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EXISTENCE OF SOLUTIONS FOR A NONLOCAL FRACTIONAL BOUNDARY VALUE PROBLEM

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ABSTRACT. We study a nonlinear fractional boundary value problem with nonlocal boundary conditions. An associated Green's function is constructed as a series of functions by the perturbation approach. Criteria for the existence of solutions are obtained based upon it.

1. Introduction. In this paper, we study the boundary value problem (BVP) consisting of the fractional differential equation

$$-D_{0+}^{\alpha}u + a(t)u = w(t)f(u,s), \quad 0<t<1,\ \alpha>2,\ \alpha\notin\mathbb{N}, \tag{1.1}$$

and the nonlocal boundary conditions (BCs)

$$u^{(k)}(0)=0,\ k=0,1,\ldots,\lfloor\alpha\rfloor-1, \qquad u(1)=\int_0^1 u(s)\,dA(s), \tag{1.2}$$

where $\lfloor\alpha\rfloor$ denotes the integer part of α, and the following assumptions are satisfied:

(i) $a\in C[0,1]$, $w\in L[0,1]$ with $w(t)\not\equiv 0$ almost everywhere on $[0,1]$, and $f\in C(\mathbb{R}\times[0,1],\mathbb{R})$.

(ii) $A:[0,1]\to\mathbb{R}$ is a function of bounded variation, and

$$\int_0^1 u(s)\,dA(s)$$

denotes the Riemann-Stieltjes integral of u with respect to A.

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(iii) For $u : [0,1] \to \mathbb{R}$, $D_{0+}^{\alpha} u$ is the αth Riemann-Liouville fractional derivative of u, defined by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(l-\alpha)} \frac{d^l}{dt^l} \int_0^t (t-s)^{l-\alpha-1} u(s)\, ds, \quad l = \lfloor \alpha \rfloor + 1,$$

provided the right-hand side exists with Γ the Gamma function.

Remark 1.1. It is easy to see that the Riemann-Stieltjes integral in BC (1.2)

$$u(1) = \int_0^1 u(s)\, dA(s)$$

covers the multi-point and integral BCs as special cases.

Fractional differential equations have extensive applications in science and engineering. Many phenomena in viscoelasticity, porous media, and other fields, can be modeled by fractional differential equations. We refer the reader to [**13, 17**], and the references therein, for some applications.

The existence of solutions is an essential problem for BVPs of fractional differential equations. It has been studied by many authors, see [**2**]–[**12, 13, 14, 15, 16, 18, 20**], and the references therein. Due to certain special properties of the fractional calculus, critical point theory can only be applied to study equations involving both the left and right Riemann-Liouville fractional derivatives, see, for example, [**3**]. To the best of our knowledge, if only the left (or right) Riemann-Liouville fractional derivatives are involved, the only feasible approach for studying the existence of solutions of a nonlinear BVP is to convert the problem to an integral equation and use various techniques to find the fixed points. In this approach, Green's function plays an important role in deriving the integral equation.

The special cases of BVP (1.1), (1.2) when $a(t) \equiv 0$ on $[0,1]$ was studied by Feng, Zhang and Ge [**4**], Zhang and Han [**20**] and Tan, Cheng and Zhang [**16**]. Henderson and Luca [**11**] further studied a system of coupled nonlocal fractional BVPs. The general case of BVP (1.1), (1.2) with $a(t) \not\equiv 0$ has not yet been considered by many scholars. Graef, Kong, Kong and Wang [**7**] studied a special case of BVP (1.1), (1.2) with $a(t) \equiv a > 0$. However, a very strong restriction on the BC is imposed in [**7**] due to some technical needs in the proofs; see [**7**] for details.

It is notable that, in much literature, Green's functions are constructed by a standard method: the solution of a linear BVP is first expressed as an integral with unknown parameters; then, the unknown parameters are determined using the BCs. Finally, one part of the integral is taken as the associated Green's function for the BVP. This method is very effective if the equation in the BVP contains exactly one term of u, i.e.,

$$-D_{0+}^{\alpha} u = 0, \tag{1.3}$$

see, for example, [**4,** Lemma 2.1]. However, if the equation contains multiple terms of u or its derivative(s), e.g.,

$$-D_{0+}^{\alpha} u + a(t)u = 0, \tag{1.4}$$

certain restrictions on BCs must be imposed so that the unknown parameters in the integral can be determined. For the general form of BC (1.2), it is difficult to construct the associated Green's function via the standard method. In fact, this is the main obstacle to studying the general case of BVP (1.1), (1.2). It is natural for us to explore new techniques for deriving Green's function.

