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ON THE DYNAMICS OF THE d-TUPLES OF m-ISOMETRIES

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ABSTRACT. A commuting d-tuple $T = (T_1, \dots, T_d)$ of bounded linear operators on a Hilbert space $\mathcal{H}$ is called a spherical m-isometry if $\sum_{j=0}^{m}(-1)^j\binom{m}{j}Q_T^j(I) = 0$, where I denotes the identity operator and $Q_T(A) = \sum_{i=1}^{d} T_i^* A T_i$ for every bounded linear operator A on $\mathcal{H}$. Also, T is called a toral m-isometry if $\sum_{p\in\mathbb{N}^d,\, 0\le p\le n}(-1)^{|p|}\binom{n}{p}T^{*p}T^p = 0$ for all $n \in \mathbb{N}^d$ with $|n| = m$. The present paper mainly focuses on the convex-cyclicity of the d-tuples of operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$. In particular, we prove that spherical m-isometries are not convex-cyclic. Also, we show that toral and spherical m-isometric operators are never supercyclic.

1. Introduction and preliminaries. Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the space of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called an m-isometry ($m \in \mathbb{N}$), if it satisfies the following property:

$$(yx-1)^m(T) := \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k = 0. \tag{1.1}$$

Since $(yx-1)^m(T)$ is a self-adjoint operator, we observe that T is an m-isometry if and only if, for each $x \in \mathcal{H}$,

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^k x\|^2 = 0. \tag{1.2}$$

It is clear that the notions of 1-isometry and isometry coincide. The m-isometric operators were introduced by Agler **[2]** and were extensively

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studied by Agler and Stankus [**3, 4, 5**]. Recently, several authors studied m-isometries. In [**41**], m-isometric composition operators were discussed. Furthermore, the authors in [**15**] proved that the class of m-isometries on a Banach space is stable under powers; and the product of m-isometries was studied in [**20**]. In addition, m-isometric weighted shift operators were considered in [**1, 18, 21, 29, 35**]. On the other hand, the dynamics of m-isometries has been studied in [**13, 14, 16, 28**], and the perturbation of m-isometries by nilpotent operators has been explored in [**17, 19, 30, 48**]. Moreover, Duggal studied the tensor product of m-isometries [**26, 27**]. There are two natural generalizations of m-isometries to the tuple of operators. The first generalization is called spherical m-isometries. An initial study of such a tuple of operators on a Hilbert space is due to Gleason and Richter [**33**]. Hoffmann and Mackey [**38**] generalized the definition of spherical m-isometries on a normed space. Also, their relation with a moment problem was studied in [**7**]. Recently, the authors of [**36**] established some basic and non-trivial properties of spherical m-isometries. They proved that spherical m-isometries are power regular and are stable under powers and products under an orthogonality condition. Moreover, they showed that, for every proper spherical m-isometry T, there are linearly independent operators $A_0, \ldots, A_{m-1}$ such that $Q_T^n(I) = \sum_{i=0}^{m-1} A_i n^i$ for every $n \geq 0$. For further references, the reader may consult [**10, 11, 22, 23, 24, 45**].

Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, we set

$$|\alpha| = \sum_{j=1}^{d} \alpha_j, \quad \alpha! = \alpha_1! \cdots \alpha_d!,$$

and $T^\alpha = T_1^{\alpha_1} \cdots T_d^{\alpha_d}$. For every tuple of commuting operators $T = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$, there is a function $Q_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by $Q_T(A) = \sum_{i=1}^{d} T_i^* A T_i$. It is easy to see that $Q_T^j(I) = \sum_{|\alpha|=j} (j!/\alpha!) T^{*\alpha} T^\alpha$, $j \geq 1$, where $T^* = (T_1^*, \ldots, T_d^*)$. For each $m \geq 0$, denote $(I - Q_T)^m(I)$ by $P_m(T)$, in other words,

$$P_m(T) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} Q_T^j(I).$$

A commuting d-tuple $T = (T_1, \ldots, T_d)$ is said to be a spherical m-

isometry, if $P_m(T) = 0$. When $m = 1$, it is called a *spherical isometry.* It is shown in **[33]** that a d-shift operator, which played a role in the dilation of d-contractions (also called row contractions), is a spherical d-isometry. Note that

$$P_{n+1}(T) = P_n(T) - Q_T(P_n(T)) \tag{1.3}$$

for all $n \geq 0$. Now, observe that, if T is a commuting tuple of operators on $\mathcal{H}$ and $P_m(T) = 0$, then $P_{m+n}(T) = 0$ for all $n \geq 0$. Hence, if T is a spherical m-isometry, then T is a spherical $(m+n)$-isometry for all $n \geq 0$. For a spherical m-isometry T, define

$$\Delta_{T,m} := (-1)^{m-1} P_{m-1}(T).$$

It is proven that, if T is a spherical m-isometry for some $m \geq 0$, then $\Delta_{T,m}$ is a positive operator (see **[33,** Proposition 2.3]).

The second generalization of m-isometries is called toral m-isometries. Let $n = (n_1, \ldots, n_d)$ and $p = (p_1, \ldots, p_d)$ be in $\mathbb{N}^d$. We write $p \leq n$ if $p_j \leq n_j$ for $j = 1, \ldots, d$, and we also let

$$\binom{n}{p} = \prod_{j=1}^{d} \binom{n_j}{p_j}.$$

A commuting d-tuple $T = (T_1, \ldots, T_d)$ is said to be a *toral m-isometry* if

$$B_{n,m}(T) := \sum_{\substack{p \in \mathbb{N}^d \\ 0 \leq p \leq n}} (-1)^{|p|} \binom{n}{p} T^{*p} T^p = 0 \tag{1.4}$$

for all $n \in \mathbb{N}^d$ with $|n| = m$. Toral m-isometries were introduced and studied in **[12, 23]**. Note that, if T is a toral m-isometry, then each T_i, $i = 1, \ldots, d$, is an m-isometry. Indeed, let n be a d-tuple of non-negative integers with m in the ith place and zeros elsewhere. Then, (1.4) shows that T_i is an m-isometry. The following example shows that the converse is not true.

Example 1.1. Let $\ell^2(\mathbb{N})$ be the Hilbert space of complex sequences indexed by $\mathbb{N}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$. For $\alpha = (\alpha_n)_{n=1}^{\infty}$ in $\ell^2(\mathbb{N})$, let T_1 be the unilateral weighted shift operator defined by $T_1 e_n = \omega_n e_{n+1}$ and T_2 the unilateral weighted shift operator defined by $T_2 e_n = \nu_n e_{n+1}$, where $\{e_n\}_{n=1}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{N})$ and

$$(\omega_n)_{n\geq 1} := \sqrt{\frac{n+1}{n}} \quad \text{and} \quad (\nu_n)_{n\geq 1} := \sqrt{\frac{n+2}{n+1}}.$$

Since

$$\|T_i^2 e_n\|^2 - 2\|T_i e_n\|^2 + 1 = 0, \quad i = 1, 2,$$

for all $n \geq 1$, we conclude that T_1 and T_2 are 2-isometry. However, simple computation shows that, for $T = (T_1, T_2)$, $\langle B_{n,2}(T)e_1, e_1\rangle = -1/4 \neq 0$, where $n = (1,1)$, and thus, T is not a toral 2-isometry.

