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HIGHER ACCURACY METHODS FOR SECOND-KIND VOLTERRA INTEGRAL EQUATIONS BASED ON ASYMPTOTIC EXPANSIONS OF ITERATED GALERKIN METHODS

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Dedicated to Professor Phil Anselone, with our best wishes

ABSTRACT. On the basis of asymptotic expansions, we study the Richardson extrapolation method and two defect correction schemes by an interpolation post-processing technique, namely, interpolation correction and iterative correction for the numerical solution of a Volterra integral equation by iterated finite element methods. These schemes are of higher accuracy than the postprocessing method and analyzed in a recent paper [**5**] by Brunner, Q. Lin and N. Yan. Moreover, we give a positive answer to a conjecture in [**5**].

1. Introduction. In this paper we are concerned with finite element methods for the Volterra integral equation of the second kind,

$$y(t) = g(t) + \int_0^t K(t,s)y(s)\,ds, \quad t \in I := [0,1], \tag{1.1}$$

where $g : I \to \mathbf{R}$ and $K : D \to \mathbf{R}$ (with $D := \{(t,s) : 0 \leq s \leq t \leq 1\}$) denote given (continuous) functions. It is well known that if $K \in C^m(D)$ and $g \in C^m(I)$, the solution y of (1.1) is in $C^m(I)$.

The study of (local) superconvergence properties of collocation methods for Volterra integral equations (1.1) (as well as for second-kind Fredholm integral equations) and of methods for accelerating the convergence orders has received considerable attention since the early 1980s, compare, for example, [**1, 2, 4, 8, 9** and **13**].

In this note we present two defect correction schemes, namely, interpolation correction and iterative correction, for the numerical solution

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of (1.1) by iterated finite element methods in certain piecewise polynomial spaces. The main motivation derives from the results given in [**5**]; we shall show that iterative correction of Galerkin finite element (rather than collocation) solutions leads to a considerably higher global rate of convergence, and we shall prove a conjecture (on the local order of convergence at the mesh points) stated in [**5**]. In Section 2 we derive the asymptotic expansion of the iterated finite element solution. Section 3 deals with Richardson extrapolation in the case of piecewise constant finite element spaces. The next two sections are concerned with interpolation correction and iterative correction, again in the case of piecewise constant elements; these investigations are partly motivated by a recent study of iterative correction schemes for collocation approximations to (1.1), see [**5**]. Finally, in Section 6, we describe some ongoing and future work on high-order correction schemes for nonlinear and weakly singular Volterra integral equations.

2. The asymptotic expansion. The weak form of (1.1) consists in finding $y \in L^2(I)$ such that

$$(y, v) = (g, v) + (Ky, v), \quad \forall\, v \in L^2(I), \tag{2.1}$$

where $(.,.)$ denotes the usual inner product in $L^2(I)$; $K : L^2(I) \to C(I)$ is the Volterra integral operator

$$(Ky)(t) := \int_0^t K(t,s)y(s)\,ds, \quad t \in I.$$

Let $T_h : 0 = t_0 < t_1 < \cdots < t_N = 1$ be a given mesh for the interval I, and denote the finite element space by

$$S_{m-1}^{(-1)}(T_h) := \{u : u|_{\sigma_k} \in P_{m-1}\ (0 \le k \le N-1)\}.$$

Here P_r denotes the space of (real) polynomials of degree not exceeding r, and we have set

$$\sigma_k := \begin{cases} [t_0, t_1] & \text{if } k = 0, \\ (t_k, t_{k+1}] & \text{if } 1 \le k \le N-1, \end{cases}$$

$h_k := t_{k+1} - t_k$, $h := \max_{(k)}\{h_k\}$.

We use the superscript (-1) in the notation for the above finite element space to emphasize that it is not a subspace of $C(I)$. Note that, for ease of notation, we have suppressed the dependence on N, the number of subintervals given by $T_h = T_h^{(N)}$, in $h_k = h_k^{(N)}$ and $h = h^{(N)}$.

Thus, the Galerkin equation for (2.1) reads: Find $u \in S_{m-1}^{(-1)}(T_h)$ such that

$$(2.2) \qquad (u, v) = (g, v) + (Ku, v), \quad \forall\, v \in S_{m-1}^{(-1)}(T_h).$$

Let $P_h : L^2(I) \to S_{m-1}^{(-1)}(T_h)$ be an L^2-projection operator, defined by

$$(y, v) = (P_h y, v), \quad \forall\, v \in S_{m-1}^{(-1)}(T_h).$$

Then the problem (2.2) can be equivalently rewritten: Find $u \in S_{m-1}^{(-1)}(T_h)$ such that

$$(2.3) \qquad u = P_h g + P_h K u.$$

Therefore, the iterated finite element solution, u_{it}, corresponding to the above finite element solution u, is given by

$$u_{it} := g(t) + (Ku)(t), \quad t \in I$$

with

$$(2.4) \qquad P_h u_{it} = u.$$

We now define an interpolatory operator $i_h^{m-1} : L^2(I) \to S_{m-1}^{(-1)}(T_h)$ of degree $(m-1)$ by

$$i_h^{m-1} u|_{\sigma_k} \in P_{m-1},$$

and

$$\int_{\sigma_k} v i_h^{m-1} u \, dt = \int_{\sigma_k} v u \, dt, \quad \forall\, v \in P_{m-1}.$$

