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APPROXIMATE AMENABILITY AND A VARIANT OF CONTINUOUS HOCHSCHILD COHOMOLOGY

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ABSTRACT. We study a variant of continuous Hochschild cohomology of a Banach algebra in connection with a higher-dimensional analogue of the approximate amenability of the algebra. Some results on higher-dimensional amenability have natural analogues in our context. Alternating cocycles, due to Johnson [**18**], are studied, and a previous result of the author [**17**] on Lipschitz algebras over compact metric spaces is improved.

1. Introduction and preliminaries. The notion of amenable Banach algebras was introduced by Johnson [**16**] and, since then, Banach algebras satisfying the amenability condition or its variants have been extensively studied. A higher-dimensional analogue of amenability was introduced and studied by Effros-Kishimoto, Paterson, Smith, and Lykova et al. [**4, 21, 22, 23**], etc. In [**5**], Ghahramani and Loy introduced the notion of approximate amenability, which was subsequently studied in [**6, 7, 8, 10**], etc.

This paper studies a variant of the continuous cohomology introduced by Pourabbas and Shirinkalam [**24**], here denoted by $\overline{\mathrm{H}}^n(A,X)_{\mathrm{pt}}$, of a Banach algebra A and a Banach A-bimodule X. The approximate amenability of A is equivalent to the condition $\overline{\mathrm{H}}^1(A,X^*)_{\mathrm{pt}}=0$ for the dual module X^* of an arbitrary Banach A-bimodule X [**24,** Example 3.2]. It is defined by taking the quotient of the space of cocycles by the *closure* of the subspace of coboundaries with respect to an appropriate topology on the cocycle space.

In Section 2, we define the group $\overline{\mathrm{H}}^n(A,X)_{\mathrm{pt}}$ for an arbitrary Banach algebra A and an arbitrary A-bimodule X, and we study some basic properties. First, we give an alternative proof of a theorem

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of **[24]**, stating that A is approximately amenable if and only if $\overline{\mathrm{H}}^n(A, X^*)_{\mathrm{pt}} = 0$ for each $n \geq 1$, and for each Banach A-bimodule X. This naturally leads to the notion of approximate n-amenability. There exists a canonical surjection $\mathrm{H}^n(A, X^*) \to \overline{\mathrm{H}}^n(A, X^*)_{\mathrm{pt}}$, and every n-amenable algebra is approximately n-amenable. We characterize unital approximately n-amenable Banach algebras by the existence of a net of certain $(n-1)$-cochains, which is a natural counterpart to the virtual n-diagonal introduced in **[17, 22]** (also see **[4, 23]**). In view of the notion of the pseudo-amenability due to Ghahramani and Zhang **[10]**, the condition so obtained could be called the "approximate n-pseudo-amenability," which is a subject of future study.

In Section 3, we consider alternating cocycles, due to Johnson **[18]**, in our context and study cohomology of the Lipschitz algebra $\mathrm{Lip}K$ over a compact metric space K. We prove that, if the space K has the property (seq) in the sense of **[19]**, then $\dim_{\mathbb{C}} \overline{\mathrm{H}}^n(\mathrm{Lip}K, (\mathrm{Lip}K)^*)_{\mathrm{pt}} = \infty$ for each $n \geq 1$. This improves a result of **[19]**.

Most of our proofs are straightforward modifications of the existing arguments; we hope that these results may shed light on approximate amenability and its higher-dimensional analogue.

The remainder of this section sets notation and recounts some basic facts. For a Banach space Z, $\|\cdot\|_Z$ denotes the norm of Z. For a Banach algebra A and a Banach A-bimodule X, recall the following inequality:

$$\|a \cdot x\|_X \leq \|a\|_A \|x\|_X, \quad a \in A,\ x \in X. \tag{1.1}$$

A Banach A-bimodule X of a Banach algebra A with unit e is said to be *unital* if $e \cdot x = x \cdot e = x$ for each $x \in X$. The dual space X^*, endowed with the operator norm, is a Banach A-bimodule, where the A-module structure is given by

$$\begin{gathered}(a \cdot \xi)(x) = \xi(xa), \qquad (\xi \cdot a)(x) = \xi(ax), \\ a \in A, \quad \xi \in X^*, \quad x \in X.\end{gathered} \tag{1.2}$$

It is sometimes convenient to use the notation

$$\langle \xi, x \rangle := \xi(x) \tag{1.3}$$

for an element x of a Banach space X and an element ξ of the dual space X^*. A continuous linear operator $D : A \to X$ is called a *derivation* if it satisfies the formula:

$$D(ab) = a \cdot Db + Da \cdot b, \quad a, b \in A. \tag{1.4}$$

For an element $x \in X$, an *inner derivation* is a derivation $\mathrm{ad}_x : A \to X$ defined by $\mathrm{ad}_x(a) = a \cdot x - x \cdot a$, $a \in A$. A Banach algebra is said to be *amenable* if each derivation $D : A \to X^*$ of A to the dual module X^* of an arbitrary Banach A-bimodule X is an inner derivation.

Definition 1.1 ([**5**])**.** A Banach algebra A is said to be *approximately amenable* if, for each derivation $D : A \to X^*$ of A to the dual module X^* of an arbitrary Banach A-bimodule X, there exists a net (ξ_ν) in X^* such that

$$\lim_\nu \|D(a) - \mathrm{ad}_{\xi_\nu}(a)\|_{X^*} = 0 \quad \text{for each } a \in A.$$

For Banach spaces X and Y, let $\mathcal{L}(X, Y)$ be the space of bounded linear operators of X to Y equipped with the operator norm. The projective tensor product of X and Y is denoted by $X \widehat{\otimes} Y$, and $X^{\widehat{\otimes}\, n}$ denotes the n-fold projective tensor product of X. For a Banach algebra A and a Banach A-bimodule X, let $C^n(A, X)$ be the space of n-cochains, that is, the space of bounded n-linear, $n \geq 1$, operators of A to X, endowed with the norm given by

$$\|f\| = \sup \left\{ \frac{\|f(a_1, \dots, a_n)\|}{\prod_{i=1}^n \|a_i\|} \;\middle|\; (a_1, \dots, a_n) \in (A \setminus \{0\})^n \right\}$$

for $f \in C^n(A, X)$. Also, let $C^0(A, X) = X$. By definition, $C^1(A, X) = \mathcal{L}(A, X)$. We recall the following basic facts (see, for example, [**15**]).

Lemma 1.2. *Let A and X be Banach spaces.*

(i) *Let $\Phi : C^{n-1}(A, \mathcal{L}(A, X)) \to C^n(A, X)$ be the linear operator defined by*

$$\Phi(f)(a_1, \dots, a_n) = (f(a_1, \dots, a_{n-1}))(a_n)$$

$$f \in C^{n-1}(A, \mathcal{L}(A, X)), \quad (a_1, \dots, a_n) \in A^n.$$

Then, Φ is an isometric isomorphism of Banach spaces.

(ii) *Let $\Psi : (A \widehat{\otimes} X)^* \to \mathcal{L}(A, X^*)$ be the linear operator defined by*

$$\langle \Psi(\xi)(a), x \rangle = \xi(a \otimes x), \quad \xi \in (A \widehat{\otimes} X)^*, \; a \in A, \; x \in X.$$

Then, Ψ *is an isometric isomorphism of Banach spaces.*

(iii) *Let* $\Theta : C^n(A,X) \to \mathcal{L}(A^{\widehat{\otimes}\, n}, X)$ *be the linear operator defined by*

$$\Theta(f)(a_1 \otimes \cdots \otimes a_n) = f(a_1, \ldots, a_n), \; a_1 \otimes \cdots \otimes a_n \in A^{\widehat{\otimes}\, n}.$$

Then, Θ *is an isometric isomorphism of Banach spaces.*

We use the A-module structure of X to define the coboundary operator $\delta^n : C^n(A,X) \to C^{n+1}(A,X)$:

$$\begin{aligned} \delta^n f(a_1, \ldots, a_{n+1}) &= a_1 \cdot (f(a_2, \ldots, a_{n+1})) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\ &\quad + (-1)^{n+1} f(a_1, \ldots, a_n) \cdot a_{n+1} \end{aligned} \tag{1.5}$$

for $f \in C^n(A,X)$ and $(a_1, \ldots a_{n+1}) \in A^{n+1}$. Then, $Z^n(A,X) = \mathrm{Ker}\ \delta^n$, $B^n(A,X) = \mathrm{Im}\ \delta^{n-1}$, and the continuous Hochschild cohomology is defined by $\mathrm{H}^n(A,X) = Z^n(A,X)/B^n(A,X)$. The first cohomology $\mathrm{H}^1(A,X)$ is isomorphic to the space of derivations modulo the inner derivations. Thus, A is amenable if and only if $\mathrm{H}^1(A,X^*) = 0$ for the dual X^* of an arbitrary Banach A-bimodule X [**15, 16**]. The operator δ^n is often denoted by δ for simplicity.