In the next proposition, we observe that a d-tuple of operators in a length of more than one cannot simultaneously be spherical and toral m-isometry. In order to see this, we need the following result, obtained in **[6]**.

Lemma 1.2 (**[6,** Theorem 3.1]). *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and $m \in \mathbb{N}$. Then, we have*

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} a_{n+k} = 0 \quad \textit{for all } n \in \mathbb{N},$$

if and only if there exists a polynomial function f of degree less than or equal to $m-1$ with $f(n) = a_n$ for all $n \in \mathbb{N}$.

Proposition 1.3. *There is no d-tuple of simultaneously spherical and toral m-isometry when $d > 1$.*

Proof. Assume that $d > 1$ and $T = (T_1, \ldots, T_d)$. To keep the exposition simple, let $d = 2$. It is straightforward to verify that

(1.5)

$$\sum_{\substack{n=(n_1,n_2)\\ |n|=m+1}} \sum_{\substack{p\in\mathbb{N}^2\\ 0\leq p=(p_1,p_2)\leq n}} (-1)^{|p|} \binom{n_1}{p_1}\binom{n_2}{p_2} T_1^{*p_1} T_2^{*p_2} T_1^{p_1} T_2^{p_2}$$
$$= \sum_{\substack{n=(n_1,n_2)\\ |n|=m}} \sum_{\substack{p=(p_1,p_2)\in\mathbb{N}^2\\ 0\leq p_1\leq n_1+1\\ 0\leq p_2\leq n_2}} (-1)^{|p|} \binom{n_1+1}{p_1}\binom{n_2}{p_2} T_1^{*p_1} T_2^{*p_2} T_1^{p_1} T_2^{p_2}$$

$$+ \sum_{\substack{n=(n_1,n_2)\\|n|=m}} \sum_{\substack{p=(p_1,p_2)\in\mathbb{N}^2\\0\le p_1\le n_1\\0\le p_2\le n_2+1}} (-1)^{|p|}\binom{n_1}{p_1}\binom{n_2+1}{p_2} T_1^{*p_1}T_2^{*p_2}T_1^{p_1}T_2^{p_2}.$$

Let $S=(1/\sqrt{2})T$. Then, an induction argument on m shows that

$$P_m(S) = \frac{1}{2^m}\sum_{|n|=m}\sum_{\substack{p\in\mathbb{N}^2\\0\le p\le n}} (-1)^{|p|}\binom{n}{p} T^{*p}T^p. \tag{1.6}$$

Indeed, the above equality holds for $m=1$. On the other hand, by (1.3), (1.5) and the induction hypothesis, we get

$$\begin{aligned}
P_{m+1}(S) &= P_m(S) - Q_S(P_m(S))\\
&= \frac{1}{2^m}\sum_{\substack{n=(n_1,n_2)\\|n|=m}}\sum_{\substack{p\in\mathbb{N}^2\\0\le p=(p_1,p_2)\le n}}\\
&\quad\cdot(-1)^{|p|}\binom{n}{p}\Big[T_1^{*p_1}T_2^{*p_2}T_1^{p_1}T_2^{p_2} - \frac{1}{2}T_1^{*p_1+1}T_2^{*p_2}T_1^{p_1+1}T_2^{p_2}\\
&\qquad\qquad - \frac{1}{2}T_1^{*p_1}T_2^{*p_2+1}T_1^{p_1}T_2^{p_2+1}\Big]\\
&= \frac{1}{2^m}\sum_{\substack{n=(n_1,n_2)\\|n|=m}}\sum_{\substack{p\in\mathbb{N}^2\\0\le p=(p_1,p_2)\le n}} (-1)^{|p|}\binom{n}{p}T_1^{*p_1}T_2^{*p_2}T_1^{p_1}T_2^{p_2}\\
&\quad+\frac{1}{2^{m+1}}\sum_{\substack{n=(n_1,n_2)\\|n|=m}}\sum_{\substack{0\le p_1\le n_1+1\\0\le p_2\le n_2}}\\
&\quad\cdot(-1)^{|p|}\binom{n_1}{p_1-1}\binom{n_2}{p_2}T_1^{*p_1}T_2^{*p_2}T_1^{p_1}T_2^{p_2}\\
&\quad+\frac{1}{2^{m+1}}\sum_{\substack{n=(n_1,n_2)\\|n|=m}}\sum_{\substack{0\le p_1\le n_1\\0\le p_2\le n_2+1}}\\
&\quad\cdot(-1)^{|p|}\binom{n_1}{p_1}\binom{n_2}{p_2-1}T_1^{*p_1}T_2^{*p_2}T_1^{p_1}T_2^{p_2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^m} \sum_{\substack{n=(n_1,n_2)\\|n|=m}} \sum_{\substack{p\in\mathbb{N}^2\\0\le p=(p_1,p_2)\le n}} (-1)^{|p|}\binom{n}{p} T_1^{*p_1}T_2^{*p_2}T_1^{p_1}T_2^{p_2} \\
&\quad + \frac{1}{2^{m+1}} \sum_{\substack{n=(n_1,n_2)\\|n|=m}} \sum_{\substack{0\le p_1\le n_1+1\\0\le p_2\le n_2}} \\
&\quad \cdot (-1)^{|p|}\left[\binom{n_1+1}{p_1} - \binom{n_1}{p_1}\right]\binom{n_2}{p_2} T_1^{*p_1}T_2^{*p_2}T_1^{p_1}T_2^{p_2} \\
&\quad + \frac{1}{2^{m+1}} \sum_{\substack{n=(n_1,n_2)\\|n|=m}} \sum_{\substack{0\le p_1\le n_1\\0\le p_2\le n_2+1}} \\
&\quad \cdot (-1)^{|p|}\binom{n_1}{p_1}\left[\binom{n_2+1}{p_2} - \binom{n_2}{p_2}\right] T_1^{*p_1}T_2^{*p_2}T_1^{p_1}T_2^{p_2} \\
&= \frac{1}{2^{m+1}} \sum_{\substack{n=(n_1,n_2)\\|n|=m+1}} \sum_{\substack{p\in\mathbb{N}^2\\0\le p\le n}} (-1)^{|p|}\binom{n}{p} T^{*p}T^{p}.
\end{aligned}
$$

Therefore, if T is a spherical and toral m-isometry, then $P_m(T) = P_m(S) = 0$, and thus,