Since $S_{m-1}^{(-1)}(T_h)$ is a discontinuous piecewise polynomial space and P_h possesses localization, we have

$$P_h = i_h^{m-1}$$

with
$$\|i_h^{m-1}y - y\|_{0,\infty} \leq Ch^m \|y\|_{m,\infty},$$
where, and elsewhere in this paper, we write, for a given nonnegative integer r,
$$\|v\|_{r,\infty} := \max_{0\leq k\leq r}\{\|v^{(k)}\|_\infty\}.$$

Here C denotes a generic constant whose subsequent meanings will become clear by the context in which it arises. In general, C will depend on the length of the (compact) subinterval $[0, b]$ on which (1.1) is to be solved; without loss of generality, we have assumed that $b = 1$. Note that, if the mesh is nonuniform, then C will also depend on the parameter(s) characterizing the degree of nonuniformity. For example, if we deal with graded meshes of the form

$$t_n = (n/N)^q, \qquad q = q(\alpha) = m/(\nu + 1 - \alpha),$$

when the (continuous) kernel in (1.1) has the form $K(t, s) = (t - s)^{\nu-\alpha}$ with $0 < \alpha < 1$, $\nu \in \mathbf{N}$, $1 \leq \nu < m$, see [**3**], then $C = C(q)$.

Let I and $R(t, s)$ denote the identity operator and the resolvent kernel associated with the given kernel $K(t, s)$ in (1.1), respectively, and let $e_{it} := y - u_{it}$ be the error corresponding to the iterated finite element solution u_{it}. In addition, set

$$\delta(t) := e(t) - (Ke)(t), \quad t \in I,$$

where $e := y - u$ is the finite element error. And thus, from (1.1) and (2.3) we get that

$$\begin{aligned} e = y - u &= (g + Ky) - (i_h^{m-1}g + i_h^{m-1}Ku) \\ &= (I - i_h^{m-1})g + K(y - u) + (Ku - i_h^{m-1}Ku) \\ &= (I - i_h^{m-1})g + Ke + (I - i_h^{m-1})Ku \\ &= (I - i_h^{m-1})u_{it} + Ke, \end{aligned}$$

or
$$e - Ke = (I - i_h^{m-1})u_{it},$$

that is,

$$\delta = (I - i_h^{m-1})u_{it}. \tag{2.5}$$

Using the well-known resolvent equation (or Fredholm identity, see [**4**]),

$$R(t,s) = -K(t,s) + \int_0^t R(\tau,s)K(t,\tau)\,d\tau, \quad (t,s) \in D,$$

it is easy to show, compare also [**2, 4**], that the iterated finite element error may be expressed in terms of the resolvent kernel of (1.1) and the defect δ

$$e_{it}(t) = -\int_0^t R(t,s)\delta(s)\,ds, \quad t \in I, \tag{2.6}$$

where $R(t,s)$ inherits the smoothness of the given kernel $K(t,s)$.

Set

$$(R_h y)(t) := \int_0^t R(t,s)(I - i_h^{m-1})y(s)\,ds. \tag{2.7}$$

Then it follows from (3.1) and (3.2) that

$$\begin{aligned} e_{it}(t) &= -(R_h u_{it})(t) = -[R_h(u_{it} - y + y)](t) \\ &= (R_h e_{it})(t) - (R_h y)(t), \end{aligned} \tag{2.8}$$

which leads to a recurrence formula, see also [**5**],

$$e_{it} = -\sum_{k=1}^{n} R_h^k y + R_h^n e_{it}. \tag{2.9}$$

Lemma 2.1. *In* (1.1), *assume that* $g \in C^m(I)$, $K \in C^m(D)$ *and that the integral operator* R_h *is given by* (2.7). *Then, for the iterated finite element error* e_{it} *we have*

$$\|e_{it}\|_{1,\infty} \le Ch^m \|y\|_{m,\infty}.$$

Proof. It follows from

$$\|R_h y\|_{1,\infty} \le C\|(I - i_h^{m-1})y\|_{0,\infty} \le Ch^m \|y\|_{m,\infty},$$

that, for sufficiently small $h > 0$,

$$\|R_h\|_{C^1\to C^1} := \sup_{y\in C^1(I)} \frac{\|R_h y\|_{1,\infty}}{\|y\|_{1,\infty}} \le Ch \le \sigma < 1.$$

Thus, we know that $(I - R_h)^{-1}$ exists and is uniformly bounded on $C^1(I)$ for all $h \in (0, \tau)$. Therefore, from (2.8) we get that

$$e_{it} = -(I - R_h)^{-1} R_h y,$$

which yields

$$\|e_{it}\|_{1,\infty} = \|(I - R_h)^{-1} R_h y\|_{1,\infty} \le C\|R_h y\|_{1,\infty} \le Ch^m \|y\|_{m,\infty}. \qquad \square$$

Theorem 2.1. *Suppose that the conditions of Lemma* 2.1 *hold. Then,*

$$\|R_h^n e_{it}\|_{0,\infty} \le Ch^{n+m+1}\|y\|_{m,\infty}, \quad n \ge 1. \tag{2.10}$$