Recently, Graef, et al., [**8, 9, 10**] proposed a "perturbation approach" for constructing the Green's functions for BVPs consisting of equation (1.4) and various separated BCs. This approach constructs the Green's functions as a function series without parameter determination, and hence, has less restrictions on the BCs. In this paper, we first utilize the perturbation approach for deriving the Green's function for BVP (1.4), (1.2). The perturbation approach is summarized as a lemma (Section 3, Lemma 3.2). With this lemma, we are able to construct the Green's functions for a large family of BVPs. This is our main contribution of the paper. Then, we study the nonlinear BVP (1.1), (1.2) by the fixed point theory.

This paper is organized as follows: after this introduction, our main results are stated in Section 2. All of the proofs are given in Section 3.

2. Main results. We first consider the Green's function for BVP (1.4), (1.2). The following notation is needed. Let $\Lambda \in \mathbb{R}$ and $\mathcal{G}_0 \in$

$C([0,1] \times [0,1], \mathbb{R})$ be defined by

$$\Lambda = \int_0^1 t^{\alpha-1}\, dA(t)$$

and

$$\mathcal{G}_0(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \le s \le t \le 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)} & 0 \le t \le s \le 1. \end{cases}$$

Then, define $\mathcal{G}_A : [0,1] \to \mathbb{R}$ by

$$\mathcal{G}_A(s) = \int_0^1 \mathcal{G}_0(t,s)\, dA(t).$$

Throughout this paper, we assume

(H1) $0 \le \Lambda < 1$ and $\mathcal{G}_A(s) \ge 0$ on $[0,1]$.

Define

$$H_0(t,s) = \frac{t^{\alpha-1}}{1-\Lambda}\mathcal{G}_A(s) + \mathcal{G}_0(t,s) \tag{2.1}$$

and

$$\overline{H}_0 = \frac{\max_{s\in[0,1]} |\mathcal{G}_A(s)|}{1-\Lambda} + \frac{1}{\Gamma(\alpha-1)}. \tag{2.2}$$

Remark 2.1. It is known that $H_0 \in C([0,1] \times [0,1], \mathbb{R})$ is the Green's function for BVP (1.3), (1.2) and $H_0(t,s) \ge 0$ on $[0,1] \times [0,1]$ when (H1) holds; the reader is referred to **[16, 20]** for more properties of H_0.

Let $H : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$H(t,s) = \sum_{n=0}^{\infty} (-1)^n H_n(t,s), \tag{2.3}$$

with H_0 defined by (2.1) and $H_n : [0,1] \times [0,1] \to \mathbb{R}$, $n = 1, 2, \ldots$, defined by

$$H_n(t,s) = \int_0^1 a(\tau) H_0(t,\tau) H_{n-1}(\tau,s)\, d\tau, \quad n \ge 1.$$

We also need the assumption:

(H2) $\overline{a} := \max_{t\in[0,1]} |a(t)| < \overline{H}_0^{-1}$ with $\overline{H}_0$ defined by (2.2).

Then, we obtain the following result.

Theorem 2.2. *Assume that* (H1) *and* (H2) *hold. Then, H, defined by* (2.3) *as a series of functions, is uniformly convergent for* $(t,s) \in [0,1] \times [0,1]$ *and continuous on* $[0,1] \times [0,1]$*. Furthermore, H is the Green's function for* BVP (1.4), (1.2) *with*

$$|H(t,s)| \leq \frac{\overline{H}_0}{1 - \overline{a}\,\overline{H}_0}$$

on $[0,1] \times [0,1]$*, where $\overline{H}_0$ is defined by* (2.2).

With Theorem 2.2, we are ready to study the nonlinear BVP (1.1), (1.2). Let

$$U = \max_{t\in[0,1]} \int_0^1 |H(t,s)w(s)|\,ds, \tag{2.4}$$

where H is defined by (2.3). For any $u \in C[0,1]$, define $\|u\| = \max_{t\in[0,1]} |u(t)|$.