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}\langle Q_T^j(I)x,x\rangle = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\langle Q_S^j(I)x,x\rangle = 0$$

for all $x \in \mathcal{H}$. Consequently,

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}\langle T_i^* Q_T^j(I)T_i x,x\rangle = 0$$

for all $x \in \mathcal{H}$ and $i = 1, 2$. By summing up these two equalities, we get

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}\langle Q_T^{j+1}(I)x,x\rangle = 0$$

for all $x \in \mathcal{H}$ and, by continuing this process, we conclude that

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}\langle Q_T^{j+k}(I)x,x\rangle = 0$$

for all $x \in \mathcal{H}$ and $k \geq 1$. Now, by Lemma 1.2, the mappings $j \to \langle Q_T^j(I)x, x\rangle$ and $j \to \langle Q_S^j(I)x, x\rangle$ are polynomials in j of degree less than or equal to $m-1$. However, since $\langle Q_S^j(I)x, x\rangle = (1/2^j)\langle Q_T^j(I)x, x\rangle$, we obtain a contradiction. $\square$

If $T = (T_1, \ldots, T_d)$ is a d-tuple of operators, we denote the semigroup generated by T by $\mathcal{F}_T = \{T_1^{k_1}T_2^{k_2}\cdots T_d^{k_d},\ k_i \geq 0,\ i = 1, \ldots, d\}$ and the orbit of x under the tuple T by $\mathrm{Orb}(T, x) = \{Sx : S \in \mathcal{F}_T\}$. A vector $x \in \mathcal{H}$ is called a *hypercyclic vector* for T if $\mathrm{Orb}(T, x)$ is dense in $\mathcal{H}$, and, in this case, the tuple T is called *hypercyclic*. Also, a vector $x \in \mathcal{H}$ is called a *supercyclic vector* for T if the set $\{\lambda Sx : \lambda \in \mathbb{C}, S \in \mathcal{F}_T\}$ is dense in $\mathcal{H}$, and, in this case, the tuple T is called *supercyclic*. These definitions generalize the hypercyclicity and supercyclicity of a single operator to a tuple of operators.

Hypercyclicity on Banach spaces was discussed in 1969 by Rolewicz [**44**] who showed that, whenever $|\lambda| > 1$, λT is hypercyclic where T is the unilateral backward shift on ℓ^p for $1 \leq p \leq \infty$. Kitai in her Ph.D. dissertation in 1982 [**39**], determined the conditions that ensure a continuous linear operator to be hypercyclic. In 1974, Hilden and Wallen [**37**] proved that every backward unilateral weighted shift is supercyclic. Moreover, they proved that no normal operator on a complex Hilbert space can be supercyclic. Later, Ansari and Bourdon [**8**] extended this to the class of all isometries on a Banach space. In 2012, Faghih-Ahmadi and Hedayatian [**28**] proved that no m-isometry can be supercyclic; they showed that the orbit of each vector is norm increasing, except possibly for a finite number of terms. Bermúdez, Marrero and Martinón [**16**] proved that, under a sufficient condition, m-isometric operators are not N-supercyclic (the operator $A \in \mathcal{B}(\mathcal{H})$ is N-supercyclic if there exists an N-dimensional subspace E of $\mathcal{H}$ such that its orbit under A is dense in $\mathcal{H}$). Eventually, Bayart [**13**] showed that m-isometric operators are never N-supercyclic.

On the other hand, the dynamics of perturbation of m-isometries by nilpotent operators were considered in [**17, 19, 30, 48**]. Hypercyclicity of tuples of operators was first investigated by Feldman [**31**]. He showed that there are no hypercyclic tuples of normal operators on an infinite-dimensional Hilbert space, and he also proved that no hypercyclic tuples of subnormal operators have a commuting normal extension on an infinite-dimensional Hilbert space. In addition, the supercyclicity of tuples of operators was first investigated by Soltani, Hedayatian and Khani-Robati [**46**]. They proved that there are no supercyclic subnormal tuples of operators in an infinite-dimensional Hilbert space. Recently, the authors in [**10**] proved that there is a supercyclic spherical isometric d-tuple on $\mathbb{C}^d$, but there is no supercyclic spherical isometry on an infinite-dimensional Hilbert space. Moreover, the supercyclicity of spherical isometries and toral 1-isometries on Banach spaces were investigated in [**9**].

In Section 3 of this paper, we will show that toral and spherical m-isometric operators are never supercyclic.

If E is a subset of $\mathcal{H}$, then the convex hull of E, denoted by $\operatorname{co}(E)$, is the set of all convex combinations of members of E, that is, all finite linear combinations of the members of E where the coefficients are non-negative and their sum is one. An operator $S \in \mathcal{B}(\mathcal{H})$ is called *convex-cyclic* if the convex hull generated by $\operatorname{Orb}(S, x)$ is dense in $\mathcal{H}$ for some $x \in \mathcal{H}$. The concept of convex-cyclicity for a single operator was introduced by Rezaei [**42**] and has been studied in [**14, 32, 40**]. In the next section, we define the concept of convex-cyclicity of tuples of operators, and we give some necessary and sufficient conditions for a d-tuple of commuting of operators on a Hilbert space $\mathcal{H}$ to be convex-cyclic. We then show that spherical m-isometries are not convex-cyclic.

2. Convex-cyclicity. In this section, we give necessary and sufficient conditions for convex-cyclicity of the d-tuple of commuting operators. Let $T = (T_1, \dots, T_d)$ be a d-tuple of bounded operators on a Hilbert space $\mathcal{H}$. The Harte spectrum of T is denoted by $\sigma(T)$; recall that $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d) \notin \sigma(T)$ if and only if there exist bounded operators $A_1, \dots, A_d$, $B_1, \dots, B_d$ on $\mathcal{H}$ such that

$$\sum_{i=1}^{d} (T_i - \lambda_i) A_i = \sum_{i=1}^{d} B_i (T_i - \lambda_i) = I.$$

Note that $\sigma(T)$ is compact and non-void. The spectral radius of T is

$$r_2(T) = \max\{\|\boldsymbol{\lambda}\|_2 : \boldsymbol{\lambda} \in \sigma(T)\},$$

where $\|\boldsymbol{\lambda}\|_2 = (\sum_{i=1}^{d} |\lambda_i|^2)^{1/2}$. Also, let

$$r_\infty(T) = \max\{\|\boldsymbol{\lambda}\|_\infty : \boldsymbol{\lambda} \in \sigma(T)\},$$

where

$$\|\boldsymbol{\lambda}\|_\infty = \|(\lambda_1, \ldots, \lambda_d)\|_\infty = \max\{|\lambda_j| : 1 \le j \le d\}.$$

The unit polydisc in $\mathbb{C}^d$ is denoted by $\mathbb{D}^d$:

$$\mathbb{D}^d = \{(z_1, \ldots, z_d) : |z_j| < 1 \text{ for } j = 1, \ldots, d\}.$$

A point $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$ of $\mathbb{C}^d$ is said to be a *joint eigenvalue* of T if there exists a non-zero vector x such that $T_i x = \lambda_i x$ for $i = 1, 2, \ldots, d$. The *joint point spectrum* of T, denoted by $\sigma_p(T)$, is defined by

$$\sigma_p(T) = \{\boldsymbol{\lambda} \in \mathbb{C}^d : \boldsymbol{\lambda} \text{ is a joint eigenvalue for } T\}.$$

Now, we define the concept of convex-cyclicity for a d-tuple of operators.