Proof. For any $t \in \sigma_k$, $0 \le k \le N - 1$, again from the definition of the operator i_h^{m-1} and Schwarz's inequality, we derive that

$$\begin{aligned}
|(R_h y)(t)| &\le \left|\sum_{i=0}^{k-1} \int_{\sigma_i} R(t,s)(I - i_h^{m-1})y(s)\,ds\right| \\
&\quad + \left|\int_{t_k}^{t} R(t,s)(I - i_h^{m-1})y(s)\,ds\right| \\
&= \left|\sum_{i=0}^{k-1} \int_{\sigma_i} (I - i_h^{m-1})R(t,s)(I - i_h^{m-1})y(s)\,ds\right| \\
&\quad + \left|\int_{t_k}^{t} R(t,s)(I - i^{m-1})y(s)\,ds\right| \\
&\le \|(I - i_h^{m-1})R\|_{0,2}\|(I - i_h^{m-1})y\|_{0,2} \\
&\quad + Ch^m\|y\|_{m,\infty}(t - t_k) \\
&\le Ch^{m+1}\|y\|_{m,\infty},
\end{aligned}$$

that is,

$$\|R_h y\|_{0,\infty} \le C h^{m+1} \|y\|_{m,\infty}. \tag{2.11}$$

Moreover,

$$\|R_h y\|_{1,\infty} \le C h \|y\|_{1,\infty},$$

which, together with (2.11), yields that

$$\begin{aligned}
\|R_h^n e_{it}\|_{0,\infty} &= \|R_h(R_h^{n-1} e_{it})\|_{0,\infty} \\
&\le Ch^2 \|R_h^{n-1} e_{it}\|_{1,\infty} \\
&= Ch^2 \|R_h(R_h^{n-2} e_{it})\|_{1,\infty} \\
&\le Ch^3 \|R_h^{n-2} e_{it}\|_{1,\infty}.
\end{aligned}$$

According to Lemma 2.1, this leads to

$$\|R_h^n e_{it}\|_{0,\infty} \le Ch^{n+1} \|e_{it}\|_{1,\infty} \le Ch^{n+m+1} \|y\|_{m,\infty}. \qquad \square$$

From (2.10) we know that the order of the term $R_h^n e_{it}$ in (2.9) can be made arbitrarily large (provided we have sufficient regularity in the data), by carrying out a suitable number of iterations.

Next we turn to discussing the main theorem in this section–the asymptotic expansion theorem.

Theorem 2.2. *If $u \in C^{m+2}(I)$ and $v \in C^{m+2}(I)$, then there exists a constant $\tilde{C} = \tilde{C}(m)$, independently of the mesh T_h, such that*

$$\int_{\sigma_k} v(t)(u - i_h^{m-1} u)(t)\, dt = \tilde{C} h_k^{2m} \int_{\sigma_k} u^{(m)}(t) v^{(m)}(t)\, dt + O(h_k^{2m+3})$$
$$0 \le k \le N-1.$$

Proof. It follows from the definition of the interpolation operator i_h^{m-1} and the Taylor expansion of $v(t)$ at the mid-point of the subinterval σ_k, $t_{k+1/2} := (t_k + t_{k+1})/2$,

$$v(t) = \sum_{j=0}^{m+1} \frac{v^{(j)}(t_{k+1/2})}{j!} (t - t_{k+1/2})^j + O(h_k^{m+2}),$$

that

$$\int_{\sigma_k} v(t)(u - i_h^{m-1}u)(t)\,dt \tag{2.12}$$
$$= \int_{\sigma_k} \frac{v^{(m)}(t_{k+1/2})}{m!}(t - t_{k+1/2})^m (u - i_h^{m-1}u)(t)\,dt$$
$$+ \int_{\sigma_k} \frac{v^{(m+1)}(t_{k+1/2})}{(m+1)!}(t - t_{k+1/2})^{m+1}$$
$$\cdot (u - i_h^{m-1}u)(t)\,dt + O(h_k^{2m+3}).$$

Set

$$E(t) := \frac{1}{2}\left[(t - t_{k+1/2})^2 - \left(\frac{h_k}{2}\right)^2\right].$$

(Since $E(t)$ reflects interpolation errors, it is referred to as the interpolation error function.) It is readily verified that

$$\frac{1}{m!}(t - t_{k+1/2})^m = \frac{2^m}{(2m)!}(E^m)^{(m)} + F_{m-2}(t) \tag{2.13}$$

holds, where $F_{m-2}(t) \in P_{m-2}$.

Note that $(E^m)^{(r)}$ vanishes at the two endpoints of σ_k when $r \leq m-1$. Then by means of integration by parts with respect to t, from (2.12) and (2.13), we further get that

$$\int_{\sigma_k} v(t)(u - i_h^{m-1}u)(t)\,dt$$
$$= \frac{2m}{(2m)!}v^{(m)}(t_{k+1/2}) \int_{\sigma_k} (E^m)^{(m)}(t)(u - i_h^{m-1}u)(t)\,dt$$
$$+ O(h_k^{2m+3}) + \frac{2^{m+1}}{(2m+2)!}v^{(m+1)}(t_{k+1/2})$$
$$\cdot \int_{\sigma_k} (E^{m+1})^{(m+1)}(t)(u - i_h^{m-1}u)(t)\,dt$$
$$= (-1)^m \frac{2^m}{(2m)!}v^{(m)}(t_{k+1/2}) \int_{\sigma_k} E^m u^{(m)}(t)\,dt$$
$$+ (-1)^{m+1}\frac{2m+1}{(2m+2)!}v^{(m+1)}(t_{k+1/2})$$

