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# A MONODROMY CRITERION FOR A TYPE OF DEGENERATE SYSTEM DEFINED BY THE SUM OF TWO HOMOGENEOUS VECTOR FIELDS

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ABSTRACT. In this paper, a type of degenerate system defined by the sum of two homogeneous vector fields is studied. By means of a blow-up technique and a classification theorem, a monodromy criterion of an isolated singular point is presented and proven. An example is given to illustrate that our result generalizes the corresponding result in **[13]**.

**1. Introduction.** Let $O(0,0)$ be an isolated singular point of the real analytic differential system

$$\begin{aligned} \dot{x} &= P(x,y) = P_m(x,y) + P_{m+1}(x,y) + \cdots\ , \\ \dot{y} &= Q(x,y) = Q_m(x,y) + Q_{m+1}(x,y) + \cdots\ , \end{aligned} \tag{1.1}$$

where $P_m^2 + Q_m^2 \not\equiv 0$, $P_k(x,y)$ and $Q_k(x,y)$ are homogeneous polynomials of degree $k \geq m$, and the dot denotes a derivative with respect to the $t$. The integer $m \geq 1$ is called the *degree* of the singular point $O(0,0)$.

One of the classical problems in the qualitative theory of (1.1) is to characterize when $O(0,0)$ is of focus-center type, and this problem is now called the focus-center problem. Recall that $O(0,0)$ is said to be of focus-center type if it is either a focus or a center. Since any orbit of (1.1) which tends to the isolated singular points $O(0,0)$ is either a spiral or tends to $O(0,0)$ in a definite direction, see **[7]** (here, and in the following, by an orbit tending to $O(0,0)$ we mean that this orbit tends to $O(0,0)$ as $t \to \infty$ or $t \to -\infty$), $O(0,0)$ is of focus-center type if and

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only if there exists no orbit which tends to $O(0,0)$ in a definite direction. This means that a monodromic Poincaré mapping can be defined in a small neighborhood of $O(0,0)$ when it is of focus-center type, and hence, $O(0,0)$ is also called a *monodromic singular point*. Therefore, the focus-center problem is also called the monodromy problem, and it is a preliminary step for solving the center problem of the vector field, one of the open classical problems in the qualitative theory of planar differential systems, see [**1, 14**]–[**18, 20**]. If the linear portion of (1.1) at $O(0,0)$ is non-degenerate, i.e., its determinant does not vanish, the characterization is well known. The problem has also been solved when the linear portion is degenerate but not identically zero, see [**6, 7**]. Hence, the main difficulty in solving the focus-center problem appears when the singular point has an identically zero linear portion.

The monodromy problem for systems of the form

$$\begin{aligned} \dot{x} &= P(x,y) = P_m(x,y) + P_M(x,y), \\ \dot{y} &= Q(x,y) = Q_m(x,y) + Q_M(x,y), \end{aligned} \tag{1.2}$$

was studied in [**13**], where $1 \leq m < M$, $P$ and $Q$ are coprime, that is, these systems are defined to be the sum of two homogeneous vector fields. The case where (1.2) is a homogeneous system, i.e., either $P_m \equiv Q_m \equiv 0$ or $P_M \equiv Q_M \equiv 0$, is already well understood, see [**9**].

As was done in [**13**], we write $P_k(\theta) = P_k(\cos\theta, \sin\theta)$ and $Q_k(\theta) = Q_k(\cos\theta, \sin\theta)$ with $k \in \{m, M\}$. Consider system (1.2), and take polar coordinates $(r,\theta)$, given by the transformation of variables $r^2 = x^2 + y^2$ and $\theta = \arctan(y/x)$. After a rescaling of time, given by $ds/dt = r^{m-1}$, we have (again denoting the derivative with respect to $s$ by a dot),

$$\begin{aligned} \dot{r} &= r[\cos\theta P_m(\theta) + \sin\theta Q_m(\theta) + r^{M-m}(\cos\theta P_M(\theta) + \sin\theta Q_M(\theta))], \\ \dot{\theta} &= \cos\theta Q_m(\theta) - \sin\theta P_m(\theta) + r^{M-m}(\cos\theta Q_M(\theta) - \sin\theta P_M(\theta)). \end{aligned} \tag{1.3}$$