2. A variant of Hochschild cohomology and approximate amenability. Let A be a Banach algebra, and let X be a Banach A-bimodule. We consider the strong operator topology on the space of n-cochains $C^n(A,X)$, that is, the coarsest topology such that the map

$$C^n(A,X) \longrightarrow X; \qquad f \longmapsto f(a_1, \ldots, a_n), \quad f \in C^n(A,X),$$

is continuous for each $a_1, \ldots, a_n \in A$. The space $C^n(A,X)$ endowed with this topology is denoted by $C^n(A,X)_{\mathrm{pt}}$ (referring to the "pointwise convergence topology"). It is a locally convex Hausdorff topological vector space, and the coboundary operator $\delta^n : C^n(A,X)_{\mathrm{pt}} \to C^{n+1}(A,X)_{\mathrm{pt}}$ is continuous. Hence, the space $Z^n(A,X)$ of the n-cocycles is closed, while the space $B^n(A,X)$ of the n-coboundaries need not be a closed subspace. Let

$$\overline{B}^n(A,X)_{\mathrm{pt}} = \text{the closure of } B^n(A,X) \text{ in } C^n(A,X)_{\mathrm{pt}},$$

and define $\overline{\mathrm{H}}^n(A,X)_{\mathrm{pt}}$ by

$$\overline{\mathrm{H}}^n(A,X)_{\mathrm{pt}} = Z^n(A,X)/\overline{B}^n(A,X)_{\mathrm{pt}}. \tag{2.1}$$

In [**24,** Definition 3.1], the above group is denoted by $\mathcal{H}^n_{\mathrm{app}}(A,X)$. In order to indicate the topology on the cocycle space under consideration, we choose the above notation. An n-cocycle $f \in Z^n(A,X)$ represents a trivial element of $\overline{\mathrm{H}}^n(A,X)_{\mathrm{pt}}$ if and only if there exists a net (g_ν) in $C^{n-1}(A,X)$ such that, for each $(a_1,\dots,a_n) \in A^n$, we have

$$\lim_\nu \|f(a_1,\dots,a_n) - \delta^{n-1}g_\nu(a_1,\dots,a_n)\|_X = 0.$$

A few observations are in order.

Remark 2.1.

(1) There exists a continuous surjection $\mathrm{H}^n(A,X) \to \overline{\mathrm{H}}^n(A,X)_{\mathrm{pt}}$, for each Banach algebra A, and for each Banach A-bimodule X.

(2) For each finite-dimensional algebra A, and for each finite-dimensional A-bimodule X, we have the equality: $\mathrm{H}^n(A,X) = \overline{\mathrm{H}}^n(A,X)_{\mathrm{pt}}$.

(3) For a finite-dimensional Banach algebra A, $\overline{\mathrm{H}}^n(A,X)_{\mathrm{pt}} = 0$ for each Banach A-bimodule X if and only if $\mathrm{H}^n(A,X) = 0$ for each Banach A-bimodule X.

Proof. Statement (1) directly follows from the definition, cf., [**24,** Propositions 2, 3]). If a Banach algebra A and a Banach A-bimodule X are finite-dimensional, then $C^n(A,X)$ is finite-dimensional, $C^n(A,X)$ with the operator-norm topology coincides with the space $C^n(A,X)_{\mathrm{pt}}$, and the space $B^n(A,X)$ is closed in $C^n(A,X)_{\mathrm{pt}}$. This implies statement (2). For statement (3), assume that A is finite-dimensional and $\overline{\mathrm{H}}^n(A,X)_{\mathrm{pt}} = 0$ for each Banach A-bimodule X. Each $f \in Z^n(A,X)$ is regarded as a cocycle $f \in Z^n(A,X_0)$ for some finite-dimensional submodule X_0 of X. Then, the assumption and (2) imply $f = \delta g$ for some $g \in C^{n-1}(A,X_0) \subset C^{n-1}(A,X)$. This proves $\mathrm{H}^n(A,X) = 0$. $\square$

For a Banach algebra A, it is known that

(2.2)
$$\begin{gathered}\mathrm{H}^n(A,X^*) = 0 \text{ for each Banach } A\text{-bimodule } X \text{ if and only if} \\ \mathrm{H}^m(A,X^*) = 0 \text{ for each Banach } A\text{-bimodule} X \text{ and for each } m \geq n.\end{gathered}$$

In particular, A is amenable if and only if $\mathrm{H}^n(A, X^*) = 0$ for each Banach A-bimodule X and for each $n \geq 1$ [**16,** pages 8–9]. The next theorem is an analogue of this result in our context.

Theorem 2.2 ([**24,** Theorem 3.8])**.** *For a Banach algebra A, the following two conditions are equivalent.*

(i) $\overline{\mathrm{H}}^n(A, X^*)_{\mathrm{pt}} = 0$ *for each Banach A-bimodule X.*
(ii) $\overline{\mathrm{H}}^m(A, X^*)_{\mathrm{pt}} = 0$ *for each $m \geq n$ and for each Banach A-bimodule X.*

Proof. Only the implication (i) $\Rightarrow$ (ii) requires a proof. We follow the proof of [**16,** pages 8-9], taking into account the topology. For a Banach A-module X, the space $\mathcal{L}(A, X)$ is endowed with the A-module structure, given by

$$(a \cdot f)(b) = a \cdot (f(b)), \qquad (f \cdot a)(b) = f(ab) - (f(a)) \cdot b$$

for $f \in \mathcal{L}(A, X)$ and $a, b \in A$. Let $\Phi : C^{n-1}(A, \mathcal{L}(A, X)) \to C^n(A, X)$ be the isometric isomorphism of Lemma 1.2 (i). First, we verify that it induces a surjection

$$\overline{\Phi} : \overline{\mathrm{H}}^{n-1}(A, \mathcal{L}(A, X))_{\mathrm{pt}} \longrightarrow \overline{\mathrm{H}}^n(A, X)_{\mathrm{pt}}. \tag{2.3}$$

Note that the isomorphism Φ, with the above A-bimodule structure on $\mathcal{L}(A, X)$, commutes with the coboundary operators. Also, we see that the bijection

$$\Phi : C^{n-1}(A, \mathcal{L}(A, X))_{\mathrm{pt}} \longrightarrow C^n(A, X)_{\mathrm{pt}}$$

is continuous: if $(f_\nu)_\nu$ is a net in $C^{n-1}(A, \mathcal{L}(A, X))_{\mathrm{pt}}$ such that

$$\lim_\nu \|f_\nu(a_1, \ldots, a_{n-1})\|_{C^{n-1}(A,\mathcal{L}(A,X))} = 0, \quad (a_1, \ldots, a_{n-1}) \in A^n,$$

then we have

$$\begin{aligned} \|\Phi(f_\nu)(a_1, \ldots, a_n)\|_X &= \|f_\nu(a_1, \ldots, a_{n-1})(a_n)\|_X \\ &\leq \|f_\nu(a_1, \ldots, a_{n-1})\|_{C^{n-1}(A,\mathcal{L}(A,X))} \|a_n\|_A, \end{aligned}$$

which implies $\lim_\nu \|\Phi(f_\nu)(a_1, \ldots, a_n)\|_X = 0$. Hence, we have the inclusion

$$\begin{aligned} \Phi(\overline{B}^n(A, \mathcal{L}(A, X))_{\mathrm{pt}}) &\subset \overline{\Phi(B^n(A, \mathcal{L}(A, X)))}^{\mathrm{pt}} \\ &= \overline{B^n(A, X)}^{\mathrm{pt}} = \overline{B}^n(A, X)_{\mathrm{pt}}, \end{aligned}$$

where $\overline{\Phi(B^n(A,\mathcal{L}(A,X)))}^{\mathrm{pt}}$ denotes the closure of $\Phi(B^n(A,\mathcal{L}(A,X)))$ in $C^n(A,X)_{\mathrm{pt}}$. Hence, the induced map $\overline{\Phi}$ is a surjection.

Next, we apply Lemma 1.2 (ii) to take the isometry $\Psi : (A\widehat{\otimes} X)^* \to \mathcal{L}(A,X^*)$, which induces an A-bimodule structure on $(A\widehat{\otimes} X)^*$ so that the map Ψ is an A-module isomorphism. It induces a topological isomorphism

$$\Psi_\sharp : C^{n-1}(A,(A\widehat{\otimes} X)^*))_{\mathrm{pt}} \longrightarrow C^{n-1}(A,\mathcal{L}(A,X^*))_{\mathrm{pt}},$$

which commutes with the coboundary operators. Hence, Ψ induces an isomorphism

$$\overline{\Psi} : \overline{\mathrm{H}}^{n-1}(A,(A\widehat{\otimes} X)^*)_{\mathrm{pt}} \longrightarrow \overline{\mathrm{H}}^{n-1}(A,\mathcal{L}(A,X))_{\mathrm{pt}}.$$

Thus, we have a surjection

$$\overline{\Phi}\circ\overline{\Psi} : \overline{\mathrm{H}}^{n-1}(A,(A\widehat{\otimes} X)^*)_{\mathrm{pt}} \longrightarrow \overline{\mathrm{H}}^{n}(A,X^*)_{\mathrm{pt}},$$

and the proof is complete by induction. □

Remark 2.3. The inverse operator

$$\Phi^{-1} : C^n(A,X) \longrightarrow C^{n-1}(A,\mathcal{L}(A,X))$$

is an isometry with respect to operator norms, while it is not necessarily a continuous operator

$$\Phi^{-1} : C^n(A,X)_{\mathrm{pt}} \longrightarrow C^{n-1}(A,\mathcal{L}(A,X))_{\mathrm{pt}}.$$

If it happens to be continuous, then we obtain the isomorphism

$$\overline{\mathrm{H}}^{n-1}(A,(A\widehat{\otimes} X)^*)_{\mathrm{pt}} \cong \overline{\mathrm{H}}^{n}(A,X^*)_{\mathrm{pt}}.$$

As a higher-dimensional analogue of the amenability, Paterson **[22]** introduced the notion of n-amenability: a Banach algebra A is said to be *n-amenable* if $\mathrm{H}^n(A,X^*)=0$ for each Banach A-bimodule X. This has a natural analogue in the present context.