$$
\begin{aligned}
&\cdot \int_{\sigma_k} E^{m+1} u^{(m+1)}\, dt + O(h_k^{2m+3}) \\
&= (-1)^m \frac{2^m}{(2m)!} \int_{\sigma_k} E^m v^{(m)}(t_{k+1/2}) u^{(m)}(t)\, dt + O(h_k^{2m+3}) \\
&= (-1)^m \frac{2^m}{(2m)!} \int_{\sigma_k} E^m v^{(m)}(t) u^{(m)}(t)\, dt \\
&\quad + (-1)^{m+1} \frac{2^m}{(2m)!} \int_{\sigma_k} \frac{1}{m+1} (E^{m+1})' v^{(m+1)}(t) u^{(m)}(t)\, dt + O(h_k^{2m+3}) \\
&= (-1)^m \frac{2^m}{(2m)!} \int_{\sigma_k} E^m v^{(m)}(t) u^{(m)}(t)\, dt \\
&\quad + (-1)^m \frac{2^m}{(2m)!(m+1)} \int_{\sigma_k} E^{m+1} [v^{(m+1)}(t) u^{(m)}(t)]'\, dt + O(h^{2m+3}) \\
&= (-1)^m \frac{2^m}{(2m)!} \int_{\sigma_k} E^m v^{(m)} u^{(m)}\, dt + O(h_k^{2m+3}).
\end{aligned}
$$

Since

$$E^m = \frac{1}{2^m} \sum_{i=0}^{m} (-1)^{m-i} C_m^i \left(\frac{h_k}{2}\right)^{2m-2i} (t - t_{k+1/2})^{2i},$$

with $C_m^i := m!/(i!(m-i)!)$, from (2.14) we derive that

$$
\begin{aligned}
(2.15)\quad &\int_{\sigma_k} v(t)(u - i_h^{m-1} u)(t)\, dt \\
&= \frac{1}{(2m)!} \left(\frac{h_k}{2}\right)^{2m} \int_{\sigma_k} v^{(m)}(t) u^{(m)}(t)\, dt - \frac{1}{2(2m-2)!} \left(\frac{h_k}{2}\right)^{2m-2} \\
&\quad \cdot \int_{\sigma_k} (t - t_{k+1/2})^2 v^{(m)} u^{(m)}\, dt \\
&\quad + \frac{1}{(2m)!} \int_{\sigma_k} \sum_{i=2}^{m} (-1)^i C_m^i \left(\frac{h_k}{2}\right)^{2m-2i} (t - t_{k+1/2})^{2i} v^{(m)} u^{(m)}\, dt \\
&=: I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$(2.16)\qquad I_1 := \frac{1}{(2m)!} \left(\frac{h_k}{2}\right)^{2m} \int_{\sigma_k} v^{(m)}(t) u^{(m)}(t)\, dt.$$

Due to

$$(t-t_{k+1/2})^2 = \frac{1}{3}(E^2)'' + \frac{1}{3}\left(\frac{h_k}{2}\right)^2,$$

again by virtue of integration by parts, we get that

$$\begin{aligned}\int_{\sigma_k}(t-t_{k+1/2})^2 v^{(m)}u^{(m)}\,dt &= \frac{1}{3}\int_{\sigma_k}E^2[v^{(m)}u^{(m)}]''\,dt \\ &\quad + \frac{1}{3}\left(\frac{h_k}{2}\right)^2\int_{\sigma_k}v^{(m)}u^{(m)}\,dt,\end{aligned} \tag{2.17}$$

and this leads to

$$\begin{aligned}I_2 &:= -\frac{1}{2(2m-2)!}\left(\frac{h_k}{2}\right)^{2m-2}\int_{\sigma_k}(t-t_{k+1/2})^2 v^{(m)}u^{(m)}\,dt \\ &= -\frac{1}{6(2m-2)!}\left(\frac{h_k}{2}\right)^{2m}\int_{\sigma_k}v^{(m)}u^{(m)}\,dt + O(h_k^{2m+3}).\end{aligned} \tag{2.18}$$

Using (2.13), (2.17) and repeated integration by parts, we find that

$$I_3 := \frac{1}{(2m)!}\int_{\sigma_k}\sum_{i=2}^{m}(-1)^i C_m^i\left(\frac{h_k}{2}\right)^{2m-2i}(t-t_{k+1/2})^{2i}v^{(m)}u^{(m)}\,dt$$

is also of the form expected in Theorem 2.2 which, together with (2.15), (2.16) and (2.18), completes the proof of Theorem 2.2. $\square$

Remark 1. Suppose that in Theorem 2.2 we have, for arbitrary nonnegative integer p, $u \in C^{m+2(p+1)}(I)$ and $v \in C^{m+2(p+1)}(I)$. Then the following holds:

$$\int_{\sigma_k} v(t)(I-i_h^{m-1})u(t)\,dt = \sum_{i=0}^{p}\alpha_i h_k^{2(m+i)} + O(h_k^{2(m+p+1)}), \tag{2.19}$$

where α_i, $0 \le i \le p$, is invariable when the mesh is refined uniformly.

3. Global extrapolation. In the following discussion we only consider the practically important case where $u \in S_0^{(-1)}(T_h)$ is the finite element solution defined by (2.2) or (2.3).