Theorem 2.3. *Assume that* (H1) *and* (H2) *hold. If there exist $M > 0$ and $\kappa \in [0, U^{-1})$ such that*

$$\max_{(x,t)\in[-M,M]\times[0,1]} |f(x,t)| \leq M/U, \tag{2.5}$$

and, for any x_1, x_2 with $|x_i| \leq M$, $i = 1,2$,

$$|f(x_1,t) - f(x_2,t)| \leq \kappa |x_1 - x_2|, \quad t \in [0,1], \tag{2.6}$$

then:

(a) BVP (1.1), (1.2) *has a unique solution $u \in C[0,1]$ with $\|u\| \leq M$.*

(b) *For any $u_0 \in C[0,1]$ with $\|u_0\| \leq M$, the sequence $\{u_n\}$ defined by*

$$u_{n+1} = \int_0^1 H(t,s)w(s)f(u_n(s),s)\,ds, \quad n = 0,1,2,\ldots,$$

satisfies $\|u_n - u\| \to 0$ as $n \to 0$.

Remark 2.4.

(a) Theorem 2.3 ensures that BVP (1.1), (1.2) has exactly one solution u with $\|u\| \le M$. BVP (1.1), (1.2) may or may not have other solutions in $\{u \in C[0,1] \mid \|u\| > M\}$.

(b) It is well known that, if f satisfies the global Lipschitz condition with respect to x, i.e., (2.6) holds for all $x_1, x_2 \in (-\infty, \infty)$, then, by the contraction mapping theorem, BVP (1.1), (1.2) has exactly one solution. Our conditions in Theorem 2.3 are different from the global Lipschitz condition. For example,

$$f(x) = x^3, \quad -\infty < x < \infty,$$

satisfies the conditions in Theorem 2.3 but not the global Lipschitz condition.

The next result on the existence of solutions is obtained when a weaker condition than Theorem 2.3 is used.

Theorem 2.5. *Assume that* (H1) *and* (H2) *hold. If there exists an* $M > 0$ *such that* (2.5) *holds, then* BVP (1.1), (1.2) *has at least one solution* $u \in C[0,1]$ *with* $\|u\| \le M$.

3. Proofs. We first present a general result to construct the Green's functions for BVPs consisting of equation (1.4) and given BCs. The following lemma on the spectral theory in Banach spaces will be needed. See [**19,** page 795, 57b, 57d] for details.

Lemma 3.1. *Let* X *be a Banach space and* $\mathcal{A} : X \to X$ *a linear operator with the operator norm* $\|\mathcal{A}\|$ *and the spectral radius* $r(\mathcal{A})$ *of* $\mathcal{A}$. *Then*:

(a) $r(\mathcal{A}) \le \|\mathcal{A}\|$;
(b) *if* $r(\mathcal{A}) < 1$, *then* $(\mathcal{I} - \mathcal{A})^{-1}$ *exists and*

$$(\mathcal{I} - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} \mathcal{A}^n,$$

where $\mathcal{I}$ *stands for the identity operator.*

In the sequel, we let $X = C[0,1]$ be the Banach space with the standard maximum norm. Consider a BVP

$$\begin{cases} -D_{0+}^{\alpha} u = 0 \quad 0 < t < 1, \\ B[u] = 0, \end{cases} \tag{3.1}$$

where $B[u] = 0$ denotes the given BCs. They may either be local or nonlocal BCs. Assume that $G_0 \in C([0,1] \times [0,1], \mathbb{R})$ is the Green's function for BVP (3.1) with

$$|G_0(t,s)| \le \phi(t)\psi(s) \quad \text{on } [0,1] \times [0,1]. \tag{3.2}$$

Define the series $G : [0,1] \times [0,1] \to \mathbb{R}$ by

$$G(t,s) = \sum_{n=0}^{\infty} (-1)^n G_n(t,s), \tag{3.3}$$

where G_0 is the Green's function for BVP (3.1) and
(3.4)

$$G_n(t,s) = \int_0^1 a(s) G_0(t,\tau) G_{n-1}(\tau,s)\, d\tau, \quad a \in C[0,1],\ n = 1,2,\ldots.$$

Let $\overline{a} = \max_{t\in[0,1]} |a(t)|$, $\overline{\phi} = \max_{t\in[0,1]} \phi(t)$ and $\overline{\psi} = \max_{s\in[0,1]} \psi(s)$. Then, we obtain the following result.