We say that $\theta = \theta_*$ is a *characteristic direction* for the origin of system (1.2) if $\cos\theta_* Q_m(\theta_*) - \sin\theta_* P_m(\theta_*) = 0$, and $\cos\theta Q_m(\theta) - \sin\theta P_m(\theta) = 0$ is called the *characteristic equation* for the origin of system (1.2). In fact, a characteristic direction for the origin of system (1.2) is merely a root of its characteristic equation. An orbit of system (1.2) which tends to the origin in a definite direction $\overline{\theta}$ is called a *characteristic orbit* for the origin of system (1.2), and, in this case,

$\overline{\theta}$ must be one of its characteristic directions, see **[7]**. However, the reciprocal is not true, and a counter-example may be found in **[12]**.

The next two conditions follow from **[13]**.

Condition (a). System (1.2) satisfies Condition (a) if there exists a neighborhood $\mathcal{U}$ of the origin of system (1.2) such that

$$\Theta(x,y) = xQ(x,y) - yP(x,y) \neq 0 \quad \text{for all } (x,y) \in \mathcal{U} \setminus \{(0,0)\}.$$

If system (1.2) satisfies Condition (a), we denote the sign of $\Theta(x,y)$ for all $(x,y) \in \mathcal{U} \setminus \{(0,0)\}$ by $\mathrm{sign}_{\vec{0}}(\Theta)$.

Condition (b). System (1.2) satisfies Condition (b) if either it has no characteristic directions, or else, if all characteristic directions are isolated and $P_m(\theta_*) = Q_m(\theta_*) = 0$ for every characteristic direction $\theta_*$.

Obviously, if system (1.2) has at least one characteristic direction and all characteristic directions are isolated, then the number of characteristic directions is finite and no larger than $2(m+1)$ on the interval $[0, 2\pi)$, see **[7, 12]**. Let $\theta_1, \theta_2, \ldots, \theta_k$ be the characteristic directions associated with system (1.2). For all $j = 1, 2, \ldots, k$, we set

$$a_j = \cos\theta_j, \qquad b_j = \sin\theta_j,$$
$$\alpha_j = \frac{d}{dz}(P_m^j(1,z))\bigg|_{z=0}$$

and

$$\beta_j = \frac{1}{2}\frac{d^2}{dz^2}(Q_m^j(1,z))\bigg|_{z=0},$$

where

(1.4)
$$P_m^j(1,z) = a_j P_m(a_j - b_j z, b_j + a_j z) + b_j Q_m(a_j - b_j z, b_j + a_j z),$$
$$Q_m^j(1,z) = -b_j P_m(a_j - b_j z, b_j + a_j z) + a_j Q_m(a_j - b_j z, b_j + a_j z).$$

A vector field $\mathcal{X} = (P_m(x,y) + P_M(x,y), Q_m(x,y) + Q_M(x,y))$ belongs to class $\mathcal{G}$ if either there are no characteristic directions, or else if, for every characteristic direction $\theta_j$, $j = 1, 2, \ldots, k$, we have $\alpha_j^2 + \beta_j^2 \neq 0$.

The next theorem **[13]** yields a monodromy criterion for system (1.2).

**Theorem 1.1.** *Let $\mathcal{X}$ be the vector field associated with system* (1.2). *Then the following statements hold.*

(i) *If the origin of* (1.2) *is a focus-center, then Conditions* (a) *and* (b) *are satisfied, and $m$ is odd. Furthermore, if system* (1.2) *has characteristic directions, then $M$ is also odd.*

(ii) *Assume that $\mathcal{X} \in \mathcal{G}$. Then the origin of system* (1.2) *is a focus-center if and only if the system satisfies Conditions* (a) *and* (b), *and for every characteristic direction $\theta_j$,*

$$\mathrm{sign}_{\vec{0}}(\Theta)((2+M-m)\alpha_j - 2\beta_j) \le 0 \quad \text{for all } j = 1,2,\ldots,k.$$

We note that Theorem 1.1 (ii) will no longer be valid when $\alpha_j^2 + \beta_j^2 = 0$ for some $j = 1,2,\ldots,k$. In this paper, we shall give another monodromy criterion for system (1.2) when $\alpha_j^2 + \beta_j^2 = 0$ for some $j = 1,2,\ldots,k$ by using the blow-up technique (see **[8]** for a brief geometric description; see **[5, 10, 19]** for a more detailed description of this technique) and the theorem of the classification of an isolated singular point, see **[11]**. Toward this aim, we firstly give the following notation and concept.