Definition 2.4. Let $n \geq 1$. A Banach algebra A is said to be *approximately n-amenable* if $\overline{\mathrm{H}}^{n}(A,X^*)_{\mathrm{pt}} = 0$ for each Banach A-bimodule X.

The approximate 1-amenability coincides with the approximate amenability.

Remark 2.5.

(1) Every n-amenable Banach algebra is approximately n-amenable. For every finite-dimensional Banach algebra, the converse holds.

(2) Every approximately n-amenable Banach algebra is approximately m-amenable for each $m \geq n$. There exists an approximately n-amenable, but not approximately $(n-1)$-amenable, Banach algebra.

(3) There exists an approximately n-amenable Banach algebra which is not n-amenable.

Proof. Statement (1) is a direct consequence of Remark 2.1. The first statement of (2) follows from Theorem 2.2. Paterson and Smith [**23,** Theorem 4.2] gave an example of a finite-dimensional matrix algebra, denoted by B_n, which is n-amenable, but not $(n-1)$-amenable. From (1), we see that B_n serves as an example for the second statement of (2). For statement (3), take a norm-closed subalgebra A of $\mathcal{L}(H, H)$ for some Hilbert space H, which is approximately amenable but not amenable, given by Choi [**2**]. From the construction, we see that the algebra A fails to have the total reduction property, see [**11**], and hence, $H^1(A, \mathcal{L}(H, H)) \neq 0$ for some A-bimodule structure on $\mathcal{L}(H, H)$, [**11,** Theorem 2.1]. Take the $(n-1)$-fold suspension $\mathcal{S}^{n-1}(A)$ in the sense of Gilfeather and Smith [**12**]. Then, we have

$$\mathrm{H}^1(A, \mathcal{L}(H, H)) \cong \mathrm{H}^n(\mathcal{S}^{n-1}(A), \mathcal{L}(H, H)) \neq 0.$$

Furthermore, the proof of [**12**] shows that there exists a continuous surjection

$$\overline{\mathrm{H}}^1(A, \mathcal{L}(H, H))_{\mathrm{pt}} \longrightarrow \overline{\mathrm{H}}^n(\mathcal{S}^{n-1}(A), \mathcal{L}(H, H))_{\mathrm{pt}}.$$

By the approximate amenability of A, we have $\overline{\mathrm{H}}^1(A, \mathcal{L}(H, H))_{\mathrm{pt}} = 0$; thus, we obtain $\overline{\mathrm{H}}^n(\mathcal{S}^{n-1}(A), \mathcal{L}(H, H))_{\mathrm{pt}} = 0$. Hence, $\mathcal{S}^{n-1}(A)$ is approximately n-amenable, but not n-amenable. $\square$

Next, we give an approximate analogue of the characterization theorem of n-amenable unital Banach algebras in terms of higher-dimensional virtual diagonals, [**4, 17, 22, 23**]. Let $C_n(A) = A^{\widehat{\otimes}\, n}$

and recall the natural isomorphism $C^n(A,X) \cong \mathcal{L}(C_n(A),X)$, Lemma 1.2 (iii). Let $\pi_n : C_n(A) \to C_{n-1}(A)$ be the map defined by

$$\pi_n(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1}(-1)^{i+1}a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \tag{2.4}$$

for $a_1 \otimes \cdots \otimes a_n \in C_n(A)$. We have $\pi_{n-1} \circ \pi_n = 0$. Let $i_k : C_k(A) \to C_k(A)^{**}$ be the canonical injection, and note that

$$\pi_n^{**} \circ i_n = i_{n-1} \circ \pi_n \tag{2.5}$$

for each $n \geq 1$.

Definition 2.6 ([**22**])**.** Let A be a Banach algebra with unit e, and let $n \geq 1$. A cocycle $M \in Z^{n-1}(A, C_{n+1}(A)^{**})$ is called a *virtual n-diagonal* if, for each $(a_1, \ldots, a_{n-1}) \in A^{n-1}$, we have

$$\pi_{n+1}^{**}M(a_1, \ldots, a_{n-1}) = i_n(\pi_{n+1}(e \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes e)).$$

For $n = 1$, this notion coincides with the virtual diagonal in the sense of Johnson [**17**]. The n-amenability of a unital Banach algebra A is equivalent to the existence of a virtual n-diagonal [**4, 17**], and a similar characterization for general, not necessarily unital, Banach algebras has been proved by Paterson [**22,** Theorem 3.2]. The latter proof relies on the machinery of homological algebra which is not currently known to be available for the present cohomology $\overline{\mathrm{H}}^n(A,X)_{\mathrm{pt}}$. Also, it is shown in [**22,** Theorem 4.2] (also, see [**23,** Theorem 3.1]) that a Banach algebra is n-amenable ($n \geq 1$) if and only if there exists a virtual $(n+1)$-diagonal which is a coboundary.

The next theorem is an approximate analogue of these theorems for unital algebras. Our proof is a straightforward modification of those of [**4,** Theorem 3.1] and [**22,** Theorem 4.2] with the aid of [**23,** Theorem 3.1]. For a net (ξ_ν) in the dual space X^* of a Banach space X, $\lim_\nu \xi_\nu = \xi$ means $\lim_\nu \|\xi_\nu - \xi\|_{X^*} = 0$, the convergence *with respect to the operator norm* on X^*.

Theorem 2.7. *Let A be a Banach algebra with unit e. For $n \geq 1$, the following conditions are equivalent*:

(i) *the algebra A is approximately n-amenable*;

(ii) *there exists a net* $(M^\nu)_\nu$ *in* $C^{n-1}(A, C_{n+1}(A)^{**})$ *such that, for each* $(a_1, \dots, a_n) \in A^n$ *and for each* $(b_1, \dots, b_{n-1}) \in A^{n-1}$*, we have*

(a) $\lim_\nu \delta M^\nu(a_1, \dots, a_n) = 0$; *and*

(b) $\lim_\nu \pi^{**}_{n+1} M^\nu(b_1, \dots, b_{n-1}) = i_n(\pi_{n+1}(e \otimes b_1 \otimes \cdots \otimes b_{n-1} \otimes e))$;

(iii) *there exists a net* $(L^\nu)_\nu$ *in* $C^n(A, C_{n+2}(A)^{**})$ *such that*

(a) *there exists a cochain* $W^\nu \in C^{n-1}(A, C_{n+2}(A)^{**})$ *such that* $\lim_\nu L^\nu(a_1, \dots, a_n) = \lim_\nu \delta W^\nu(a_1, \dots, a_n)$ *for each* $(a_1, \dots, a_{n-1}) \in A^{n-1}$;

(b) $\lim_\nu \pi^{**}_{n+2} L^\nu(b_1, \dots, b_n) = i_{n+1}(\pi_{n+2}(e \otimes b_1 \otimes \cdots \otimes b_n \otimes e))$ *for each* $(b_1, \dots, b_n) \in A^n$*.*

We follow the proofs of [**4,** Theorem 3.1] and [**22,** Theorem 4.2]. For a Banach algebra A with unit e and, for $k \geq 1$, we define a cochain $h_k \in C^k(A, C_{k+2}(A)) \cong \mathcal{L}(C_k(A), C_{k+2}(A))$ by

$$h_k(a_1 \otimes \cdots \otimes a_k) = e \otimes a_1 \otimes \cdots \otimes a_k \otimes e \tag{2.6}$$

for $a_1 \otimes \cdots \otimes a_k \in C_k(A)$. In addition, let $h_0 = e \otimes e \in C_2(A)$. As proven in [**4**], we have

Lemma 2.8 ([**4,** page 272])**.** *For each* $m \geq 1$*, we have*

$$\pi_{m+2}(h_m) = \delta h_{m-1} \in C^m(A, C_{m+1}(A)).$$

Proof of Theorem 2.7*.*

(i) $\Leftrightarrow$ (ii). First, assume that A is approximately n-amenable, and consider the cochain $h_{n-1} \in C^{n-1}(A, C_{n+1}(A))$. From Lemma 2.8, we have $\pi^{**}_{n+1}(i_{n+1}(\delta h_{n-1})) = (i_n \circ \pi_{n+1})(\pi_{n+2}(h_n)) = 0$, and thus, we have

$$i_{n+1}(\delta h_{n-1}) \in C^n(A, \operatorname{Ker} \pi^{**}_{n+1}) \subset C^n(A, C_{n+1}(A)^{**}).$$

Let $\overline{\operatorname{Im} \pi^*_{n+1}}$ be the closure of the subspace $\operatorname{Im} \pi^*_{n+1}$ of $C_{n+1}(A)^*$ with respect to the operator-norm topology, and let $V = C_{n+1}(A)^*/\overline{\operatorname{Im} \pi^*_{n+1}}$ for which we have $V^* \cong \operatorname{Ker} \pi^{**}_{n+1}$. Since A is approximate n-amenable, we have a net $(f^\nu) \in C^{n-1}(A, V^*) = C^{n-1}(A, \operatorname{Ker} \pi^{**}_{n+1})$ such that $\lim_\nu \delta f^\nu(a_1, \dots, a_n) = i_{n+1}(\delta h_{n-1})$ for each $(a_1, \dots, a_n) \in A^n$. Let

M^ν be the $(n-1)$-cochain, defined by

$$M^\nu = i_{n+1}(h_{n-1}) - f^\nu \in C^{n-1}(A, C_{n+1}(A)^{**}).$$

Then, we have

$$\lim_\nu \delta M^\nu = i_{n+1}(\delta(h_{n-1})) - \lim_\nu \delta f^\nu = 0,$$

and

$$\begin{aligned}\lim_\nu \pi^{**}_{n+1} M^\nu &= \pi^{**}_{n+1} i_{n+1}(h_{n-1}) - \lim_\nu \pi^{**}_{n+1} f^\nu \\ &= \pi^{**}_{n+1}(i_{n-1}(h_{n-1})) = i_n(\pi_{n+1}(h_{n-1})).\end{aligned}$$

Thus, (M^ν) is the desired net, cf., [**4**].