Lemma 3.2. *Assume that $\overline{a} < (\overline{\phi\psi})^{-1}$. Then, the series G defined by (3.3) is continuous and uniformly convergent on $[0,1] \times [0,1]$. Furthermore, G is the Green's function for the* BVP

$$\begin{cases} -D_{0+}^{\alpha} u + a(t)u = 0 \quad 0 < t < 1, \\ B[u] = 0, \end{cases} \tag{3.5}$$

and

$$|G(t,s)| \le \frac{\overline{\phi\psi}}{1 - \overline{a}\,\overline{\phi\psi}} \quad \text{on } [0,1] \times [0,1]. \tag{3.6}$$

Proof. For any $h \in C[0,1]$, assume that u is the solution of the BVP consisting of the equation

$$-D_{0+}^{\alpha} u + a(t)u = h(t) \tag{3.7}$$

and the BC

$$B[u] = 0.$$

Then, u must satisfy

$$u(t) = \int_0^1 G_0(t,s)[h(s) - a(s)u(s)]\,ds, \tag{3.8}$$

where G_0 is the Green's function for BVP (3.1).

Define $\mathcal{A}$ and $\mathcal{B} : X \to X$ by

$$\begin{aligned} (\mathcal{A}\mu)(t) &= \int_0^1 a(s)G_0(t,s)\mu(s)\,ds, \\ (\mathcal{B}\mu)(t) &= \int_0^1 G_0(t,s)\mu(s)\,ds. \end{aligned} \tag{3.9}$$

Then, equation (3.8) becomes

$$(\mathcal{I} + \mathcal{A})u = \mathcal{B}h. \tag{3.10}$$

Note that, for any $\mu \in X$ with $\|\mu\| = 1$ and $t \in [0,1]$, by (3.2),

$$|(\mathcal{A}\mu)(t)| = \left| \int_0^1 a(s)G_0(t,s)\mu(s)\,ds \right| \le \int_0^1 \overline{a}\phi(t)\psi(s)\|\mu\|\,ds \le \overline{a}\overline{\phi\psi}.$$

Therefore, $\|\mathcal{A}\| < 1$ when $\overline{a} < (\overline{\phi\psi})^{-1}$. From (3.10) and Lemma 3.1,

$$u = \sum_0^\infty (-1)^n \mathcal{A}^n \mathcal{B}h. \tag{3.11}$$

We claim that

$$(\mathcal{A}^n\mathcal{B}h)(t) = \int_0^1 G_n(t,s)h(s)\,ds, \quad n = 0,1,2,\ldots, \tag{3.12}$$

where G_n is defined by (3.4). It is easy to see that (3.12) holds when $n = 0$. Assume that (3.12) holds for $n = m - 1$. Then, by (3.4), (3.9), (3.12) and Fubini's theorem,

$$\begin{aligned} (\mathcal{A}^m\mathcal{B}h)(t) &= \mathcal{A}(\mathcal{A}^{m-1}\mathcal{B}h)(t) \\ &= \int_0^1 a(\tau)G_0(t,\tau) \int_0^1 G_{m-1}(\tau,s)h(s)\,ds\,d\tau \end{aligned}$$

$$= \int_0^1 \int_0^1 a(\tau) G_0(t,\tau) G_{m-1}(\tau, s)\, d\tau h(s)\, ds$$
$$= \int_0^1 G_m(t,s) h(s)\, ds.$$

By induction, (3.12) holds for all $n = 0, 1, 2, \ldots$.

Similarly, we can prove by induction that

$$|G_n(t,s)| \le \overline{a}^n (\overline{\phi\psi})^{n+1}, \quad n = 0, 1, 2, \ldots.$$

Since $\overline{a}\overline{\phi\psi} < 1$, for any $(t,s) \in [0,1] \times [0,1]$, we have

$$\sum_{n=0}^{\infty} |(-1)^n G_n(t,s)| \le \sum_{n=0}^{\infty} \overline{a}^n (\overline{\phi\psi})^{n+1} = \frac{\overline{\phi\psi}}{1 - \overline{a}\overline{\phi\psi}}. \tag{3.13}$$

Therefore, G is absolutely convergent on $[0,1] \times [0,1]$. Then, by (3.3), (3.11) and (3.12),

$$u(t) = \sum_0^{\infty} (-1)^n \int_0^1 G_n(t,s) h(s)\, ds = \int_0^1 G(t,s) h(s)\, ds. \tag{3.14}$$

On the other hand, assume that u is defined by (3.14). Then, we may reverse the above procedure to show that u is a solution of the BVP consisting of equation (3.7) and the BC

$$B[u] = 0.$$

Therefore, G, defined by (3.3), is the Green's function for the BVP (3.5). Inequality (3.6) follows from (3.3) and (3.13). $\square$

Remark 3.3. It is easy to see that Lemma 3.2 is independent of the BCs. As long as the Green's function for BVP (3.1) is known, the Green's function for BVP (3.5) may be constructed as a series of functions by Lemma 3.2.