Lemma 3.1. *In* (1.1) *assume that* $g \in C^{2p+1}(I)$, $K \in C^{2p+1}(D)$ *for any positive integer* $p \geq 1$, *and that the integral operator* R_h *is given by* (2.7). *Then, for the solution* y *of* (1.1), *there exists the following asymptotic expansion at the points of the mesh* T_h*:*

$$(R_h y)(t_n) = \sum_{i=1}^{p} \beta_i(t_n) h^{2i} + O(h^{2(p+1)}), \quad 1 \leq n \leq N, \tag{3.1}$$

where $\beta_i(t) \in C^{2p+1}(I)$ *does not change when the mesh is refined uniformly.*

Proof. It follows from (2.19) and $\sum_{k=0}^{n-1} h_k \leq 1$ that

$$\begin{aligned}(R_h y)(t_n) &= \int_0^{t_n} R(t_n, s)(y - i_h^0 y)(s)\, ds \\ &= \sum_{k=0}^{n-1} \int_{\sigma_k} R(t_n, s)(y - i_h^0 y)(s)\, ds \\ &= \sum_{k=0}^{n-1} \sum_{i=1}^{p} \tilde{\alpha}_i(t_n) h_k^{2i} + \sum_{k=0}^{n-1} O(h_k^{2p+3}) \\ &= \sum_{i=1}^{p} \tilde{\alpha}_i(t_n) h^{2i} \left(\sum_{k=0}^{n-1} \left(\frac{h_k}{h} \right)^{2i} \right) + O(h^{2p+2}),\end{aligned}$$

where $\tilde{\alpha}_i(t) \in C^{2p+1}(I)$ remains invariant when the mesh is refined uniformly.

Letting

$$\beta_i(t) := \tilde{\alpha}_i(t) \sum_{k=0}^{n-1} \left(\frac{h_k}{h} \right)^{2i},$$

we have

$$(R_h y)(t_n) = \sum_{i=1}^{p} \beta_i(t_n) h^{2i} + O(h^{2p+2}),$$

where $\beta_i(t) \in C^{2p+1}(I)$ does not change when the mesh is refined uniformly. □

Lemma 3.2. *If $v \in C^{2p}(I)$, $u \in C^{2p+1}(D)$ and $y \in C^{2p+1}(I)$ for any nonnegative integer p, then $\int_{\sigma_k} v(t) \int_0^t u(t,s)(I - i_h^0)y(s)\,ds$ has an asymptotic expansion similar to* (2.19).

Proof. Exchanging the order of integration for t and s, we find that

$$\begin{aligned}\int_{\sigma_k} v(t) \int_0^t u(t,s)(I - i_h^0)y(s)\,ds = &\int_0^{t_k} (I - i_h^0)y(s)\,ds \\ &\cdot \int_{\sigma_k} v(t)u(t,s)\,dt \\ &+ \int_{\sigma_k} (I - i_h^0)y(s)\,ds \\ &\cdot \int_s^{t_{k+1}} v(t)u(t,s)\,dt,\end{aligned}$$

which, together with (2.19), yields the conclusion claimed in Lemma 3.2. □

Observing that $(i_h^0 y)' = 0$ in σ_k, $0 \leq k \leq N-1$, we know from (2.19) and Lemma 3.2 that $(R_h^j y)(t_n)$, $j \geq 2$, is of order h^{2j}, and the asymptotic expansion analogous to the one in Lemma 3.1 holds, where R_h is the integral operator defined by (2.7). Therefore, combining Theorem 2.1 with Lemma 3.1, we have

Theorem 3.1. *Assume that the conditions of Lemma* 3.1 *hold. Then, choosing the number of iterations as $n = 2p$ in* (2.10) *we obtain, for the iterated finite element error e_{it}, the following asymptotic expansion at the points of the mesh T_h:*

$$e_{it}(t_n) = \sum_{j=1}^{p} \gamma_j(t_n)h^{2j} + O(h^{2p+2}), \quad 1 \leq n \leq N, \tag{3.2}$$

where $\gamma_j(t) \in C^{2p+1}(I)$ does not vary when the mesh is refined uniformly.

To use Theorem 3.1 we employ, in addition to $S_0^{-1}(T_h)$, the piecewise-constant finite element space gained by dividing each subinterval σ_k

into two halves. Thus we define $S_0^{-1}(T_{h/2})$ to be the piecewise-constant finite element space with nodal points at the interior points of the partition

$$T_{h/2} : 0 = t_0 < t_{1/2} < t_1 < t_{3/2} < \cdots < t_{N-1/2} < t_N = 1.$$

Here again, for $0 \le k \le N-1$, let $t_{k+1/2} := (t_k + t_{k+1})/2$ be the midpoint of σ_k. Denoting by $u^{1/2}$ and $u_{it}^{1/2}$ the finite element and iterated finite element approximations, respectively, with respect to this new partition, we derive, from Theorem 3.1,

$$y(t_n) - u_{it}^{1/2}(t_n) = \sum_{j=1}^{p} \gamma_j(t_n)\left(\frac{h}{2}\right)^{2j} + O(h^{2p+2}).$$