The proof of the implication (ii) $\Rightarrow$ (i) is divided into two steps.

Step 1. For an arbitrary unital Banach A-bimodule V, we construct a net $(P^\nu : C^{n-1}(A, V^*) \to C^{n-1}(A, V^*))$ of bounded linear operators which satisfies the following conditions:

(a) for each $(a_1, \dots, a_n) \in A^n$, we have $\lim_\nu \delta P^\nu(a_1, \dots, a_n) = 0$; and

(b) for each $f \in Z^{n-1}(A, V^*)$, and for each $(b_1, \dots, b_{n-1}) \in A^{n-1}$, we have $f(b_1, \dots, b_{n-1}) = \lim_\nu P^\nu f(b_1, \dots, b_{n-1})$.

For a cochain $f \in C^{n-1}(A, V^*)$, we follow [**4,** page 273] to define an $(n+1)$-cochain $F_f \in C^{n+1}(A, V^*)$ by

$$\begin{aligned}F_f(a_1, \dots, a_{n+1}) &= a_1 \cdot (f(a_2, \dots, a_n)) \cdot a_{n+1}, \\ &(a_1, \dots, a_{n+1}) \in A^{n+1}.\end{aligned} \tag{2.7}$$

For $v \in V$, let $F^v_f \in C_{n+1}(A)^*$ be the element defined by

$$(F^v_f)(a_1 \otimes \cdots \otimes a_{n+1}) = \langle F_f(a_1, \dots, a_{n+1}), v\rangle. \tag{2.8}$$

For each $a \in A$ and for each $v \in V$, we have the equalities:

$$F^{a\cdot v}_f = a \cdot F^v_f, \ \ F^{v\cdot a}_f = F^v_f \cdot a. \tag{2.9}$$

The operator $P^\nu : C^{n-1}(A, V^*) \to C^{n-1}(A, V^*)$ is then defined by

$$\langle P^\nu(f)(a_1, \dots, a_{n-1}), v\rangle = \langle M^\nu(a_1, \dots, a_{n-1}), F^v_f\rangle, \quad v \in V. \tag{2.10}$$

We verify that (P^ν) satisfies conditions (a) and (b). Direct computation using (2.9) reveals

$$\langle \delta P^\nu(f)(a_1,\ldots,a_n), v\rangle = \langle \delta M^\nu(a_1,\ldots,a_n), F_f^v\rangle \tag{2.11}$$

for each $(a_1,\ldots,a_n) \in A^n, v \in V$. It follows from this that

$$\begin{aligned} |\langle \delta P^\nu(f)(a_1,\ldots,a_n), v\rangle| &\le \|\delta M^\nu(a_1,\ldots,a_n)\|_{C_{n+1}(A)^{**}} \\ &\quad \cdot \|F_f^v\|_{C_{n+1}(A)^*} \\ &\le \|\delta M^\nu(a_1,\ldots,a_n)\|_{C_{n+1}(A)^{**}} \\ &\quad \cdot \|f\|_{C^{n-1}(A,V^*)} \cdot \|v\|_V, \end{aligned}$$

where we use (2.7), (2.8) and (1.1) for the second inequality. This and condition (ii) (a) of the hypothesis imply:

$$\begin{aligned} &\lim_\nu \|\delta P^\nu(f)(a_1,\ldots,a_n)\|_{V^*} \\ &\qquad \le \|f\|_{C^{n-1}(A,V^*)}\Big(\lim_\nu \|\delta M^\nu(a_1,\ldots,a_n)\|_{C_{n+1}(A)^{**}}\Big) = 0, \end{aligned}$$

and hence, condition (a) follows. For condition (b), we introduce another cochain $G_f \in C^n(A, V^*)$ by

$$G_f(a_1,\ldots,a_n) = f(a_1,\ldots,a_{n-1}) \cdot a_n, \quad (a_1,\ldots,a_n) \in A^n. \tag{2.12}$$

For each $v \in V$, let $G_f^v \in C_{n+1}(A)^*$ be the element given by

$$G_f^v(a_1,\ldots,a_n) = \langle G_f(a_1,\ldots,a_n), v\rangle.$$

Take a cocycle $f \in Z^{n-1}(A, C_{n+1}(A)^{**})$. As in [**4,** page 274], we obtain

$$F_f^v = \pi_{n+1}^* G_f^v \tag{2.13}$$

by making use of the cocycle condition $(\delta f)(a_1,\ldots,a_n) = 0$. Then, we have

$$\begin{aligned} \langle P^\nu(f)(a_1,\ldots,a_{n-1}), v\rangle &= \langle M^\nu(a_1,\ldots,a_{n-1}), F_f^v\rangle \\ &= \langle \pi_{n+1}^{**} M^\nu(a_1,\ldots,a_{n-1}), G_f^v\rangle. \end{aligned} \tag{2.14}$$

Since V is a unital module, we may use the cocycle condition

$$(\delta f)(e, a_1,\ldots,a_{n-1}) = 0$$

to obtain the equality

$$\text{(2.15)}\quad \sum_{i=1}^{n-2}(-1)^{i+1}f(e,a_1,\dots,a_ia_{i+1},\dots,a_{n-1}) \\ + (-1)^n f(e,a_1,\dots,a_{n-2})\cdot a_{n-1} = 0$$

for each $(a_1,\dots,a_{n-1}) \in A^{n-1}$. Combining (2.15) with Lemma 2.8, we have

$$\begin{aligned} &G_f(\pi_{n+1}(h_{n-1})(a_1,\dots,a_{n-1})) \\ &\quad = G_f(\delta h_{n-2}(a_1,\dots,a_{n-1})) \\ &\quad = G_f(a_1,\dots a_{n-1},e) + \sum_{i=1}^{n-2}(-1)^i G_f(a_1,\dots,a_ia_{i+1},\dots a_{n-1}) \\ &\qquad + (-1)^{n-1}G_f(e,a_1,\dots,a_{n-1}) \\ &\quad = f(a_1,\dots,a_{n-1}) + \sum_{i=1}^{n-2}(-1)^i f(e,a_1,\dots,a_ia_{i+1},\dots,a_{n-1}) \\ &\qquad + (-1)^{n-1}f(e,a_1,\dots,a_{n-2})\cdot a_{n-1} \\ &\quad = f(a_1,\dots,a_{n-1}) \quad \text{(by (2.15))}. \end{aligned}$$

In other words, $\langle G_f, \pi_{n+1}(h_{n-1})\rangle = f$, and thus,

$$\text{(2.16)}\qquad \begin{aligned} \langle i_n\pi_{n+1}(h_{n-1}), G_f\rangle &= \langle \pi_{n+1}^{**}i_{n+1}(h_{n-1}), G_f\rangle \quad \text{(by (2.5))} \\ &= \langle i_{n+1}(h_{n-1}), \pi_{n+1}^{*}G_f\rangle \\ &= \langle G_f, \pi_{n+1}(h_{n-1})\rangle = f. \end{aligned}$$

Using (2.10) and (2.16) we have, for $b_1,\dots,b_{n-1}\in A$,

$$\begin{aligned} &\|P^\nu(f)(b_1,\dots,b_{n-1}) - f(b_1,\dots,b_{n-1})\|_{V^*} \\ &\quad = \|\langle \pi_{n+1}^{**}M^\nu(b_1,\dots,b_{n-1}) \\ &\qquad\qquad - i_{n+1}(\pi_{n+1}(h_{n-1}(b_1,\dots,b_{n-1}))), G_f\rangle\|_{V^*} \\ &\quad \le \|\pi_{n+1}^{**}M^\nu(b_1,\dots,b_{n-1}) \\ &\qquad\qquad - i_{n+1}(\pi_{n+1}(h_{n-1}(b_1,\dots,b_{n-1})))\|_{C_{n+1}(A)^{**}}\|G_f\|_{C_{n+1}(A)^*}. \end{aligned}$$

By condition (ii) (b) of the hypothesis, the last term converges to 0 and

$$\lim_\nu P^\nu(f)(b_1,\dots,b_{n-1}) = f(b_1,\dots,b_{n-1}), \quad (b_1,\dots,b_{n-1})\in A^{n-1}.$$

Thus, we obtain condition (b).