Definition 2.1. The polynomial

$$p(x_1, \ldots, x_d) = \sum_{k=0}^{n} \sum_{k_1+\cdots+k_d=k} a_{k_1,\ldots,k_d} x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}$$

of d variables $x_1, \ldots, x_d$ is a convex polynomial if the coefficients $a_{k_1,\ldots,k_d}$ are non-negative and

$$\sum_{k=0}^{n} \sum_{k_1+\cdots+k_d=k} a_{k_1,\ldots,k_d} = 1.$$

If $T = (T_1, \ldots, T_d)$ is a d-tuple of operators, then the convex hull of an orbit is $\operatorname{co}(\operatorname{Orb}(T, x)) = \{p(T)x : p \text{ is a convex polynomial}\}$. We say that T is convex-cyclic if $\operatorname{co}(\operatorname{Orb}(T, x))$ is dense in $\mathcal{H}$ for some $x \in \mathcal{H}$.

The proof of the next result relies on the following lemma.

Lemma 2.2. *If $\mathcal{H}$ is a Hilbert space and $y, z \in \mathcal{H}$ are linearly independent, then the linear map $\Lambda : \mathcal{H} \to \mathbb{C}^2$, defined by $\Lambda(x) = (\langle x, y\rangle, \langle x, z\rangle)$, is continuous and onto.*

Proof. By the Cauchy-Schwarz inequality, Λ is continuous. We can assume that $\|y\| = 1$. Moreover, suppose that $y^\perp \subset z^\perp$; therefore, if $\langle x, y\rangle \ne 0$, then $x/\langle x, y\rangle - y \in y^\perp$. Thus, $\langle x, z\rangle = \langle x, \langle z, y\rangle y\rangle$. In addition, the last equality holds if $x \in y^{\perp}$. Hence, $z = \langle z, y\rangle \cdot y$, that is, y and z are linearly dependent. Therefore, $y^{\perp} \not\subset z^{\perp}$, and this implies that there is a $v \in \mathcal{H}$ such that $\langle v, y\rangle = 0$ and $\langle v, z\rangle = 1$. Similarly, there is a $w \in \mathcal{H}$ such that $\langle w, y\rangle = 1$ and $\langle w, z\rangle = 0$. Now, if $(\lambda_1, \lambda_2) \in \mathbb{C}^2$, then $\Lambda(x) = (\lambda_1, \lambda_2)$, where $x = \lambda_1 w + \lambda_2 v$, and the lemma follows. □

Theorem 2.3. *Suppose that $T = (T_1, \ldots, T_d)$ is a convex-cyclic commuting d-tuple of operators on a Hilbert space $\mathcal{H}$. Then, the following hold.*

(a) *The joint ℓ_∞-spherical radius of T, i.e., $r_\infty(T)$ is greater than or equal to one. Consequently, $\sigma(T) \cap (\mathbb{C}^d \setminus \mathbb{D}^d)$ is non-empty.*
(b) $\sigma_p(T^*) \cap (\overline{\mathbb{D}}^d \cup \mathbb{R}^d) = \emptyset$.
(c) *If $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d)$ are in $\sigma_p(T^*)$, then there exists a $1 \leq i \leq d$ such that $\lambda_i \neq \overline{\gamma_i}$.*
(d) *T is not self-adjoint.*

Proof.

(a) Following [**47**],

$$r_\infty(T) = \lim_{k\to\infty} \|T^k\|_\infty^{1/k},$$

where

$$\|T^k\|_\infty = \max\{\|T_1^{k_1} \cdots T_d^{k_d}\| : k_1 + \cdots + k_d = k\}.$$

Let $\mathrm{co}(T_1, \ldots, T_d) = \{p(T_1, \ldots, T_d) : p \text{ is a convex polynomial}\}$. It follows that $\{\|S\| : S \in \mathrm{co}(T_1, \ldots, T_d)\}$ is bounded if $r_\infty(T) < 1$. Hence, $r_\infty(T) \geq 1$ if T is convex-cyclic.

For simplicity, we only prove our results in (b), (c) and (d) for the case $d = 2$; for other ds, the proof is similar. We also assume that x is a convex-cyclic vector for T.

(b) Assume to the contrary that $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \sigma_p(T^*) \cap (\overline{\mathbb{D}}^2 \cup \mathbb{R}^2)$. Then, there exists a non-zero vector $y \in \mathcal{H}$ such that $(T_i^* - \lambda_i)y = 0$ for $i = 1, 2$. Hence, for every convex polynomial p,

$$\begin{aligned}\langle y, p(T_1, T_2)x\rangle &= \langle p(T_1, T_2)^* y, x\rangle = \langle p(T_1^*, T_2^*)y, x\rangle \\ &= \langle p(\lambda_1, \lambda_2)y, x\rangle = p(\lambda_1, \lambda_2)\langle y, x\rangle.\end{aligned}$$

Since $\langle y, \cdot\rangle : \mathcal{H} \to \mathbb{C}$ is continuous and onto, it maps the dense set $\mathrm{co}(\mathrm{Orb}(T, x))$ onto a dense subset of $\mathbb{C}$. However, for any convex polynomial p, $p(\mathbb{R}^2) \subseteq \mathbb{R}$ and $p(\overline{\mathbb{D}}^2) \subseteq \overline{\mathbb{D}}$. It follows that

$$\{p(\lambda_1, \lambda_2)\langle y, x\rangle : p \text{ is a convex polynomial}\}$$

is not dense in $\mathbb{C}$, and this is a contradiction.

