stability of these numerical methods can be investigated similarly to the above considerations.

Let $\gamma = \gamma(s)$, $s \in \mathbf{R}$, be some 1-periodic parametrization of the simple closed curve Γ. For simplicity, we assume that γ is twice continuously differentiable on $\mathbf{R}$. We consider the following sequence of the meshes $\{\mathcal{P}_n\}_{n \in \mathbf{N}}$, $\mathcal{P}_n := \{t_0^{(n)}, t_1^{(n)}, \ldots, t_{n-1}^{(n)}\}$,

$$t_j^{(n)} := \gamma\left(\frac{j}{n}\right), \quad j = 1, 2, \ldots, n-1. \tag{4.1}$$

For the point t^* of Γ we choose the first condensation arc $l_0(t^*)$ such that its end points coincide with some points $t_{j_0}^{(n)}$ and $t_{j_1}^{(n)}$ of the mesh (4.1). Let $s_0, s_1 \in (0, 1]$ be such that

$$\gamma(s_0) = t_{j_0}^{(n)}, \qquad \gamma(s_1) = t_{j_1}^{(n)}.$$

Without loss of generality we can assume $s_1 = 1$. Otherwise, we use the parametrization $\gamma_1(s) = \gamma(s + s_1)$. Further, since $s_0 \in \{1/n, 2/n, \ldots, (n-1)/n\}$ there exist the relatively prime numbers k_0 and m_0 such that $s_0 = k_0/m_0$. In the sequel we will consider only the subsequence $\{\mathcal{P}_{lm_0}\}_{l \geq l_0}$ of $\{\mathcal{P}_n\}_{n \in \mathbf{N}}$, where $l_0 = n/m_0$. Now we put $n = lm_0$, $l \geq l_0$. Then, on the arcs $[\gamma(0), \gamma(s_0)) \subset \Gamma$ and $[\gamma(s_0), \gamma(1)) \subset \Gamma$ there are situated pn and qn points of $\mathcal{P}_n$, respectively. It is clear that $p + q = 1$, and the numbers p and q are independent of n. We extend the number of points on the arc $[\gamma(s_0), \gamma(1))$ $r+1$ times with r such that $rq + 1$ is a natural number. For given $n = lm_0$ the mesh on the arc $[\gamma(0), \gamma(s_0))$ is retained.

Now we need a suitable parametrization of Γ which would reflect the changes we made. Such a parametrization can be given, for example, by the following relations

$$\tilde{\gamma}(s) = \begin{cases} \gamma((rq+1)s) & \text{if } s \in [0, s_0/(rq+1)) \\ \gamma\left(\dfrac{(rq+1)}{r+1}s + \dfrac{r(1-q)}{r+1}\right) & \text{if } s \in [s_0/(rq+1), 1]. \end{cases} \tag{4.2}$$

Since the function γ is 1-periodic we can extend the function $\tilde{\gamma}$ to a 1-periodic continuous function on $\mathbf{R}$ which is continuously differentiable on $\mathbf{R}$ except at the points k, $k + s_0$, with $k \in \mathbf{Z}$.

We construct a new sequence of meshes $\{\mathcal{P}^{(1)}_{N_n}\}$. This one has the property that the meshes $\mathcal{P}^{(1)}_{N_n}$ and $\mathcal{P}_n$ coincide on the arc $[\gamma(0), \gamma(s_0))$. However, on the arc $[\gamma(s_0), \gamma(1))$ the mesh $\mathcal{P}^{(1)}_{N_n}$ has $r+1$ times as many points as the mesh $\mathcal{P}_n$ has. We put

$$N_n = (rq+1)n \tag{4.3}$$

and

$$\tilde{t}^{(N_n)}_j := \tilde{\gamma}\left(\frac{j}{(rq+1)n}\right), \quad j = 1, 2, \dots, N_n - 1. \tag{4.4}$$

Then the mesh $\mathcal{P}^{(1)}_{N_n}$ consists of $(rq+1)n$ points and

$$\tilde{t}^{(N_n)}_j = t^{(n)}_j \in [\gamma(0), \gamma(s_0))$$

if $j/n < s_0$. All other points of $\mathcal{P}^{(1)}_{N_n}$ belong to the arc $[\gamma(s_0), \gamma(1))$.

Now we are in a position to formulate the main result about the stability of the adaptive quadrature method. Note that a numerical method is said to be an adaptive one if the meshes defined in (4.3)–(4.4) are used instead of the original ones. It should be mentioned, however, that the concept of adaptivity includes some rules on how to select the meshes in order to get any improvement of the convergence rate. But such problems are not studied in this paper.

Let $A_1^{\rho,p,\omega}(\alpha,\beta)$ refer to the operator (1.17)–(1.22) with $a=\alpha$ and $b=\beta$, and let $\nu=\nu(t)=|t-\gamma(s_0)|^{\rho_0}|t-\gamma(1)|^{\rho_1}$, $\rho_0,\rho_1\in(-1/2,1/2)$.

Theorem 4. *Let $a,b\in C(\Gamma)$ and $k\in C(\Gamma\times\Gamma)$. The adaptive quadrature method* (3.1) *based on meshes* (4.3)–(4.4) *is stable if and only if the operators $A\in\mathcal{L}(L^2_\nu(\Gamma))$, $A_1^{\rho,r+1,\pi}(a(\gamma(s_0)),b(\gamma(s_0)))\in\mathcal{L}(\tilde{l}^2_{-\rho_0})$ and $A_1^{\rho,1/(r+1),\pi}(a(\gamma(1)),b(\gamma(1)))\in\mathcal{L}(\tilde{l}^2_{-\rho_1})$ are invertible.*

Proof. Since the sequence of operators $\{A_{N_n}\}$ which corresponds to the quadrature method under consideration is a subsequence of the sequence $\{A_n\}$ from Section 3, then the sufficiency follows from Theorem 1. To prove the necessity, we use Theorem 1 and the fractal property of the sequence $\{A_n\}$, see [**19**].

Concluding remarks. The reader of this paper could get the impression that the results presented above are of no practical interest. Actually, some conditions of the stability for the methods under consideration are formulated in terms which cannot be effectively verified at present. However, we believe that such a categorical statement is far from being true. In reality, our related investigations, see, e.g., [**12, 17** and references therein], show that objects arising in numerical treatment of singular integral equations have a highly complicated structure in order to be handled in a simple manner. In addition, we are sure that the previous investigations build a foundation for further progress and give us some hints on how to construct more practicable methods. One of the possible strategies consists in using the cut-off techniques, cf. [**12, 18**]. The second one is based on special regularization of the approximation operators. This technique appeared quite recently and it has the advantage of requiring that the corresponding operators A_τ have to be Fredholm only. However, in order to make such ideas transparent and applicable, there is still much to do.

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