Step 2. In order to prove $\overline{\mathrm{H}}^n(A, X^*)_{\mathrm{pt}} = 0$, the standard reduction [**18,** page 12] allows us to assume that the Banach A-bimodule X is unital. We take the composition of the linear isomorphisms of Lemma 1.2:

$$\omega = \Psi \circ \Phi : C^n(A, X^*) \longrightarrow C^{n-1}(A, \mathcal{L}(A, X^*)) \longrightarrow C^{n-1}(A, (A \widehat{\otimes} X)^*)$$
$$\langle \omega(f)(a_1, \ldots, a_{n-1}), a \otimes x \rangle = \langle f(a_1, \ldots, a_{n-1}, a), x \rangle$$
$$f \in C^n(A, X^*), \quad (a_1, \ldots, a_{n-1}) \in A^{n-1}, \quad a \otimes x \in A \widehat{\otimes} X.$$

The map ω is an A-bimodule isomorphism where the A-bimodule structure of $(A \widehat{\otimes} X)^*$ is given by

$$\langle a \cdot \xi \cdot b, c \otimes x \rangle = \langle \xi, b \cdot c \otimes x \cdot a \rangle - \langle \xi, b \otimes c \cdot x \cdot a \rangle$$

for $a, b \in A$, $\xi \in (A \widehat{\otimes} X)^*$, $c \otimes x \in A \widehat{\otimes} X$ so that the isometry Ψ is an A-module isomorphism (see the beginning of the proof of Theorem 2.2). We apply Step 1 to $V = A \widehat{\otimes} X$ and obtain a net $(P^\nu : C^{n-1}(A, (A \widehat{\otimes} X)^*) \to C^{n-1}(A, (A \widehat{\otimes} X)^*))_\nu$, satisfying conditions (a) and (b).

For each $f \in Z^n(A, X^*)$ we have $\omega(f) \in Z^{n-1}(A, (A \widehat{\otimes} X)^*)$. From condition (b), we see

$$\begin{aligned} \omega(f)(a_1, \ldots, a_{n-1}) &= \lim_\nu P^\nu(f)(a_1, \ldots, a_{n-1}) \\ &= \lim_\nu \langle M^\nu(a_1, \ldots, a_{n-1}), F_{\omega(f)} \rangle \end{aligned}$$

for $(a_1, \ldots, a_{n-1}) \in A^{n-1}$. For $a \in A$, define $\varepsilon(a) : (A \widehat{\otimes} X)^* \to X^*$ by

$$\langle \varepsilon(a)(\gamma), x \rangle = \langle \gamma, a \otimes x \rangle$$

for $\gamma \in (A \widehat{\otimes} X)^*$, $x \in X$. We follow the computation of [**4,** page 275] to have:

$$\begin{aligned} f(a_1, \ldots, a_n) &= \lim_\nu \varepsilon(a_n)(\langle M^\nu(a_1, \ldots, a_{n-1}), F_{\omega(f)} \rangle) \\ &= \lim_\nu \langle M^\nu(a_1, \ldots, a_{n-1}), \varepsilon(a_n) \circ F_{\omega(f)} \rangle. \end{aligned} \tag{2.17}$$

Let $K_f \in C^{n+1}(A, X^*)$ be the cochain defined by

$$K_f(a_1, \ldots, a_{n+1}) = a_1 \cdot (f(a_2, \ldots, a_{n+1})), \quad (a_1, \ldots, a_{n+1}) \in A^{n+1},$$

and, for $v \in X$, let K_f^v be the element of $C_{n+1}(A)^{**}$ given by

$$K_f^v(a_1, \ldots, a_{n+1}) = \langle K_f(a_1, \ldots, a_{n+1}), v \rangle.$$

We obtain, as in [**4,** page 275], that

$$\begin{aligned}\varepsilon(a_n)\circ F_{\omega(f)}(b_1,\dots,b_{n+1}) \\ = (a_n\cdot K_f)(b_1,\dots,b_{n+1}) - K_f(b_1,\dots,b_{n+1})\cdot a_n, \\ (b_1,\dots,b_{n+1})\in A^{n+1}.\end{aligned}\tag{2.18}$$

Using (2.18), we obtain from (2.17)

$$\begin{aligned} f(a_1,\dots,a_n) &= \lim_\nu \big\{\langle M^\nu(a_1,\dots,a_{n-1}), a_n\cdot K_f\rangle \\ &\qquad - \langle M^\nu(a_1,\dots,a_{n-1}), K_f\cdot a_n\rangle\big\} \\ &= \lim_\nu \big\{\langle M^\nu(a_1,\dots,a_{n-1})\cdot a_n, K_f\rangle \\ &\qquad - \langle M^\nu(a_1,\dots,a_{n-1}), K_f\rangle\cdot a_n\big\}.\end{aligned}\tag{2.19}$$

Let $g_f^\nu\in C^{n-1}(A,X^*)$ be the cochain defined by

$$\langle g_f^\nu(a_1,\dots,a_{n-1}),v\rangle = \langle M^\nu(a_1,\dots,a_{n-1}), K_f^v\rangle, \\ v\in X,\quad (a_1,\dots,a_{n-1})\in A^{n-1}.$$

In what follows, we compare $(-1)^n f$ with δg^ν. First, it follows directly from the definition that

$$K_f^{v\cdot a} = K_f^v\cdot a,\quad v\in X,\ a\in A,$$

from which we conclude

$$a_1\cdot\langle M^\nu(a_2,\dots,a_n),K_f\rangle = \langle a_1\cdot M^\nu(a_2,\dots,a_n),K_f\rangle.\tag{2.20}$$

Using (2.20) with the coboundary formula:

$$\begin{aligned}(\delta M^\nu)(a_1,\dots,a_n) &= a_1\cdot(M^\nu(a_2,\dots,a_n)) \\ &+ \sum_{i=1}^{n-1}(-1)^i M^\nu(a_1,\dots,a_ia_{i+1},\dots,a_n) \\ &+ (-1)^n M^\nu(a_1,\dots,a_{n-1})\cdot a_n,\end{aligned}$$

we obtain, as in [**22,** page 276],

$$\begin{aligned}&(-1)^n f(a_1,\dots,a_n) \\ &\quad = \lim_\nu\langle(\delta M^\nu)(a_1,\dots,a_n),K_f\rangle\end{aligned}$$

$$
\begin{aligned}
&-\lim_{\nu}\Big\{a_1\cdot\langle M^\nu(a_2,\dots,a_n),K_f\rangle \\
&\qquad\qquad+\sum_{i=1}^{n-1}(-1)^i\langle M^\nu(a_1,\dots,a_ia_{i+1},\dots,a_n),K_f\rangle \\
&\qquad\qquad\qquad\qquad+(-1)^n\langle M^\nu(a_1,\dots,a_{n-1}),K_f\rangle\cdot a_n\Big\} \\
&=\lim_{\nu}\langle(\delta M^\nu)(a_1,\dots,a_n),K_f\rangle-\lim_{\nu}\delta g^\nu(a_1,\dots,a_n) \\
&=-\lim_{\nu}\delta g^\nu(a_1,\dots,a_n).
\end{aligned}
$$

Therefore, $f\in\overline{B}^n(A,X^*)_{\mathrm{pt}}$. This proves $\overline{\mathrm{H}}^n(A,X^*)_{\mathrm{pt}}=0$. □

For the proof of equivalence (i) $\Leftrightarrow$ (iii), we follow the argument [**22,** Theorem 4.2] with the aid of [**23,** Theorem 3.1].

Lemma 2.9 ([**23,** Theorem 3.1])**.** *Let A be a unital Banach algebra. For $n\geq 1$, there exists a cochain $Z\in C^{n-1}(A,C_{n+2}(A)^{**})$ such that*

$$\pi_{n+2}^{**}((\delta Z)(a_1,\dots,a_n)-e\otimes a_1\otimes\cdots\otimes a_{n-1}\otimes e\otimes a_n)=0,$$

for each $(a_1,\dots,a_n)\in A^n$.

For $F\in C_k(A)^{**}$ and $a\in A$, let $F\otimes a\in C_{k+1}(A)^{**}$ be the element defined by

$$\langle F\otimes a,\xi\rangle=\langle F,\gamma_{a,\xi}\rangle,\quad \xi\in C_k(A)^*,$$

where $\gamma_{a,\xi}\in C_k(A)^*$ is the element given by

$$\langle w,\gamma_{a,\xi}\rangle=\langle w\otimes a,\xi\rangle,\ w\in C_k(A).$$

For a cochain $N\in C^n(A,C_{n+2}(A)^{**})$ and $a\in A$, let $N\otimes a\in C^n(A,C_{n+3}(A)^{**})$ be the cochain defined by

$$(N\otimes a)(a_1,\dots,a_n)=N(a_1,\dots,a_n)\otimes a$$

for $(a_1,\dots,a_n)\in A^n$.