A first application of Richardson extrapolation yields a new approximation which is of higher accuracy:

$$\frac{4u_{it}^{1/2}(t_n) - u_{it}(t_n)}{3} = y(t_n) + O(h^4). \tag{3.3}$$

This process can obviously be continued to generate approximations of higher and higher order. In fact, let

$$u_k^{(e)} := \frac{2^{2k} u_{k-1}^{(e)/2} - u_{k-1}^{(e)}}{2^{2k} - 1}$$

with

$$u_0^{(e)} := u_{it} \quad \text{and} \quad u_0^{(e)/2} := u_{it}^{1/2}.$$

Then

$$y(t_k) - u_n^{(e)}(t_k) = O(h^{2n+2}), \quad 1 \le k \le N. \tag{3.4}$$

Note that the extrapolation property (3.3) (or (3.4)) holds only at the points of the mesh T_h. Thus it is necessary for us to establish that the global extrapolation approximation possesses a higher accuracy. By virtue of (3.3), we can obtain the global extrapolation approximation of order 4 by an interpolation post-processing method [**10**]. To this end, we assume that T_h has been gained from T_{3h} with mesh size $3h$

by subdividing each element of T_{3h} into three elements, so that the elements number N for T_h is a multiple of 3. Then, we need to define a higher interpolation operator I^3_{3h} of degree 3 associated with T_{3h} according to the following conditions:

$$I^3_{3h}u|_{\sigma_k} \in P_3, \quad k = 3l+1, \quad l = 0, 1, \dots, N/3-1$$

and

$$I^3_{3h}u(t_i) = u(t_i), \quad i = k-1, k, k+1, k+2, \quad 1 \le k \le N-2.$$

Theorem 3.2. *In (1.1), assume that $g \in C^4(I)$ and $K \in C^4(D)$. Then there exists*

$$\|I^3_{3h}u_1^{(e)} - y\|_\infty \le Ch^4.$$

Proof. Denoting the basis functions corresponding to $\{t_j\}$ by $\{\varphi_j\}$, $1 \le j \le N$, we have

$$I^3_{3h}(y - u_1^{(e)})(t) = \sum_{j=1}^{N}(y - u_1^{(e)})(t_j)\varphi_j(t),$$

which, by means of (3.3) and the uniform boundedness of $\{\varphi_j\}_1^N$, leads to

$$\|I^3_{3h}(y - u_1^{(e)})\|_\infty \le \sum_{j=1}^{N} Ch^4\|\varphi_j\|_\infty \le Ch^4.$$

Using the result

$$\|I^3_{3h}y - y\|_\infty \le Ch^4,$$

we finally obtain that

$$\|I^3_{3h}u_1^{(e)} - y\|_\infty \le \|I^3_{3h}(u_1^{(e)} - y)\|_\infty + \|I^3_{3h}y - y\|_\infty \le Ch^4. \qquad \square$$

Similarly,

$$\|I^{2n+1}_{(2n+2)h}u_n^{(e)} - y\|_\infty \le Ch^{2n+2},$$

where $I^{2n+1}_{(2n+2)h}$ is an interpolation operator associated with the mesh $T_{(2n+2)h}$ with mesh size $(2n+2)h$ obtained by a combination of each $(2n+2)$-element in T_h.

4. Interpolation correction.

In this section we propose an interpolation correction scheme [**11, 12**] for the iterated finite element solution produced by the piecewise-constant finite element solution to obtain the approximation of the same accuracy as that in the once step extrapolation described in the last section. First, we have

Theorem 4.1. *In* (1.1), *assume that* $K \in C^4(D)$ *and* $g \in C^4(I)$. *Then, for each* $t \in I$ *we have*

$$y(t) - I^3_{3h} u_{it}(t) = h^2 \gamma_1(t) + O(h^4).$$

Proof. From (3.2) we get, by arguments similar to those for Theorem 3.2, that

$$I^3_{3h}(y - u_{it} - h^2\gamma_1)(t) = O(h^4).$$

This leads to the global expansion

$$\begin{aligned} y - I^3_{3h} u_{it} &= h^2 I^3_{3h}\gamma_1 + (y - I^3_{3h} y) + O(h^4) \\ &= h^2\gamma_1 + h^2(I^3_{3h}\gamma_1 - \gamma_1) + O(h^4) \\ &= h^2\gamma_1 + O(h^4), \end{aligned}$$

since $\|I^3_{3h}\gamma_1 - \gamma_1\|_\infty \le Ch^2 \|\gamma_1\|_{2,\infty}$. $\square$

Let u_{it} be the iterated finite element solution of problem (1.1), defined by

$$u_{it} := g + Ku,$$

where $u \in S_0^{-1}(T_h)$ is the finite element solution of (1.1). It follows from $(I-K)y = g$ and (2.4) that

$$Ki^0_h u_{it} = Ku - u_{it} - g,$$

or

$$(I - Ki^0_h)u_{it} = (I-K)y. \tag{4.1}$$

Assume that 1 is not an eigenvalue of K, so that $(I-K)^{-1}$ always exists, the inverse operator $(I-Ki_h^0)^{-1}$ exists and is uniformly bounded on $C(I)$ for all $h \in (0,\sigma)$ with $\sigma > 0$ sufficiently small. Then, setting $Q_h y := u_{it}$ we have, according to (4.1), that

$$Q_h y = (I-Ki_h^0)^{-1}(I-K)y. \tag{4.2}$$

From (4.2) we know that Q_h is a linear and bounded operator, and Q_h is the iterated finite element solution of the problem (1.1) if y is the solution of (1.1).

Theorem 4.2. *Under the conditions of Theorem* 4.1*,*

$$\|y - u_{it}^{(c)}\|_\infty \le Ch^4,$$

where $u_{it}^{(c)} := I_{3h}^3 u_{it} + i_h^1 u_{it} - i_h^1 Q_h I_{3h}^3 u_{it}$.