(c) Assume, to the contrary, that $\boldsymbol{\lambda}, \boldsymbol{\gamma} \in \sigma_p(T^*)$ and $\lambda_i = \overline{\gamma_i}$ for $i = 1, 2$. Let $(\beta_1, \beta_2) = \boldsymbol{\beta} = \boldsymbol{\lambda} = \overline{\boldsymbol{\gamma}}$. Then, $\boldsymbol{\beta}, \overline{\boldsymbol{\beta}} \in \sigma_p(T^*)$. Thus, there are y and z in $\mathcal{H}$ such that $T_i^* y = \beta_i y$ and $T_i^* z = \overline{\beta_i} z$ for $i = 1, 2$. However, by part (b), $\boldsymbol{\beta} \notin \mathbb{R}^2$. Thus, $\boldsymbol{\beta} \neq \overline{\boldsymbol{\beta}}$. Hence, y and z are linearly independent vectors. Now, for every convex polynomial p, we have

$$\langle p(T_1, T_2)x, y\rangle = \langle x, p(T_1^*, T_2^*)y\rangle = \langle x, p(\beta_1, \beta_2)y\rangle = p(\overline{\beta}_1, \overline{\beta}_2)\langle x, y\rangle.$$

Also,

$$\langle p(T_1, T_2)x, z\rangle = \langle x, p(T_1^*, T_2^*)z\rangle = \langle x, p(\overline{\beta}_1, \overline{\beta}_2)z\rangle = p(\beta_1, \beta_2)\langle x, z\rangle.$$

On the other hand, by Lemma 2.2 the linear map $\Lambda : \mathcal{H} \to \mathbb{C}^2$ defined by $\Lambda(h) = (\langle h, y\rangle, \langle h, z\rangle)$ is continuous and onto, so it maps the dense set $\{p(T_1, T_2)x : p \text{ is a convex polynomial}\}$ onto a dense subset of $\mathbb{C}^2$. It follows that

$$\big\{\big(p(\overline{\beta}_1, \overline{\beta}_2)\langle x, y\rangle, p(\beta_1, \beta_2)\langle x, z\rangle\big) : p \text{ is a convex polynomial}\big\}$$

must be dense in $\mathbb{C}^2$, and thus, $\langle x, y\rangle$ and $\langle x, z\rangle$ are non-zero. Hence, for every z_1 and z_2 in $\mathbb{C}$, there exists a convex polynomial p_n such that

$$p_n(\overline{\beta}_1, \overline{\beta}_2) \longrightarrow \frac{z_1}{\langle x, y\rangle} \quad \text{and} \quad p_n(\beta_1, \beta_2) \longrightarrow \frac{z_2}{\langle x, z\rangle}.$$

Put $z_1 = z_2 \in \mathbb{R}$. Therefore, $\langle x, z\rangle = \overline{\langle x, y\rangle}$, and consequently, $\overline{z}_2 = z_1$ for all z_1 and z_2 in $\mathbb{C}$, a contradiction. Hence, (c) holds.

(d) Assume that T is a self-adjoint 2-tuple. We have

$$\begin{aligned}\langle x, p(T_1, T_2)x\rangle &= \langle p(T_1, T_2)^* x, x\rangle = \langle p(T_1^*, T_2^*)x, x\rangle \\ &= \langle p(T_1, T_2)x, x\rangle = \overline{\langle x, p(T_1, T_2)x\rangle}\end{aligned}$$

for every convex polynomial p. This implies that $\{\langle x, p(T_1, T_2)x\rangle : p \text{ is a convex polynomial}\}$ is not dense in $\mathbb{C}$, a contradiction. Therefore, (d) holds. $\square$

Remark 2.4. In part (a) of the above theorem, $\sigma(T)$ can be replaced by the Taylor spectrum or the joint approximate point spectrum of T since the convex hull of all of these spectra coincide [**25**].

We say that $T = (T_1, \ldots, T_d)$ is convex-transitive if, for all non-empty open subsets U and V of $\mathcal{H}$, there exists a d-variable convex polynomial p such that $p(T_1, \ldots, T_d)(U) \cap V \neq \emptyset$. Moreover, T satisfies the convex-cyclicity criterion if there exist two dense subsets Y and Z in $\mathcal{H}$, a sequence $\{p_k\}$ of d-variable convex polynomials, and a sequence of maps $s_k : Z \to \mathcal{H}$ such that

(a) $p_k(T_1, \ldots, T_d)y \to 0$ for every $y \in Y$;
(b) $s_k z \to 0$ for every $z \in Z$;
(c) $p_k(T_1, \ldots, T_d)s_k z \to z$ for every $z \in Z$.

In the next theorem, we will consider the relationship between convex-transitivity and convex-cyclicity criterion with convex-cyclicity.

Theorem 2.5. *Suppose that $T = (T_1, \ldots, T_d)$ is a commuting d-tuple of operators on $\mathcal{H}$. Then, the following hold.*

(a) *If T satisfies the convex-cyclicity criterion, then T is convex-transitive.*
(b) *If T is convex-transitive, then T is convex-cyclic.*
(c) *If T is convex-cyclic and $\sigma_p(T_i^*) = \emptyset$ for $i = 1, \ldots, d$, then T is convex-transitive.*

Proof.

(a) Let U and V be two non-empty open subsets in $\mathcal{H}$, and let Y, Z, p_k and s_k be those obtained by the property of the convex-cyclicity criterion for T. Pick $y \in Y \cap U$ and $z \in Z \cap V$. Then,

$$y_k := y + s_k(z) \longrightarrow y \in U$$

and

$$p_k(T_1, \ldots, T_d)y_k \longrightarrow z \quad \text{as } k \to \infty.$$

Therefore, $p_k(T_1, \ldots, T_d)(U) \cap V \neq \emptyset$, if k is large enough. This implies that T is convex-transitive.

(b) Let $(V_j)_{j \in \mathbb{N}}$ be a countable basis for the topology of $\mathcal{H}$. Each $x \in \mathcal{H}$ is a convex-cyclic vector for T if $\{p(T_1, \ldots, T_d)x : p \in \mathcal{P}\}$ is dense in $\mathcal{H}$, where $\mathcal{P}$ is a collection of convex polynomials in d-variables, that is,

$$x \in \bigcap_{j\in\mathbb{N}} \bigcup_{p\in\mathcal{P}} p(T_1,\ldots,T_d)^{-1}(V_j).$$

The convex-transitivity of T implies that, for every non-empty open set U, there exists a convex polynomial $p \in \mathcal{P}$ such that $p(T_1,\ldots,T_d)^{-1}(V_j) \cap U \neq \emptyset$, for any $j \in \mathbb{N}$. It follows that, for each $j \in \mathbb{N}$,

$$\bigcup_{p\in\mathcal{P}} p(T_1,\ldots,T_n)^{-1}(V_j)$$

is a dense open subset in $\mathcal{H}$. Now, by the Baire category theorem,

$$\bigcap_{j\in\mathbb{N}} \bigcup_{p\in\mathcal{P}} p(T_1,\ldots,T_d)^{-1}(V_j)$$

is dense in $\mathcal{H}$, which implies that T is convex-cyclic.