Lemma 2.10 (cf. [**22,** Proposition 4.1])**.** *Let $n\geq 1$, and let (N^ν) be a net in $C^{n-1}(A,C_{n+1}(A)^{**})$ satisfying conditions* (ii) (a) *and* (ii) (b) *of Theorem* 2.7*. Then, we have the equality*

$$
\begin{aligned}
\lim_{\nu}\pi_{n+2}^{**}\big\{\delta(N^\nu\otimes e)(a_1,\dots,a_n)+(-1)^n i_{n+2}(e\otimes a_1\otimes\cdots\otimes a_n\otimes e)\big\}& \\
=(-1)^n i_{n+1}(\pi_{n+2}(e\otimes a_1\otimes\cdots a_{n-1}\otimes e\otimes a_n))&
\end{aligned}
$$

for each $(a_1,\dots,a_n)\in A^n$.

Proof. The net $(N^\nu)_\nu \subset C^{n-1}(A, C_{n+1}(A)^{**})$ satisfies the following conditions:

(i) $\lim_\nu (\delta N^\nu)(a_1, \ldots, a_n) = 0$ for each $(a_1, \ldots, a_n) \in A^n$;
(ii) $\lim_\nu \pi_{n+1}^{**} N^\nu(b_1, \ldots, b_{n-1}) = \pi_{n+1}(e \otimes b_1 \otimes \cdots \otimes b_{n-1} \otimes e)$ for each $b_1, \ldots, b_{n-1} \in A$.

As in [**22,** Proposition 4.1 (4.1)], we have

$$\delta(N^\nu \otimes e)(a_1, \ldots, a_n) = (\delta N^\nu)(a_1, \ldots, a_n) \otimes e$$
$$+ (-1)^{n+1} N^\nu(a_1, \ldots, a_{n-1}) \cdot a_n \otimes e + (-1)^n N^\nu(a_1, \ldots, a_{n-1}) \otimes a_n.$$

By direct computation, we obtain

$$\gamma_{a, \pi_{n+2}^* \xi} = \pi_{n+1}^* \gamma_{a,\xi} + (-1)^n a \cdot \xi, \quad a \in A, \ \xi \in C_{n+1}(A)^*.$$

It then follows that

$$\pi_{n+2}^{**}(L \otimes a) = (\pi_{n+1}^{**} L) \otimes a + (-1)^n L \cdot a, \qquad L \in C_{n+1}(A)^{**}, \quad a \in A. \tag{2.21}$$

Applying (2.21) and using that π_{n+1}^{**} is an A-module homomorphism, we obtain

(2.22)
$$\begin{aligned} &\pi_{n+2}^{**}(N^\nu(a_1, \ldots, a_{n-1}) \cdot a_n \otimes e) \\ &\quad = \pi_{n+1}^{**}(N^\nu(a_1, \ldots, a_{n-1}) \cdot a_n) \otimes e + (-1)^n N^\nu(a_1, \ldots, a_{n-1}) \cdot a_n \\ &\quad = \pi_{n+1}^{**}(N^\nu(a_1, \ldots, a_{n-1})) \cdot a_n \otimes e + (-1)^n N^\nu(a_1, \ldots, a_{n-1}) \cdot a_n. \end{aligned}$$

We use (2.21) to obtain

$$\begin{aligned} &\pi_{n+2}^{**}(N^\nu(a_1, \ldots, a_{n-1}) \otimes a_n) \\ &\quad = \pi_{n+1}^{**}(N^\nu(a_1, \ldots, a_{n-1})) \otimes a_n + (-1)^n N^\nu(a_1, \ldots, a_{n-1}) \cdot a_n. \end{aligned} \tag{2.23}$$

Then, by (2.22) and (2.23), we have

$$\begin{aligned} &\pi_{n+2}^{**}(\delta(N^\nu \otimes e))(a_1, \ldots, a_n) \\ &\quad = \pi_{n+2}^{**}((\delta N^\nu)(a_1, \ldots, a_n) \otimes e) \\ &\qquad + (-1)^{n+1}\big\{\pi_{n+1}^{**}(N^\nu(a_1, \ldots, a_{n-1})) \cdot a_n \\ &\qquad\qquad \otimes e - \pi_{n+1}^{**}(N^\nu(a_1, \ldots, a_{n-1})) \otimes a_n\big\} \end{aligned}$$

for each $(a_1, \dots, a_n) \in A^n$. Taking the limit and using condition (a), we have

$$
\begin{aligned}
(2.24)\qquad &(-1)^{n+1} \lim_{\nu} \pi_{n+2}^{**}(\delta(N^\nu \otimes e)(a_1, \dots, a_n)) \\
&= \lim_{\nu} \big\{\pi_{n+1}^{**}(N^\nu(a_1, \dots, a_{n-1}) \cdot a_n \otimes e) \\
&\qquad\qquad - \pi_{n+1}^{**}(N^\nu(a_1, \dots, a_{n-1})) \otimes a_n)\big\} \\
&= i_n \pi_{n+1}(e \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes e) \\
&\cdot a_n \otimes e - i_n \pi_{n+1}(e \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes e) \otimes a_n.
\end{aligned}
$$

We carry out the same computation as that in **[22,** Proposition 4.1] to see that the last term of (2.24) is equal to:

$$
\begin{aligned}
(2.25)\qquad & i_{n+1}\pi_{n+2}(e \otimes a_1 \otimes \cdots \otimes a_n \otimes e) \\
&\qquad - i_{n+1}(\pi_n(e \otimes a_1 \otimes \cdots \otimes a_{n-1}) \otimes e \otimes a_n).
\end{aligned}
$$

On the other hand, we see by direct computation:

$$
\begin{aligned}
(2.26)\qquad & \pi_{n+2}(e \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes e \otimes a_n) \\
&\qquad = \pi_n(e \otimes a_1 \otimes \cdots \otimes a_{n-1}) \otimes e \otimes a_n.
\end{aligned}
$$

Using (2.26) in (2.25), we obtain the following:

$$
\begin{aligned}
&\lim_{\nu} \pi_{n+2}^{**}(\delta(N^\nu \otimes e)(a_1, \dots, a_n)) \\
&\quad = (-1)^{n+1}\big\{i_{n+1}\pi_{n+2}(e \otimes a_1 \otimes \cdots \otimes a_n \otimes e) \\
&\qquad\qquad - i_{n+1}(\pi_{n+2}(e \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes e \otimes a_n))\big\},
\end{aligned}
$$

which completes the proof of Lemma 2.10. □

Proof of Theorem 2.7.

(i) $\Leftrightarrow$ (iii). Assume first that A is approximately n-amenable. By equivalence (i) $\Leftrightarrow$ (ii), we may choose a net $(M^\nu)_\nu$ in $C^{n-1}(A, C_{n+1}(A)^{**})$ such that, for each $(a_1, \dots, a_n) \in A^n$, $(b_1, \dots, b_{n-1}) \in A^{n-1}$, we have

(i) $\lim_\nu \delta N^\nu(a_1, \dots, a_n) = 0$; and
(ii) $\lim_\nu \pi_{n+1}^{**} N^\nu(b_1, \dots, b_{n-1}) = i_n(\pi_{n+1}(e \otimes b_1 \otimes \cdots \otimes b_{n-1} \otimes e))$.

Let $Z \in C^{n-1}(A, C_{n+2}(A)^{**})$ be the cochain as in Lemma 2.9:

$$(2.27)\qquad \pi_{n+2}^{**}((\delta Z)(a_1, \dots, a_n) - e \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes e \otimes a_n) = 0.$$

Let $W^\nu = (-1)^{n+1}N^\nu \otimes e + Z \in C^{n-1}(A, C_{n+2}(A)^{**})$, and let $L^\nu = \delta W^\nu \in B^n(A, C_{n+2}(A)^{**})$. We show that (L^ν) is the desired net. It suffices to verify condition (iii) (b). We use Lemma 2.10 and (2.27) to obtain

$$\begin{aligned}
&\lim_\nu \pi_{n+2}^{**} L^\nu(b_1, \ldots, b_n) \\
&\quad = (-1)^{n+1} \lim_\nu \pi_{n+2}^{**}\delta(N^\nu \otimes e)(b_1, \ldots, b_n) + \pi_{n+2}^{**}(\delta Z)(b_1, \ldots, b_n) \\
&\quad = \pi_{n+2}(e \otimes b_1 \otimes \cdots \otimes b_{n-1} \otimes e \otimes b_n),
\end{aligned}$$

for each $(b_1, \ldots, b_n) \in A^n$. This verifies condition (iii) (b) and proves statement (iii).

Conversely, assume that there exists a net $(N^\nu)_\nu \in C^n(A, C_{n+2}(A)^{**})$ such that

(iii) there exists a cochain $W^\nu \in C^{n-1}(A, C_{n+2}(A)^{**})$ such that

$$\lim_\nu N^\nu(a_1, \ldots, a_n) = \lim_\nu \delta W^\nu(a_1, \ldots, a_n)$$

for each $a_1, \ldots, a_n \in A$; and

(iv) $\lim_\nu \pi_{n+1}^{**} N^\nu(b_1, \ldots, b_n) = i_n(\pi_{n+2}(e \otimes b_1 \otimes \cdots \otimes b_n \otimes e))$ for each $(b_1, \ldots, b_n) \in A^n$.