Let

$$\begin{aligned} \alpha_j^{(l)} &= \frac{1}{l!}\frac{d^l}{dz^l}(P_m^j(1,z))\bigg|_{z=0}, \\ \beta_j^{(l)} &= \frac{1}{(l+1)!}\frac{d^{l+1}}{dz^{l+1}}(Q_m^j(1,z))\bigg|_{z=0} \end{aligned} \tag{1.5}$$

for $l = 1,2,\ldots$.

**Definition 1.2.** A vector field $\mathcal{X} = (P_m(x,y) + P_M(x,y), Q_m(x,y) + Q_M(x,y))$ is said to belong to class $\mathcal{G}_1$ if one of the following conditions holds:

(1) there are no characteristic directions for the origin of system (1.2);

(2) all of the characteristic directions are isolated, and, for any such characteristic direction $\theta_j$, $j = 1,2,\ldots,k$, either

$$(\alpha_j^{(1)})^2 + (\beta_j^{(1)})^2 \neq 0, \quad \text{or}$$

$$(\alpha_j^{(1)})^2 + (\beta_j^{(1)})^2 = 0;$$

however,

$$(\alpha_j^{(3)})^2 + (\beta_j^{(3)})^2 \neq 0.$$

Now, we give the main result of this paper as follows.

**Theorem 1.3.** *Let $\mathcal{X}$ be the vector field associated with system* (1.2), *and $\mathcal{X} \in \mathcal{G}_1$. Then the origin of system* (1.2) *is a focus-center if and only if it satisfies the following conditions*:

(i) *system* (1.2) *satisfies Conditions* (a) *and* (b);

(ii) *$m$ is odd. Furthermore, if system* (1.2) *has characteristic directions, then $M$ is also odd*;

(iii) *for every isolated characteristic direction $\theta_j$, $j = 1, 2, \ldots, k$,*

$$\mathrm{sign}_{\vec{0}}(\Theta)((2 + M - m)\alpha_j^{(1)} - 2\beta_j^{(1)}) \leq 0$$

*holds when $(\alpha_j^{(1)})^2 + (\beta_j^{(1)})^2 \neq 0$;*

$$\mathrm{sign}_{\vec{0}}(\Theta)((4 + M - m)\alpha_j^{(3)} - 4\beta_j^{(3)}) \leq 0$$

*holds when $(\alpha_j^{(1)})^2 + (\beta_j^{(1)})^2 = 0$; however, $(\alpha_j^{(3)})^2 + (\beta_j^{(3)})^2 \neq 0$.*

It is obvious that $\mathcal{G} \subseteq \mathcal{G}_1$. At the end of this paper we give an example to show that $\mathcal{G} \neq \mathcal{G}_1$. Hence, our result generalizes the corresponding result in [**13**], and our monodromy conditions are also easily verified as in [**13**].

There are many papers dealing with the monodromy problem for system (1.1). In [**14, 20**], necessary monodromy conditions as well as sufficient monodromy conditions for a large class of degenerate singular points of planar differential systems (1.1) are given; however, it is merely a genericity result, and, in order to find the monodromy of the origin of system (1.2) with $M - m$ large, the number of conditions needed are greater than those given in Theorem 1.3. A necessary condition in the monodromy problem for (1.1) is given in [**12**]. Tang, et al. [**21**] used a method of generalized normal sectors for finding orbits in exceptional directions near high degenerate equilibria. In [**1**], the monodromy problem for quasi-homogeneous polynomial systems is studied through the conservative/dissipative splitting. In **[2]**, whether an isolated singular point of the vector field is monodromic or has a characteristic trajectory is determined by using the Newton diagram and the conservative/dissipative splitting for the lowest-degree quasi-homogeneous terms of an analytic planar vector field. Recently, a new algorithmic criterion that determines whether an isolated degenerate singular point of a system of differential equations on the plane is monodromic was given in **[3]** by using conservative and dissipative parts associated to the edges and vertices of the Newton diagram of the planar analytic differential system. The monodromy problem for nilpotent systems and a wide family of systems with a degenerate singular point, so-called generalized nilpotent cubic systems, is solved in **[4]** through the Newton diagram and the conservative/dissipative splitting for the lowest-degree quasi-homogeneous terms of such systems.