(c) Let T be convex-cyclic with convex-cyclic vector x. Since $\sigma_p(T_i^*) = \emptyset$ for $i = 1,\ldots,d$, $p(T_1,\ldots,T_d)$ has a dense range, and $p(T_1,\ldots,T_d)x$ is a convex-cyclic vector for every convex polynomial p. It follows that T has a dense subset of convex-cyclic vectors in $\mathcal{H}$. Now, let U and V be two non-empty open subsets of $\mathcal{H}$. Choose a convex-cyclic vector $x \in U$ such that $p(T_1,\ldots,T_d)x \in V$ for some convex polynomial p. Hence, $p(T_1,\ldots,T_d)U \cap V \neq \emptyset$, and thus, T is convex-transitive. $\square$

Corollary 2.6. *If $T = (T_1,\ldots,T_d)$ satisfies the convex-cyclicity criterion, then $S = (T_1 \oplus T_1,\ldots,T_d \oplus T_d)$ also satisfies the convex-cylicity criterion. Hence, S is convex-cyclic.*

Remark 2.7. Let a and b be relatively prime integers, both greater than 1, and $T = (T_1,T_2,T_3) = (aI_1, 1/bI_1, e^{i\theta}I_1)$, where I_1 is the identity operator on $\mathbb{C}$ and θ is an irrational multiple of π. Then, T is convex-transitive on $\mathbb{C}$, but T does not satisfy the convex-cyclicity criterion. Indeed, let U and V be two non-empty open sets in $\mathbb{C}$. Let z_0 be a non-zero vector in U. Since

$$\left\{\frac{a^n}{b^k}e^{im\theta}z_0 : n,k,m \in \mathbb{N}\right\}$$

is dense in $\mathbb{C}$, there are $n_0, k_0, m_0 \in \mathbb{N}$ such that

$$\frac{a^{n_0}}{b^{k_0}} e^{im_0\theta} z_0 \in V.$$

Put $p(z_1, z_2, z_3) = z_1^{n_0} z_2^{k_0} z_3^{m_0}$. Then, p is a convex polynomial and $p(T)(U) \cap V \neq \emptyset$. It follows that T is convex-transitive. However, T does not satisfy the convex-cyclicity criterion since $(T_1 \oplus T_1, T_2 \oplus T_2, T_3 \oplus T_3) = (aI_2, 1/bI_2, e^{i\theta} I_2)$ is not convex-cyclic on $\mathbb{C}^2$, where I_2 is the identity operator on $\mathbb{C}^2$.

Corollary 2.8. *Suppose that A and B are convex-cyclic operators and C is an operator that commutes with B and $\sigma_p(C^*) = \emptyset$. If $T_1 = A \oplus C$ and $T_2 = I \oplus B$, then the pair (T_1, T_2) is convex-cyclic.*

Proof. Let x be a convex-cyclic vector for A and y a convex-cyclic vector for B. We show that $x \oplus y$ is a convex-cyclic vector for the pair (T_1, T_2). Let U and V be two non-empty open sets. There is a convex polynomial p_0 such that $p_0(A)x \in U$. Moreover, $p_0(C)$ has dense range. Indeed, we can factor $p_0(C)$ as $p_0(C) = a(C - \mu_1) \cdots (C - \mu_d)$, where $a \neq 0$ and $\mu_1, \ldots, \mu_d \in \mathbb{C}$. Since $\sigma_p(C^*) = \emptyset$, each $C - \mu_i$ has dense range, and hence, $p_0(C)$ has dense range as well. Thus, $p_0(C)^{-1}(V)$ is a non-empty open set, so there exists a convex-polynomial p_1 such that $p_1(B)(y) \in p_0(C)^{-1}(V)$. Hence,

$$p_0(T_1)p_1(T_2)(x \oplus y) = p_0(A)x \oplus p_0(C)p_1(B)y \in U \times V.$$

This implies that (T_1, T_2) is convex-cyclic. $\square$

Remark 2.9. The authors showed in [**14**] that there is a convex-cyclic operator S such that $\sigma_p(S^*) = \emptyset$, but S^2 is not convex-cyclic. Let $A = B = S$ and $C = S^2$ in the above corollary. Thus, we obtain a convex-cyclic pair (T_1, T_2) such that T_1 and T_2 are not convex-cyclic. As another example, let $C = M_\varphi^*$, where M_φ is the multiplication operator by φ on the Hardy space $\mathcal{H}^2(\mathbb{D})$ with $\varphi(z) = z$. Moreover, $A = B = M_\psi^*$ is convex-cyclic where $\psi(z) = 2z$. Indeed, M_ψ^* is hypercyclic [**34,** Theorem 4.5]. Hence, T_1 and T_2 are not convex-cyclic, but the pair (T_1, T_2) is convex-cyclic.

The following result is a generalization of [**14,** Proposition 2.4].

Proposition 2.10. *If $T = (T_1, \dots, T_d)$ is a convex-cyclic d-tuple of commuting operators on $\mathcal{H}$ and $c_i > 1$ for $i = 1, \dots, d$, then $S = (c_1T_1, \dots, c_dT_d)$ is also convex-cyclic.*

Proof. Let x be a convex-cyclic vector for T and y any non-zero vector in $\mathcal{H}$. Since, $c_i > 1$ we have

$$\sup\{\mathrm{Re}\langle Ax, y\rangle : A \in \mathcal{F}_S\} \geq \sup\{\mathrm{Re}\langle Ax, y\rangle : A \in \mathcal{F}_T\}.$$

Now, the Riesz representation theorem coupled with [**14,** Proposition 2.1] completes the proof. □

The next theorem states that spherical m-isometries are not convex-cyclic.

Theorem 2.11. *Let $T = (T_1, \dots, T_d)$ be a d-tuple of commuting operators on $\mathcal{H}$. If T is a spherical m-isometry, then T is not convex-cyclic.*

Proof. We proceed by induction on m. If $m = 1$, then T is a spherical isometry, and so, $\mathrm{co}(\mathrm{Orb}(T, x))$ lies in $\mathrm{ball}(0, \|x\|)$, and hence, is not dense in $\mathcal{H}$. Therefore, T cannot be convex-cyclic. Let $m \geq 2$ and $T = (T_1, \dots, T_d)$ be a spherical m-isometry. We consider the semi-inner product

$$\langle\langle x, y\rangle\rangle = \langle P_{m-1}(T)x, y\rangle, \quad x, y \in \mathcal{H},$$

with semi-norm $|||\cdot|||$. Note that $\Delta_{T,m} = (-1)^{m-1}P_{m-1}(T)$ is a positive operator [**33,** Proposition 2.3] , and let

$$N = \{x \in \mathcal{H} : \langle\langle x, x\rangle\rangle = 0\} = \ker P_{m-1}(T).$$

Moreover, define the inner product $\langle\cdot,\cdot\rangle'$ on $\mathcal{H}/N$ by

$$\langle x + N, y + N\rangle' = \langle\langle x, y\rangle\rangle.$$

In order to see that $\langle\cdot,\cdot\rangle'$ is well defined, suppose that $x_1 + N = x_2 + N$ and $y_1 + N = y_2 + N$. Hence,

$$\begin{aligned}\langle P_{m-1}(T)x_1, y_1\rangle &= \langle P_{m-1}(T)x_1 - P_{m-1}(T)x_2 + P_{m-1}(T)x_2, y_1\rangle \\ &= \langle x_2, P_{m-1}(T)y_1\rangle = \langle x_2, P_{m-1}(T)y_2\rangle \\ &= \langle P_{m-1}(T)x_2, y_2\rangle.\end{aligned}$$