By the continuity of π_{n+2}^{**}, (iii) and (iv) imply

$$\lim_\nu \pi_{n+2}^{**}(\delta W^\mu)(b_1, \ldots, b_n) = i_n(\pi_{n+2}(e \otimes b_1 \otimes \cdots \otimes b_n \otimes e)) \tag{2.28}$$

for each $(b_1, \ldots, b_n) \in A^n$. Recall the cochain $h_{n-1} \in C^{n-1}(A, C_{n+1}(A)^{**})$ from (2.6). By Lemma 2.8, we have

$$\delta(h_{n-1}) = \pi_{n+2}(h_n). \tag{2.29}$$

Let $V^\nu \in C^{n-1}(A, C_{n+1}(A)^{**})$ be the cochain defined by

$$\begin{aligned}
V^\nu(a_1, \ldots, a_{n-1}) = &-\pi_{n+2}^{**}(W^\nu(a_1, \ldots, a_{n-1})) \\
&+ i_{n+1}(h_{n-1}(a_1, \ldots, a_{n-1})), (a_1, \ldots, a_n) \in A^n.
\end{aligned}$$

We have from (2.28) and (2.29)

$$\begin{aligned}
\lim_\nu (\delta V^\nu)(a_1, \ldots, a_n) = &-\lim_\nu \pi_{n+2}^{**}(\delta W^\nu)(a_1, \ldots, a_n) \\
&+ i_{n+1}(\delta(h_{n-1})(a_1, \ldots, a_n)) = 0
\end{aligned}$$

for each $(a_1, \ldots, a_n) \in A^n$. Also, we have, using $\pi_{n+1}^{**} \circ \pi_{n+2}^{**} = 0$,

$$\begin{aligned}\lim_{\nu} \pi_{n+1}^{**} V^{\nu}(a_1, \ldots, a_{n-1}) &= \pi_{n+1} h_{n-1}(a_1, \ldots, a_{n-1}) \\ &= i_n(\pi_{n+1}(e \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes e)).\end{aligned}$$

Therefore, again by equivalence (i) $\Leftrightarrow$ (ii), we see that A is approximately n-amenable. This completes the proof of Theorem 2.7. □

In a similar manner, we can prove the following theorem.

Theorem 2.11. *Let A be a Banach algebra with unit e. For $n \geq 1$, the following conditions are equivalent.*

(i) $\overline{\mathrm{H}}^n(A, X)_{\mathrm{pt}} = 0$ *for each Banach A-bimodule X;*

(ii) *there exists a net $(L^\nu)_\nu$ in $C^{n-1}(A, C_{n+1}(A))$ such that, for each $(a_1, \ldots, a_n) \in A^n$ and for each $(b_1, \ldots, b_{n-1}) \in A^{n-1}$, we have*

(a) $\lim_\nu \|\delta L^\nu(a_1, \ldots, a_n)\|_{C_{n+1}(A)} = 0$; *and*

(b) $\lim_\nu \|\pi_{n+1} L^\nu(b_1, \ldots, b_{n-1}) - \pi_{n+1}(e \otimes b_1 \otimes \cdots \otimes b_{n-1} \otimes e)\|_{C_{n+1}(A)} = 0.$

For $n = 1$, conditions (ii) (a) and (ii) (b) reduce to approximate contractibility in the sense of Ghahramani and Loy [**5,** Definition 1.3]. It is proven in [**7,** Theorem 2.1] that approximate contractibility is equivalent to approximate amenability. It is natural to ask whether the same equivalence holds for $n \geq 2$. If the space $C^{n-1}(A, C_{n+1}(A))$ is weak*-dense in the space $C^{n-1}(A, C_{n+1}(A)^{**})$, then the proof of [**7,** Theorem 2.1] works to prove the desired equivalence. The general case is unknown to the author. Note that $C_{n+1}(A)$ is weak*-dense in $C_{n+1}(A)^{**}$ by the Goldstein theorem.

Sketch of proof. The proof of Theorem 2.11 is almost identical to that of equivalence (i) $\Leftrightarrow$ (ii) of Theorem 2.7. For the proof of implication (ii) $\Rightarrow$ (i), we define, for a Banach A-bimodule V, a net $(\widetilde{P}^\nu)$ of bounded linear operators $C^{n-1}(A, V) \to C^{n-1}(A, V)$, which satisfies:

($\widetilde{\mathrm{a}}$) for each $(a_1, \ldots, a_n) \in A^n$, we have $\lim_\nu \delta \widetilde{P}^\nu(a_1, \ldots, a_n) = 0$; and

($\widetilde{\mathrm{b}}$) for each $f \in Z^{n-1}(A,V)$, and for each $(b_1,\dots,b_{n-1}) \in A^{n-1}$, we have $f(b_1,\dots,b_{n-1}) = \lim_\nu \widetilde{P}^\nu f(b_1,\dots,b_{n-1})$.

For a cochain $f \in C^{n-1}(A,V)$, we define a cochain $\widetilde{F}_f \in C^{n+1}(A,V)$ by the same formula as that of (2.7): $\widetilde{F}_f(a_1,\dots,a_{n+1}) = a_1 \cdot (f(a_2,\dots,a_n)) \cdot a_{n+1}$. For a net (L^ν) in $C^{n-1}(A, C_{n+1}(A))$ we define $\widetilde{P}^\nu : C^{n-1}(A, C_{n+1}(A)) \to C^{n-1}(A, C_{n+1}(A))$ by

$$\widetilde{P}^\nu(f)(a_1,\dots,a_{n-1}) = \widetilde{F}_f(L^\nu(a_1,\dots,a_{n-1})),$$
$$f \in C^{n-1}(A, C_{n+1}(A)), \quad (a_1,\dots,a_{n-1}) \in A^{n-1}.$$

We may proceed in the same way as that of Step 1 in Theorem 2.7, by replacing P^ν with $\widetilde{P}^\nu$. A cocyle $\widetilde{G}_f \in C^n(A,V)$ is defined in a similar manner to (2.12) so that $\widetilde{P}^\nu(f) = \widetilde{G}_f \circ \pi_{n+1} \circ L^\nu$. Using these cocycles, we can show that $\widetilde{P}^\nu(f)$ satisfies conditions ($\widetilde{\mathrm{a}}$) and ($\widetilde{\mathrm{b}}$). A similar modification enables us to carry out the same proof as that of Step 2 in Theorem 2.7 to derive the desired conclusion. We omit the details. □

3. Alternating cocycles in Lipschitz algebras. A complex-valued function $f : K \to \mathbb{C}$ defined on a compact metric space (K,d) is called a *Lipschitz function* if the *Lipschitz constant* $L(f)$ of f is finite:

$$L(f) := \sup\left\{ \frac{|f(x)-f(y)|}{d(x,y)} \,\middle|\, x,y \in X,\ x \neq y \right\} < \infty.$$

The *Lipschitz algebra*, the Banach algebra of all Lipschitz functions over (K,d) with the pointwise addition/multiplication, with the norm defined by

$$\|f\|_L = \sup_{p\in K} |f(p)| + L(f), \quad f \in \mathrm{Lip}K,$$

is denoted by $\mathrm{Lip}K$. The *little Lipschitz algebra* $\mathrm{lip}K$ is the subalgebra of $\mathrm{Lip}K$, defined by

$$\mathrm{lip}K = \left\{ f \in \mathrm{Lip}K \,\middle|\, \lim_{d(x,y)\to 0} \frac{f(x)-f(y)}{d(x,y)} = 0 \right\}.$$

The amenability of $\mathrm{Lip}K$ and $\mathrm{lip}K$ have been studied in [**1, 3, 13, 18, 25**], etc. For an infinite compact metric space K, $\mathrm{Lip}K$ is not amenable [**13, 25**], and $\mathrm{lip}K$ is not approximately amenable [**3**]. Here, we improve a result of [**19**] by proving $\dim_{\mathbb{C}} \overline{\mathrm{H}}^n(\mathrm{Lip}K, (\mathrm{Lip}K)^*)_{\mathrm{pt}} =$

∞ for each $n \geq 1$ if K is a compact metric space which has the property (seq) at a point of K (see definition below). In particular, Lip$K$ for such a space K is not approximately amenable. As in **[19]**, we construct alternating cocyles **[18]** which remain non-trivial in $\overline{\mathrm{H}}^n(\mathrm{Lip}K, (\mathrm{Lip}K)^*)_{\mathrm{pt}}$.

The symmetric group $\mathfrak{S}_n$ acts on the space of n-cochains $C^n(A, X)$ of a Banach algebra A and its bimodule X by

$$(\sigma \cdot f)(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}),$$
$$\sigma \in \mathfrak{S}_n, \quad f \in C^n(A, X), \quad (a_1, \ldots, a_n) \in A^n.$$

Definition 3.1 (**[18]**)**.** A cochain $f \in C^n(A, X)$ is said to be *alternating* if

$$f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = (\mathrm{sgn}\ \sigma) f(a_1, \ldots a_n), \quad (a_1, \ldots, a_n) \in A^n,$$

for each $\sigma \in \mathfrak{S}_n$, where sgn σ denotes the signature of $\sigma \in \mathfrak{S}_n$. The space of all alternating n-cochains is denoted by $C^n_{\mathrm{alt}}(A, X)$. In addition, let $Z^n_{\mathrm{alt}}(A, X) := Z^n(A, X) \cap C^n_{\mathrm{alt}}(A, X)$ be the space of all alternating cocycles.

The following is an analogue of **[18,** Proposition 2.9]. An A-bimodule X of a Banach algebra A is said to be *symmetric* if $a \cdot x = x \cdot a$ for each $(a, x) \in A \times X$.