**2. The proof of the main result.** In order to prove our main result, we restate three lemmas from **[13]**.

**Lemma 2.1.** *If the origin of* (1.2) *is a focus-center, then Conditions* (a) *and* (b) *are satisfied.*

**Lemma 2.2.** *If the origin of* (1.2) *is a focus-center, then $m$ is odd. Furthermore, if system* (1.2) *has characteristic directions, then $M$ is also odd.*

Since the types of (1.2) are preserved under rotations, we use the results for case $\theta = 0$ to study sufficient conditions for the non-existence of characteristic orbits tending to the origin of a system of type (1.2), and we let

$$\alpha^{(l)} = \frac{1}{l!}\frac{d^l}{dz^l}(P_m^j(1,z))\bigg|_{z=0},$$
$$\beta^{(l)} = \frac{1}{(l+1)!}\frac{d^{l+1}}{dz^{l+1}}(Q_m^j(1,z))\bigg|_{z=0},$$

for $l = 1, 2, \ldots$.

**Lemma 2.3.** *Suppose that $\theta = 0$ is a characteristic direction of the origin of system* (1.2), *and* $(\alpha^{(1)})^2 + (\beta^{(1)})^2 \neq 0$. *Then, system* (1.2) *has no characteristic orbit tending to the origin in direction $\theta = 0$ if and only if it satisfies the following conditions*:

(i) *there exist two real numbers $\overline{r} > 0$ and $\delta > 0$, such that*

$$\Theta(r,\theta)=\cos\theta Q_m(\theta)-\sin\theta P_m(\theta)+r^{M-m}(\cos\theta Q_M(\theta)-\sin\theta P_M(\theta))\neq 0,$$

*for* $(r,\theta) \in \{(0,\overline{r}] \times (-\arctan(\delta), \arctan(\delta))\} \setminus \{(0,0)\}$;

(ii) $P_m(0) = Q_m(0) = 0$;

(iii) *$m$ and $M$ are both odd*;

(iv) $\mathrm{sign}_{\vec{0}}(\Theta)((2 + M - m)\alpha^{(1)} - 2\beta^{(1)}) \leq 0$.

From the above lemmas, it is sufficient to consider the case when $\theta = 0$ is a characteristic direction of the origin of system (1.2),

$$(\alpha^{(1)})^2 + (\beta^{(1)})^2 = 0, \qquad (\alpha^{(3)})^2 + (\beta^{(3)})^2 \neq 0$$

and

$$\mathrm{sign}_{\vec{0}}(\Theta)((4 + M - m)\alpha_j^{(3)} - 4\beta_j^{(3)}) \leq 0.$$

**Lemma 2.4.** *Suppose that $\theta = 0$ is a characteristic direction of the origin of system* (1.2). *Assume that* $(\alpha^{(1)})^2 + (\beta^{(1)})^2 = 0$, *and* $(\alpha^{(3)})^2 + (\beta^{(3)})^2 \neq 0$. *Then, system* (1.2) *has no characteristic orbit tending to the origin in the direction $\theta = 0$ if and only if it satisfies the following conditions*:

(i) *there exist two real numbers $\overline{r} > 0$ and $\delta > 0$, such that*

$$\Theta(r,\theta)=\cos\theta Q_m(\theta)-\sin\theta P_m(\theta)+r^{M-m}(\cos\theta Q_M(\theta)-\sin\theta P_M(\theta))\neq 0,$$

*for* $(r,\theta) \in \{(0,\overline{r}] \times (-\arctan(\delta), \arctan(\delta))\} \setminus \{(0,0)\}$;

(ii) $P_m(0) = Q_m(0) = 0$;

(iii) *$m$ and $M$ are both odd*;

(iv) $\alpha^{(2)} = \beta^{(2)} = 0$;

(v) $\mathrm{sign}_{\vec{0}}(\Theta)((4 + M - m)\alpha^{(3)} - 2\beta^{(3)}) \leq 0$.