Proof. Using the result of Theorem 4.1, we derive

$$\begin{aligned}(I - i_h^1 Q_h)(y - I_{3h}^3 u_{it}) &= h^2 (I - i_h^1 Q_h)\gamma_1 + O(h^4) \\ &= h^2(\gamma_1 - i_h^1\gamma_1) + h^2 i_h^1(\gamma_1 - Q_h\gamma_1) + O(h^4).\end{aligned}$$

It follows from the interpolation approximation and the global superconvergence of the iterated finite element solution [**6**] that

$$\|\gamma_1 - i_h^1\gamma_1\|_\infty \le Ch^2\|\gamma_1\|_{2,\infty}$$

and

$$\|\gamma_1 - Q_h\gamma_1\|_\infty \le Ch^2\|\gamma_1\|_{1,\infty}.$$

This yields

$$(I - i_h^1 Q_h)(y - I_{3h}^3 u_{it}) = O(h^4),$$

where the left-hand side is simply

$$(I - i_h^1 Q_h)(y - I_{3h}^3 u_{it}) = y - u_{it}^{(c)}. \qquad \square$$

5. Iterative correction. Here, based on asymptotic expansion, an iterative correction method for the interpolation post-processing of the iterated finite element solution u_{it} corresponding to the piecewise constant finite element solution is given. Compared with [**5**], the iterative technique here is of higher precision in that the $(n-1)$-fold application of the iterative correction method leads to a global convergence rate of $O(h^{2n})$ instead of $O(h^n)$ in [**5**].

Along technical lines similar to those for Theorem 4.1 we can gain, without any difficulty,

Theorem 5.1. *In* (1.1), *if* $K \in C^{2n+2}(D)$ *and* $g \in C^{2n+2}(I)$, *the following holds*

$$y(t) - I_{(2n+2)h}^{2n+1} u_{it}(t) = \sum_{j=1}^{n} h^{2j}\gamma_j(t) + O(h^{2n+2}), \tag{5.1}$$

where $\gamma_j \in C^{2n+2}(I)$.

As an immediate consequence of this result, we find

Theorem 5.2. *Under the conditions of Theorem* 5.1, *the* $(n-1)st$ *correction* $u_n^{(c)}$ *of the interpolation post-processing of the iterated finite element solution* u_{it} *corresponding to the finite element solution* $u \in S_0^{(-1)}(T_h)$ *has the property that*

$$\|y - u_n^{(c)}\|_\infty \le Ch^{2n},$$

where

$$u_n^{(c)} := \sum_{k=1}^{n} (-1)^{k-1} C_n^k (I_{(2n+2)h}^{2n+1} Q_h)^k y, \quad n \ge 2.$$

Proof. From Theorem 5.1 we derive that

$$(I - I_{(2n+2)h}^{2n+1} Q_h) y(t) = \sum_{j=1}^{n} h^{2j}\gamma_j(t) + O(h^{2n+2}),$$

which yields

$$(I - I^{2n+1}_{(2n+2)h}Q_h)^2 y = \sum_{j=1}^{n} h^{2j}(I - I^{2n+1}_{(2n+2)h}Q_h)\gamma_j + O(h^{2n+2}).$$

Since $\gamma_j \in C^{(2n+2)}(I)$, $1 \le j \le n$, and $Q_h\gamma_j$ can be regarded as the exact solution and the iterated finite element solution, respectively, of the auxiliary problem,

$$(I - K)y = f \quad \text{with} \quad f = (I - K)\gamma_j,$$

(5.1) is true for γ_j, that is,

$$(I - I^{2n+1}_{(2n+2)h}Q_h)\gamma_j = \sum_{i=1}^{n} h^{2i}\tilde{\gamma}_{ij} + O(h^{2n+2}),$$

with $\tilde{\gamma}_{ij} \in C^{2n+2}(I)$, $1 \le i \le n$. Therefore,

$$(I - I^{2n+1}_{(2n+2)h}Q_h)^2 y = \sum_{j=1}^{n} h^{2j} \sum_{i=1}^{n} h^{2i}\tilde{\gamma}_{ij} + O(h^{2n+2}).$$

Repeating the above process, we finally get

$$(I - I^{2n+1}_{(2n+2)h}Q_h)^n y = O(h^{2n}),$$

and the left-hand side is just

$$(I - I^{2n+1}_{(2n+2)h}Q_h)^n y = y - u_n^{(c)}. \qquad \square$$

As another application of the asymptotic expansion (3.1), we prove that the $(n-1)$-fold iterative correction of the iterated finite element solution, u_{it}, corresponding to the piecewise-constant finite element solution u, also possesses a local superconvergence rate of $O(h^{2n})$ at the mesh points $t_1, t_2, \dots, t_N$; in other words, Conjecture 2.4 in [**5**] is true because, for the collocation method with the collocation points taken to be the zeros of the Legendre polynomial of degree m linearly mapped to each subinterval σ_k, the similar asymptotic expansion to that established in Theorem 2.2 holds, see [**9**] for details.

Theorem 5.3. *In (1.1), suppose that $g \in C^{2n+1}(I)$ and $K \in C^{2n+1}(D)$. Then the $(n-1)$st correction of the iterated finite element solution u_{it} corresponding to the finite element solution $u \in S_0^{(-1)}(T_h)$ exhibits higher-order convergence*:

$$\max_{1\le k\le N} |y(t_k) - \tilde{u}_n(t_k)| \le Ch^{2n}, \quad n \ge 1.$$

Here $\tilde{u}_n := \sum_{k=1}^{n}(-1)^{k-1}C_n^k Q_h^k y$.