The next lemma on the bounds of the Green's function H_0 will be necessary to prove Theorem 2.2, see **[16,** Lemma 2.4] for details.

Lemma 3.4. *Assume that* (H1) *holds. Then, for any* $(t,s) \in [0,1] \times [0,1]$,

$$0 \le \frac{t^{\alpha-1}\mathcal{G}_A(s)}{1-\Lambda} \le H_0(t,s) \le \overline{H}_0 t^{\alpha-1}, \tag{3.15}$$

where H_0 *and* $\overline{H}_0$ *are defined by* (2.1) *and* (2.2).

Proof of Theorem 2.2. Theorem 2.2 immediately follows from Lemma 3.2 by Remark 2.1 and (3.15) with $\overline{\phi} = 1$ and $\overline{\psi} = \overline{H}_0$. □

In order to prove Theorem 2.3, we will need the following lemma. The reader is referred to [**19,** page 17, Theorem 1.A] for details.

Lemma 3.5. *Suppose that*

(i) K *is a closed nonempty set in a complete metric space* (X,d).

(ii) *There exists an operator* $T : K \to K$, *i.e.,* K *is mapped onto itself by* T.

(iii) T *is* κ*-contractive, i.e.,*

$$d(Tx, Ty) \le \kappa d(x,y)$$

for all $x, y \in K$ *and for a fixed* $\kappa \in [0,1)$. *Then, we may conclude the following*:

(a) T *has exactly one fixed point* x *in* K, *i.e.,*

$$Tx = x.$$

(b) *The sequence* $\{x_n\}$ *of successive approximations*

$$x_{n+1} = Tx_n, \quad x_0 \in K,\ n = 0, 1, 2, \ldots,$$

converges to the fixed point x *for an arbitrary choice of initial point* x_0 *in* K.

Proof of Theorem 2.3. For any $u \in X$, define $T : X \to X$ by

$$(Tu)(t) = \int_0^1 H(t,s)w(s)f(u(s),s)\,ds, \tag{3.16}$$

where H is defined by (2.3). It is easy to see that T is completely continuous and u is a fixed point of T if and only if u is a solution of BVP (1.1), (1.2).

Let $K = \{u \in X \mid \|u\| \le M\}$. For any $u \in K$ and $t \in [0,1]$,

$$\begin{aligned}|(Tu)(t)| &= \left| \int_0^1 H(t,s)w(s)f(u(s),s)\,ds \right| \\ &\le \int_0^1 |H(t,s)w(s)||f(u(s),s)|\,ds.\end{aligned}$$

From (2.4) and (2.5),

$$|(Tu)(t)| \le \int_0^1 |H(t,s)w(s)|\,\frac{M}{U}\,ds = M.$$

Therefore, $\|Tu\| \le M$, i.e., $TK \subset K$.

For any u_1 and $u_2 \in K$, $t \in [0,1]$,

$$\begin{aligned}|(Tu_1)(t) - (Tu_2)(t)| &= \left| \int_0^1 H(t,s)w(s)[f(u_1(s),s)\,ds - f(u_2(s),s)]\,ds \right| \\ &\le \int_0^1 |H(t,s)w(s)||f(u_1(s),s) - f(u_2(s),s)|\,ds.\end{aligned}$$

By (2.6),

$$\begin{aligned}|(Tu_1)(t) - (Tu_2)(t)| &\le \int_0^1 |H(t,s)w(s)|\kappa|u_1(s) - u_2(s)|\,ds \\ &\le U\kappa\|u_1 - u_2\| < \|u_1 - u_2\|.\end{aligned}$$

Hence, $\|Tu_1 - Tu_2\| < \|u_1 - u_2\|$. Then, parts (a) and (b) follow from Lemma 3.5. □

Theorem 2.5 is proven by Schauder's fixed point theorem. We omit the details.

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