*Proof.* Lemma 2.3 shows that conditions (i)–(iii) are necessary to ensure that there is no characteristic orbit tending to the origin in the direction $\theta = 0$. In the proof of the sufficiency of the five conditions, we can see that conditions (iv) and (v) are also necessary.

Without loss of generality, we assume that $\mathrm{sign}_{\vec{0}}(\Theta) = 1$, and thus, $(4 + M - m)\alpha^{(3)} - 2\beta^{(3)} \leq 0$. Condition (iii) implies that $P_m(1,0) = Q_m(1,0) = 0$. From condition (i), we have that

$$F(z) := Q_m(1,z) - zP_m(1,z) \geq 0,$$

and thus, the first non-vanishing derivative of $F$ at the origin is of even order. Taking into account $F^{(n)}(z) = Q_m^{(n)} - nP_m^{(n-1)}(1,z) - zP_m^{(m)}(1,z)$, we have

$$\begin{aligned}
F(0) &= Q_m(1,0) = 0,\\
F'(0) &= Q_m'(1,0) - P_m(1,0) = 0,\\
F''(0) &= Q_m''(1,0) - 2P_m'(1,0) = 2!(\beta^{(1)} - \alpha^{(1)}) = 0,\\
F'''(0) &= Q_m'''(1,0) - 3P_m''(1,0) = 3!(\beta^{(2)} - \alpha^{(2)}) = 0,\\
F^{(4)}(0) &= Q_m^{(4)}(1,0) - 4P_m'''(1,0) = 4!(\beta^{(3)} - \alpha^{(3)}) \geq 0.
\end{aligned}$$

In addition, $\gamma := Q_M(1,0) > 0$.

We make the following transformation of variables

$$T_1 : x_1 = x^{M-m}, y_1 = \frac{y}{x}.$$

This is not a global transformation of coordinates in $\mathbb{R}^2 \setminus \{(0,0)\}$, but it is a good transformation on $\{x > 0\}$, see **[13]**. In what follows, we only consider the transformation $T_1$ in the region $\{x > 0\}$; however, the results obtained are also valid for $\{x < 0\}$ since system (1.3) satisfies

$$(\dot{r}(r,\theta+\pi), \dot{\theta}(r,\theta+\pi)) = (\dot{r}(r,\theta), \dot{\theta}(r,\theta))$$

when $m$ and $M$ are both odd. After rescaling, we obtain (the dot always denotes the derivative with respect to a new time parameter)

$$\begin{aligned}
\dot{x}_1 &= kx_1[P_m(1,y_1) + x_1P_M(1,y_1)],\\
\dot{y}_1 &= Q_m(1,y_1) - y_1P_m(1,y_1) + x_1[Q_M(1,y_1) - y_1P_M(1,y_1)],
\end{aligned} \tag{2.1}$$

where $k = M - m$. Obviously, system (1.2) has an orbit tending to the origin $O(0,0) \in \mathbb{R}^2_{xy}$ in the direction $\theta = 0$ if and only if system (2.1) has an orbit tending to the origin $O_1(0,0) \in \mathbb{R}^2_{x_1y_1}$. The differential matrix of (2.1) associated with $(x_1,y_1) = (0,0)$ is given by $\left(\begin{smallmatrix} 0 & 0\\ \gamma & 0\end{smallmatrix}\right)$. This singularity is a nilpotent and, to desingularize it, we must continue the blow-up process.

Now, we consider system (2.1) and make the following transformation of variables

$$T_2 : x_2 = \frac{x_1}{y_1}, y_1 = y_1.$$

After a reparametrization of time, we have

(2.2)

$$\begin{aligned}\dot{x}_2 &= x_2\Big[(k+1)P_m(1,y_1) - \frac{Q_m(1,y_1)}{y_1} \\ &\qquad + x_2((k+1)y_1P_M(1,y_1) - Q_M(1,y_1))\Big], \\ \dot{y}_1 &= Q_m(1,y_1) - y_1P_m(1,y_1) + x_2y_1[Q_M(1,y_1) - y_1P_M(1,y_1)].\end{aligned}$$