Proposition 3.2. *Let A be a commutative Banach algebra, and let X be a symmetric Banach A-bimodule. Let $q^n : Z^n(A, X) \to \overline{\mathrm{H}}^n(A, X)_{\mathrm{pt}}$ be the projection of the space of n-cocycles onto $\overline{\mathrm{H}}^n(A, X)_{\mathrm{pt}}$. Then, the restriction $q^n | Z^n_{\mathrm{alt}}(A, X)$ is injective.*

We follow the proof of Johnson **[18,** Theorem 2.9]. For a cochain $f \in C^n(A, X)$, let $z_n f \in C^n_{\mathrm{alt}}(A, X)$ be the cochain defined by

$$(z_n f)(a_1, \ldots, a_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (\mathrm{sgn}\ \sigma) f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}),$$
$$(a_1, \ldots, a_n) \in A^n.$$

Then, we have the next lemma by direct computation:

Lemma 3.3 ([**18,** Theorem 2.9])**.** *Let A be a commutative Banach algebra, and let X be a symmetric Banach A-bimodule.*

(i) *For each $\sigma \in \mathfrak{S}_n$ we have*:

$$\sigma(z_n f) = z_n(\sigma f) = (\mathrm{sgn}\ \sigma) z_n f$$

for each $f \in C^n(A, X)$. In particular, $z_n f$ is alternating.

(ii) *A cochain $f \in C^n(A, X)$ is alternating if and only if $z_n f = f$.*

Lemma 3.4 ([**18,** Proof of Theorem 2.9])**.** *Let $f \in C^{n-1}(A, X)$ be an $(n-1)$-cochain. Then, we have $z_n(\delta f) = 0$.*

Proof. By the coboundary formula, we have

$$\begin{aligned}(\delta f)(a_1, \dots a_n) &= a_1 \cdot f(a_2, \dots a_n) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i f(a_1, \dots, a_i a_{i+1}, \dots a_n) \\ &\quad + (-1)^n f(a_1, \dots, a_n) \cdot a_n\end{aligned} \tag{3.1}$$

for $f \in C^n(A, X)$ and $(a_1, \dots a_{n+1}) \in A^{n+1}$. Let K_f be the $(n+1)$-cochain, defined by

$$K_f(a_1, \dots, a_n) = a_1 \cdot (f(a_2, \dots, a_n)), \quad (a_1, \dots, a_n) \in A^n.$$

Let $\tau = (n, n-1, \dots, 2, 1)$ be the cyclic permutation $n \mapsto n-1, \dots, 2 \mapsto 1, 1 \mapsto n$ which has the signature $(-1)^{n+1}$. Since X is a symmetric A-bimodule, we see

$$\begin{aligned}a_1 \cdot (f(a_2, \dots, a_n)) + (-1)^n f(a_1, \dots, a_{n-1}) \cdot a_n \\ = (K_f + (-1)^n (\tau K_f))(a_1, \dots, a_n).\end{aligned} \tag{3.2}$$

and thus, by Lemma 3.3 (i), we have

$$\begin{aligned}(z_n(K_f + (-1)^n \tau K_f))(a_1, \dots, a_n) \\ = (z_n + (-1)^{2n+1} z_n) K_f(a_1, \dots, a_n) = 0.\end{aligned} \tag{3.3}$$

For $i = 1, \dots, n-1$, let $U_f^i(a_1, \dots, a_n) = f(a_1, \dots, a_i a_{i+1}, \dots, a_n)$, and let $\sigma_i = (i, i+1)$ be the transposition of i and $i+1$. Since A is commutative, we see that $\sigma_i U_f^i = U_f^i$. Using Lemma 3.3 (ii), we have

$z_n U_f^i = z_n(\sigma_i U_f^i) = (\mathrm{sgn}\ \sigma_i) z_n U_f^i = -z_n U_f^i$. Thus, we obtain

$$z_n U_f^i = 0. \tag{3.4}$$

By (3.3) and (3.4), we conclude that

$$z_n(\delta f) = z_n(K_f + (-1)^n K_f) + \sum_{i=1}^{n-1}(-1)^i z_n U_i^f = 0. \qquad \square$$

Proof of Proposition 3.2. Our proof follows that of [**18,** Theorem 2.9]. Take an alternating cocycle $f \in Z_{\mathrm{alt}}^n(A, X)$, and assume that there exists a net $(g^\nu)_\nu$ in $C^{n-1}(A, X)$ such that $f(a_1, \ldots, a_n) = \lim_\nu (\delta g^\nu)(a_1, \ldots, a_n)$ for each $(a_1, \ldots, a_n) \in A^n$. By Lemma 3.4, we see

$$(z_n f)(a_1, \ldots, a_n) = \lim_\nu (z_n(\delta g^\nu))(a_1, \ldots, a_n) = 0.$$

Since f is alternating, we have $f = z_n f$ by Lemma 3.3 (i), and hence, $f = 0$. $\square$

We say that a sequence $\{r_n \mid n \geq 1\}$ on the real line $\mathbb{R}$ has Property ($\sharp$) if $\lim_n r_n = 0$, and it satisfies the following two conditions:

(s) (i) $r_1 > r_2 > \cdots > r_n > r_{n+1} > \cdots \to 0$,

(s) (ii) $|r_1 - r_2| > |r_2 - r_3| > \cdots > |r_n - r_{n+1}| > |r_{n+1} - r_{n+2}| > \cdots \to 0$.

A metric space (K, d) is said to have the property (*seq*) at a point $p \in K$ if there exists an isometric embedding $\alpha : \{r_n \mid n \geq 1\} \cup \{0\} \to K$ of a sequence $\{r_n \mid n \geq 1\}$ with Property ($\sharp$) such that $\alpha(0) = p$. Riemannian manifolds and the Cantor ternary set in $[0, 1] \subset \mathbb{R}$ are examples of metric spaces with the property (seq) at some points. In [**19**], we proved that, for each compact metric space K with the property (seq), $\dim_{\mathbb{C}} \mathrm{H}^n(\mathrm{Lip}K, \mathbb{C}_p) = \infty$ and $\dim_{\mathbb{C}} \mathrm{H}^n(\mathrm{Lip}K, (\mathrm{Lip}K)^*) = \infty$ by applying [**18,** Proposition 2.9]. Here, $\mathbb{C}_p$ denotes the complex number field which is a Banach $\mathrm{Lip}(K)$-bimodule with the module structure:

$$f \cdot z = z \cdot f = f(p)z, \quad f \in \mathrm{Lip}K,\ z \in \mathbb{C}.$$

The result may be improved as follows, by replacing [**18,** Proposition 2.9] with Proposition 3.2.

Theorem 3.5. *Let (X, d) be a compact metric space with the property* (seq) *at a point p. Then, we have*

$$\dim_{\mathbb{C}} \overline{\mathrm{H}}^n(\mathrm{Lip}K, \mathbb{C}_p)_{\mathrm{pt}} = \infty,$$

and

$$\dim_{\mathbb{C}} \overline{\mathrm{H}}^n(\mathrm{Lip}K, (\mathrm{Lip}K)^*)_{\mathrm{pt}} = \infty.$$

Proof. It is shown in [**19,** Theorem 3.5] that $\dim_{\mathbb{C}} Z^n_{\mathrm{alt}}(\mathrm{Lip}K, \mathbb{C}_p) = \infty$, and also, there exists an injection

$$E_\sharp : Z^n_{\mathrm{alt}}(\mathrm{Lip}K, \mathbb{C}_p) \longrightarrow Z^n_{\mathrm{alt}}(\mathrm{Lip}K, (\mathrm{Lip}K)^*).$$

By Proposition 3.2, the natural projection

$$\overline{q}^n : Z^n_{\mathrm{alt}}(\mathrm{Lip}K, (\mathrm{Lip}K)^*) \longrightarrow \overline{\mathrm{H}}^n(\mathrm{Lip}K, (\mathrm{Lip}K)^*)_{\mathrm{pt}}$$

is injective. Hence, the composition $q^n \circ E_\sharp$ injects $Z^n_{\mathrm{alt}}(\mathrm{Lip}K, \mathbb{C}_p)$ into $\overline{\mathrm{H}}^n(\mathrm{Lip}K, (\mathrm{Lip}K)^*)_{\mathrm{pt}}$. Thus, the conclusion follows. □

It is not currently known whether there exists an infinite compact metric space K such that $\dim_{\mathbb{C}} \mathrm{H}^n(\mathrm{Lip}K, \mathbb{C}_p) < \infty$ for some $n \geq 1$ and for a point $p \in K$. Likewise, it is unknown whether there exists a space L such that $\dim_{\mathbb{C}} \overline{\mathrm{H}}^n(\mathrm{Lip}L, \mathbb{C}_q)_{\mathrm{pt}} < \infty$ for some $n \geq 1$ and for a point $q \in L$.

In addition, at the time of this writing, it is unknown whether the present cohomology $\overline{\mathrm{H}}^n(A, X)_{\mathrm{pt}}$ is well behaved so that the machinery of homological algebra is applicable. For example, we do not know whether cohomology long exact sequence is available in its full generality (cf., [**20,** Example 1.19]). It seems that an additional argument might be required in order to complete the proof of [**24,** Theorem 3.5]. Also, tensor products of approximately amenable algebras need not be approximately amenable [**6**], which suggests that the Künneth formula [**14**] might not hold in general.

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