Then, $\mathcal{H}/N$ equipped with $\langle \cdot, \cdot \rangle'$ is a Hilbert space (we can consider its completion, if needed). Now, if $\widetilde{T} = (\widetilde{T}_1, \ldots, \widetilde{T}_d)$ is the tuple induced by T on $\mathcal{H}/N$, then, by [**33,** Proposition 2.4], $T_i(\ker P_{m-1}(T)) \subseteq \ker P_{m-1}(T)$ for each T_i, and thus, $\widetilde{T}$ is well defined. Furthermore, $\widetilde{T}$ is a spherical isometry on $\mathcal{H}/N$ equipped with the norm $|\cdot|'$. In fact, since

$$P_m(T) = P_{m-1}(T) - Q_T(P_{m-1}(T)),$$

we have

$$\begin{aligned} \sum_{j=1}^{d} |\widetilde{T}_j(x+N)|'^2 &= \sum_{j=1}^{d} \langle P_{m-1}(T)T_j x, T_j x\rangle \\ &= \sum_{j=1}^{d} \langle T_j^* P_{m-1}(T)T_j x, x\rangle \\ &= \langle Q_T(P_{m-1}(T))x, x\rangle \\ &= \langle (P_{m-1}(T) - P_m(T))x, x\rangle \\ &= \langle P_{m-1}(T)x, x\rangle = |x+N|'^2 \end{aligned}$$

On the contrary, suppose that $x \in \mathcal{H}$ is a convex-cyclic vector for T. Now, if p is a convex-cyclic polynomial and $y \in \mathcal{H}$, then

$$\begin{aligned} |\widetilde{y} - p(\widetilde{T})\widetilde{x}|' &= |y - p(T)x + N|' = |||y - p(T)x||| \\ &\le \|P_{m-1}(T)\|^{1/2} \|y - p(T)x\|. \end{aligned}$$

This implies that $\widetilde{x} = x + N$ is a convex-cyclic vector for $\widetilde{T}$, a contradiction. $\square$

Note that, since the weak closure and norm closure of a convex set coincide, we have the following result.

Corollary 2.12. *No spherical m-isometry is weakly hypercyclic.*

3. Supercyclicity. The norm of every convex-cyclic operator is greater than one [**42,** Proposition 3.2]. Thus, if an operator $T \in \mathcal{B}(\mathcal{H})$ is supercyclic, then the operator $T/1 + \|T\|$ is supercyclic but not convex-cyclic. On the other hand, Bermúdez, et al. [**14**] have shown that certain diagonalizable normal operators are convex-cyclic while they are never supercyclic [**37**]. It was proven in Theorem 2.11 that spherical m-isometries are not convex-cyclic. Thus, they are not hypercyclic. It is natural to seek their supercyclicity. In the following, we prove that toral and spherical m-isometric operators are not supercyclic. Note that, for $S \in \mathcal{B}(\mathcal{H})$, we define $\beta_\ell(S) = (1/\ell!)(yx-1)^\ell(S)$ for $\ell \geq 0$. Using the notion $\beta_\ell(S)$, if $S \in \mathcal{B}(\mathcal{H})$ is an m-isometry,

$$\|S^k x\|^2 = \sum_{\ell=0}^{m-1} k^{(\ell)} \langle \beta_\ell(S)x, x\rangle,$$

where $k^{(\ell)} = k \cdot (k-1) \cdots (k-\ell+1)$ for $\ell \geq 1$, $k \geq 0$ and $k^{(0)} = 1$ (see [**3,** equation (1.3)]).

Theorem 3.1. *Let $T = (T_1, \ldots, T_d)$ be a d-tuple of commuting operators on $\mathcal{H}$. If*

(a) *T is a toral m-isometry,*

or

(b) *T is a spherical m-isometry,*

then T is not supercyclic.

Proof.

(a) If T is a toral m-isometry, then each T_i, $i = 1, \ldots, d$, is an m-isometry. Note that we can assume that the T_is are also invertible since if, for example, T_1 is not invertible, then $(T_1, \ldots, T_d)$ and $(T_2, \ldots, T_d)$ are either both supercyclic or are non-supercyclic. Indeed, every m-isometric operator is injective and has closed range [**3**]; consequently, $\overline{\mathrm{ran} T_1} = \mathrm{ran} T_1 \neq \mathcal{H}$. Suppose that x_0 is a supercyclic vector for T, and let

$$A = \{\lambda T_1^{k_1} T_2^{k_2} \cdots T_d^{k_d} x_0 : \lambda \in \mathbb{C},\ k_1 > 0,\ k_i \geq 0,\ i = 2, 3, \ldots d\}$$

and

$$B = \{\lambda T_2^{k_2} \cdots T_d^{k_d} x_0 : \lambda \in \mathbb{C},\ k_i \geq 0,\ i = 2, 3, \ldots d\}.$$

Hence, $\mathcal{H} = \overline{A \cup B}$ and $\mathrm{int}(A) = \emptyset$; thus, $\mathcal{H} = \overline{B}$.

To simplify notation, assume that $d = 2$. On the contrary, suppose that $T = (T_1, T_2)$ is supercyclic with supercyclic vector x. Therefore, for $y \in \mathcal{H}$, there are two sequences of non-negative integers $(k_i)_i$ and

$(s_i)_i$ and one sequence of scalars $(\lambda_i)_i$ such that $\lambda_i T_1^{k_i} T_2^{s_i} x \to y$. Since $\|T_1^{k_i} x\|^2 = \sum_{\ell=0}^{m-1} k_i^{(\ell)} \langle \beta_\ell(T_1)x, x\rangle$, we have

$$\begin{aligned}
\|T_1^{k_i} T_2^{s_i} x\|^2 &= \sum_{\ell=0}^{m-1} k_i^{(\ell)} \frac{1}{\ell!} \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \langle (T_1^j)^* T_1^j T_2^{s_i} x, T_2^{s_i} x\rangle \\
&= \sum_{\ell=0}^{m-1} \sum_{j=0}^{\ell} k_i^{(\ell)} \frac{1}{\ell!} (-1)^{\ell-j} \binom{\ell}{j} \|T_2^{s_i} T_1^j x\|^2 \qquad (3.1)\\
&= \sum_{\ell=0}^{m-1} \sum_{j=0}^{\ell} \sum_{\ell'=0}^{m-1} k_i^{(\ell)} \frac{1}{l!} (-1)^{\ell-j} \binom{\ell}{j} s_i^{(\ell')} \langle \beta_{\ell'}(T_2) T_1^j x, T_1^j x\rangle \\
&= \sum_{\ell=0}^{m-1} \sum_{j=0}^{\ell} \sum_{\ell'=0}^{m-1} \sum_{n=0}^{\ell'} k_i^{(\ell)} s_i^{(\ell')} \frac{1}{\ell!} \frac{1}{\ell'!} \\
&\quad \cdot (-1)^{\ell-j} (-1)^{\ell'-n} \binom{\ell}{j} \binom{\ell'}{n} \|T_2^n T_1^j x\|^2.
\end{aligned}$$