The singular point $(x_1, y_1) = (0, 0)$ of system (2.1) is blown up to some singular points of system (2.2) lying on $\{y_1 = 0\}$. Since

$$\begin{aligned}\dot{x}_2|_{y_1=0} &= -x_2Q_M(1,0), \\ \dot{y}_1|_{y_1=0} &= 0\end{aligned}$$

and

$$Q_M(1,0) \neq 0,$$

the only singular point on $\{y_1 = 0\}$ is $(x_2, y_1) = (0, 0)$, and its differential matrix is $\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right)$. This singularity is degenerate; thus, we must continue the blow-up process. On the other hand, since the transformation $T_2$ only blows up to the singular point $(x_1, y_1) = (0, 0)$ of system (2.1) in the $y_1$-direction, we must consider the blow-up in the $x_1$-direction. For system (2.1), we make the new transformation

$$T_3 : x_1 = x_1, y_2 = \frac{y_1}{x_1}.$$

This yields

$$\begin{aligned}\dot{x}_1 &= kx_1[P_m(1,x_1y_2) + x_1P_M(1,x_1y_2)], \\ \dot{y}_2 &= -y_2(1+k)\Big[P_m(1,x_1y_2) + \frac{Q_m(1,x_1y_2)}{x_1} \\ &\qquad - x_1y_2(k+1)P_M(1,x_1y_2) + Q_M(1,x_1y_2)\Big].\end{aligned} \tag{2.3}$$

The singular point $(x_1, y_1) = (0, 0)$ of system (2.1) is blown up to some singular points of system (2.3) lying on $\{x_1 = 0\}$. Since

$$\dot{x}_1|_{x_1=0} = 0,$$
$$\dot{y}_2|_{x_1=0} = \gamma,$$

$\gamma \neq 0$, system (2.3) has no singular point on $\{x_1 = 0\}$. Hence, in this direction, the singular point $(x_1, y_1) = (0, 0)$ has been desingularized.

Now, we consider system (2.2) and make the transformation

$$T_4 : x_3 = \frac{x_2}{y_1}, y_1 = y_1.$$

We obtain

$$\begin{aligned} \dot{x}_3 &= x_3 \Bigg[ (k+2)\frac{P_m(1,y_1)}{y_1} - 2\frac{Q_m(1,y_1)}{y_1^2} \\ &\qquad + x_3((k+2)y_1 P_M(1,y_1) - 2Q_M(1,y_1)) \Bigg], \\ \dot{y}_1 &= \frac{Q_m(1,y_1)}{y_1} - P_m(1,y_1) + x_3 y_1 [Q_M(1,y_1) - y_1 P_M(1,y_1)]. \end{aligned} \tag{2.4}$$

The singular point $(x_1, y_1) = (0, 0)$ of system (2.2) is blown up to some singular points of system (2.4) lying on $\{y_1 = 0\}$. Since

$$\dot{x}_3|_{y_1=0} = -2x_3^2 Q_M(1, 0),$$
$$\dot{y}_1|_{y_1=0} = 0$$

and

$$Q_M(1, 0) \neq 0,$$

the only singular point on $\{y_1 = 0\}$ is $(x_3, y_1) = (0, 0)$, and its differential matrix is $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Therefore, this singularity is degenerate, and we must continue the blow-up process.

On the other hand, since the transformation $T_4$ blows up only to the singular point $(x_2, y_1) = (0, 0)$ of system (2.2) in the $y_1$-direction, we must consider blow-up in the $x_2$-direction. For system (2.2), we make the new transformation

$$T_5 : x_2 = x_2, y_3 = \frac{y_1}{x_2}.$$

This yields

$$\tag{2.5}
\begin{aligned}
\dot{x}_2 &= (k+1)P_m(1,x_2y_3) - \frac{Q_m(1,x_2y_3)}{x_2y_3}\\
&\quad + x_2^2y_3(k+1)P_M(1,x_2y_3) - x_2Q_M(1,x_2y_3),\\
\dot{y}_3 &= -y_3(2+k)\frac{P_m(1,x_2y_3)}{x_2} + 2\frac{Q_m(1,x_2y_3)}{x_2^2}\\
&\quad - x_2y_3^2(k+2)[P_M(1,x_2y_3) + 2y_3Q_M(1,x_2y_3)].
\end{aligned}$$