Proof. From (2.9) and Theorem 2.1, we know

$$(I - Q_h)y = -\sum_{k=1}^{2n-2} R_h^k y + O(h^{2n}). \tag{5.2}$$

Thus, by the boundedness of the linear operator $(I - Q_h)$, we have

$$(I - Q_h)^2 y = -\sum_{k=1}^{2n-2} (I - Q_h)R_h^k y + O(h^{2n}).$$

On the basis of (5.2), using the same argument as in the proof of Theorem 5.2, we can now obtain

$$(I - Q_h)^2 y = \sum_{k=1}^{2n-2}\sum_{i=1}^{2n-2} R_h^{i+k} y + O(h^{2n})$$

which, together with

$$(R_h^j y)(t_k) = O(h^{2j}), \quad j \ge 1; 1 \le k \le N,$$

demonstrated in Section 3, leads to

$$((I - Q_h)^2 y)(t_k) = O(h^4).$$

Inductively, we can eventually gain

$$((I - Q_h)^n y)(t_k) = O(h^{2n}),$$

and the lefthand side is exactly

$$(I - Q_h)^n y = y - \tilde{u}_n. \qquad \square$$

6. Future work. An obvious extension of the techniques and results presented in this paper is to Volterra integral equations (1.1) with weakly singular kernels of the form $K(t,s) = p_\alpha(t-s)H(t,s)$, with smooth H and

$$p_\alpha := \begin{cases} (t-s)^{-a} & \text{if } 0 < \alpha < 1, \\ \log(t-s) & \text{if } \alpha = 1. \end{cases}$$

Here the use of a special class of nonuniform meshes, namely graded meshes, see [**3, 4, 7**], is of particular interest. Work on this topic is currently being carried out by a number of researchers, including the authors. Analogous research on iterated correction methods for collocation solutions to such equations is being done by H. Brunner and N. Yan.

However, three important aspects of these postprocessing methods remain to be studied. They are

- Iterated correction methods and extrapolation for nonlinear (weakly singular) Volterra integral equations;

- Analysis of iterated correction methods in the (practically important) case where the integrals in (2.3) (and in the corresponding expression for the iterated finite element solution u_{it}) are not evaluated exactly but approximated by appropriate numerical quadrature rules, compare also [**1**];

- Long time integration of (1.1) and its weakly singular counterpart. In many applications the solutions of second-kind Volterra integral equations exhibit certain asymptotic properties (boundedness or asymptotic stability), and they are to be approximated on $\mathbf{R}^+$. For example, it is known that the solution of

$$y(t) = 1 + \lambda \int_0^t (t-s)^{-\alpha} y(s)\, ds, \quad t \in \mathbf{R}^+, \tag{6.1}$$

satisfies $y(t) \to 0$ as $t \to \infty$, whenever $0 < \alpha < 1$ and $\lambda < 0$. Thus error estimates for the (Galerkin or collocation based) approximation

u_{it} and its iterated corrections for (6.1) and its more general versions on $\mathbf{R}^+$ are of considerable theoretical and practical interest.

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REFERENCES

1. K. Atkinson and J. Flores, *The discrete collocation method for nonlinear integral equations*, IMA J. Numer. Anal. **13** (1993), 195–213.

2. H. Brunner, *Iterated collocation methods and their discretization for Volterra integral equations*, SIAM J. Numer. Anal. **21** (1984), 1132–1145.

3. ———, *The approximate solution of Volterra equations with nonsmooth solutions*, Utilitas Math. **27** (1985), 57–95.

4. H. Brunner and P.J. van der Houwen, *The numerical solution of Volterra equations*, CWI Monographs, **3**, North-Holland, Amsterdam, 1986.

5. H. Brunner, Q. Lin and N. Yan, *The iterative correction method for Volterra integral equations*, BIT **36** (1996), 221–228.

6. H. Brunner, Y. Lin and S. Zhang, *Global and local superconvergence and iterative correction methods for second-kind Volterra integral equations by iterated finite element methods*, to appear.

7. H. Brunner, A. Pedas and G. Vainikko, *The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations*, Math. Comp., to appear.

8. H. Brunner and N. Yan, *On global superconvergence of iterated collocation solutions to linear second-kind Volterra integral equations*, J. Comp. Appl. Math. **67** (1996), 185–189.

9. Q. Lin, I.H. Sloan and R. Xie, *Extrapolation of the iterated-collocation method for integral equations of the second kind*, SIAM J. Numer. Anal. **27** (1990), 1535–1541.

10. Q. Lin and S. Zhang, *An immediate analysis for global superconvergence for integrodifferential equations*, Appl. Math. **1** (1997), 1–21.

11. Q. Lin, S. Zhang and N. Yan, *Methods for improving approximate accuracy for hyperbolic integrodifferential equations*, Systems Sci. Math. Sci. **10** (1997), 282–288.

12. ———, *An acceleration method for integral equations by using interpolations post-processing*, Adv. Comput. Math., to appear.

13. I.H. Sloan, *Superconvergence*, in *Numerical solution of integral equations* (M.A. Golberg, ed.), Plenum Press, New York, 1990.

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