This shows that $\|T_1^{k_i} T_2^{s_i} x\|^2$ is a polynomial of two variables, k_i and s_i, with leading coefficient

$$\sum_{j=0}^{m-1} \sum_{n=0}^{m-1} \frac{(-1)^{n+j}}{((m-1)!)^2} \binom{m-1}{j} \binom{m-1}{n} \|T_2^n T_1^j x\|^2.$$

Therefore,

$$0 \le \lim_{i\to\infty} \frac{\|T_1^{k_i} T_2^{s_i} x\|^2}{k_i^{(m-1)} s_i^{(m-1)}} = \sum_{j=0}^{m-1} \sum_{n=0}^{m-1} \frac{(-1)^{n+j}}{((m-1)!)^2} \binom{m-1}{j} \binom{m-1}{n} \|T_2^n T_1^j x\|^2.$$

On the other hand, (3.1) implies that

$$\begin{aligned}
&\|T_1^{k_i+1} T_2^{s_i} x\|^2 - \|T_1^{k_i} T_2^{s_i} x\|^2 \\
&\quad = \sum_{\ell=0}^{m-1} \sum_{j=0}^{\ell} \sum_{\ell'=0}^{m-1} [(k_i+1)^{(\ell)} - k_i^{(\ell)}] \frac{1}{\ell!} (-1)^{\ell-j} \binom{\ell}{j} s_i^{(\ell')} \langle \beta_{\ell'}(T_2) T_1^j x, T_1^j x\rangle \\
&\quad = \sum_{\ell=0}^{m-1} \sum_{j=0}^{\ell} \sum_{\ell'=0}^{m-2} [(k_i+1)^{(\ell)} - k_i^{(\ell)}] \frac{1}{\ell!} (-1)^{\ell-j} \binom{\ell}{j} s_i^{(\ell')} \langle \beta_{\ell'}(T_2) T_1^j x, T_1^j x\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\ell=0}^{m-2}\sum_{j=0}^{\ell}[(k_i+1)^{(\ell)}-k_i^{(\ell)}]\frac{1}{\ell!} \\
&\cdot(-1)^{\ell-j}\binom{\ell}{j}s_i^{(m-1)}\langle\beta_{m-1}(T_2)T_1^jx,T_1^jx\rangle \\
&+\sum_{j=0}^{m-1}[(k_i+1)^{(m-1)}-k_i^{(m-1)}]\frac{1}{(m-1)!} \\
&\cdot(-1)^{m-1-j}\binom{m-1}{j}s_i^{(m-1)}\langle\beta_{m-1}(T_2)T_1^jx,T_1^jx\rangle.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\lim_{i\to\infty}\frac{\|T_1^{k_i+1}T_2^{s_i}x\|^2-\|T_1^{k_i}T_2^{s_i}x\|^2}{s_i^{(m-1)}[(k_i+1)^{(m-1)}-k_i^{(m-1)}]} \\
&\quad=\sum_{j=0}^{m-1}\frac{1}{(m-1)!}(-1)^{m-1-j}\binom{m-1}{j}\langle\beta_{m-1}(T_2)T_1^jx,T_1^jx\rangle \\
&\quad=\sum_{j=0}^{m-1}\sum_{n=0}^{m-1}\frac{(-1)^{n+j}}{((m-1)!)^2}\binom{m-1}{j}\binom{m-1}{n}\|T_2^nT_1^jx\|^2 \\
&\quad\geq 0.
\end{aligned}
$$

Set

$$
a_i=\frac{\|T_1^{k_i+1}T_2^{s_i}x\|^2-\|T_1^{k_i}T_2^{s_i}x\|^2}{s_i^{(m-1)}[(k_i+1)^{(m-1)}-k_i^{(m-1)}]};
$$

therefore, $(a_i)_i$ has a subsequence $(a_{i_j})_j$ such that the entire sequence $(a_{i_j})_j$ is negative or non-negative. Without loss of generality, we denote this subsequence by $(a_i)_i$. Now, if all a_is are negative, then

$$
\|y\|=\lim_{i\to\infty}|\lambda_i|\|T_1^{k_i}T_2^{s_i}x\|\geq\lim_{i\to\infty}|\lambda_i|\|T_1^{k_i+1}T_2^{s_i}x\|=\|T_1y\|.
$$

This shows that T_1 is a contraction, and thus, T_1 is an isometry (see **[28,** Corollary 1]). On the other hand, if all a_is are non-negative, then

$$
\|y\|=\lim_{i\to\infty}|\lambda_i|\|T_1^{k_i}T_2^{s_i}x\|\leq\lim_{i\to\infty}|\lambda_i|\|T_1^{k_i+1}T_2^{s_i}x\|=\|T_1y\|.
$$

Since the inverse of every m-isometric operator is an m-isometry, the above relation shows that T_1^{-1} is a contraction m-isometry, which, in turn, implies that T_1 is an isometry. A similar argument shows that T_2 is also an isometry, which is a contradiction (see **[9]** or **[10,** Proposition 1]).

(b) If T is a spherical m-isometry, then, by (1.6), $\sqrt{d}T$ is a toral m-isometry. The proof follows immediately from part (a). $\square$

Since the unilateral weighted backward shift operators are supercyclic, the following corollary is a consequence of Theorem 3.1.

Corollary 3.2. *Let B be a weighted backward shift on $\mathcal{H}$ and $T = (B, T_1, \ldots, T_d)$. Then, T is neither spherical m-isometry nor toral m-isometry.*

Remark 3.3. Suppose that the d-tuple $T = (T_1, \ldots, T_d)$ is convex-cyclic and each T_i is an m-isometry. Similar to the proof for supercyclicity in Theorem 3.1 (a), we can assume that each T_i is invertible. Suppose that each T_i, $i = 1, \ldots, d$, is a 2-isometry; thus, $\|T_i^2 x\|^2 - 2\|T_i x\|^2 + \|x\|^2 = 0$ for all $x \in \mathcal{H}$. By replacing x by $T_i^{-1}x$, we conclude that T_i^{-1} is also a 2-isometry. Hence, $\|T_i x\| \geq \|x\|$ and $\|T_i^{-1}x\| \geq \|x\|$ for all $x \in \mathcal{H}$ (see **[3]** or **[43,** Lemma 1]) which, in turn, imply that each T_i is an isometry. This contradicts the convex-cyclicity of T. A question remains: if each T_i, $i = 1, \ldots, d$, is an m-isometry for some $m > 2$, is the d-tuple $T = (T_1, \ldots, T_d)$ convex-cyclic?

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