The singular point $(x_2, y_1) = (0,0)$ of system (2.2) is blown up to some singular points of system (2.5) lying on $\{x_2 = 0\}$. Since

$$\begin{aligned}
\dot{x}_2|_{x_2=0} &= 0,\\
\dot{y}_3|_{x_2=0} &= 2\gamma y_3,
\end{aligned}$$

system (2.5) has a singular point $(x_2, y_3){=}(0,0)$ on $\{x_2{=}0\}$, and its differential matrix is $\left(\begin{smallmatrix} -\gamma & 0\\ 0 & 2\gamma \end{smallmatrix}\right)$. Therefore, this singular point is a saddle, and, in this direction, the singular point $(x_2, y_3) = (0,0)$ has been desingularized.

Now consider system (2.4), and make the transformation

$$T_6 : x_4 = \frac{x_3}{y_1}, y_1 = y_1.$$

This yields

$$\tag{2.6}
\begin{aligned}
\dot{x}_4 &= x_4\Bigg[(k+3)\frac{P_m(1,y_1)}{y_1^2} - 3\frac{Q_m(1,y_1)}{y_1^3}\\
&\qquad + x_4((k+3)y_1P_M(1,y_1) - 3Q_M(1,y_1))\Bigg],\\
\dot{y}_1 &= \frac{Q_m(1,y_1)}{y_1^2} - \frac{P_m(1,y_1)}{y_1} + x_4y_1[Q_M(1,y_1) - y_1P_M(1,y_1)].
\end{aligned}$$

The singular point $(x_3, y_1) = (0,0)$ of system (2.4) is blown up to some singular points of system (2.6) on $\{y_1 = 0\}$, and

$$\begin{aligned}
\dot{x}_4|_{y_1=0} &= x_4(k\alpha^{(2)} - 3\gamma x_4),\\
\dot{y}_1|_{y_1=0} &= 0.
\end{aligned}$$

Now, we show that $\alpha^{(2)} = 0$, and hence, $\beta^{(2)} = 0$ by

$$F'''(0) = 3!(\beta^{(2)} - \alpha^{(2)}) = 0.$$

Suppose that $\alpha^{(2)} \neq 0$. Then system (2.6) has two singular points $(x_4, y_1) = (0, 0)$ and $(x_4, y_1) = (k\alpha^{(2)}/3\gamma, 0)$ on $\{y_1 = 0\}$, and the differential matrix at $(x_4, y_1) = (0, 0)$ is

$$\begin{pmatrix} k\alpha^{(2)} & 0 \\ 0 & 0 \end{pmatrix},$$

while the differential matrix at $(x_4, y_1) = (k\alpha^{(2)}/3\gamma, 0)$ is

$$\begin{pmatrix} k\alpha^{(2)} & \star \\ 0 & k\alpha^{(2)}/3\gamma \end{pmatrix}.$$

Thus, we know that the singular point $(x_4, y_1) = (k\alpha^{(2)}/3\gamma, 0)$ of system (2.6) is a node, and hence, system (2.6) has infinite orbits tending to the singular point $(x_4, y_1) = (k\alpha^{(2)}/3\gamma, 0)$, which is a contradiction. Therefore, $\alpha^{(2)} = 0$. This implies that condition (iv) is necessary. When $\alpha^{(2)} = 0$, system (2.6) has a unique singular point $(x_4, y_1) = (0, 0)$, and its differential matrix is $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. This singularity is degenerate, and we must continue the blow-up process.

On the other hand, since the transformation $T_6$ blows up only to the singular point $(x_3, y_1) = (0, 0)$ of system (2.4) in the $y_1$-direction, we must consider the blow-up in the $x_3$-direction. For system (2.4), we make the new transformation

$$T_7 : x_3 = x_3, y_4 = \frac{y_1}{x_3}.$$

This yields

$$\begin{aligned} \dot{x}_3 &= (k+2)\frac{P_m(1, x_3y_4)}{x_3y_4} - 2\frac{Q_m(1, x_3y_4)}{x_3^2y_4^2} \\ &\quad + x_3^2y_4(k+2)P_M(1, x_3y_4) - 2x_3Q_M(1, x_3y_4), \\ \dot{y}_4 &= -(3+k)\frac{P_m(1, x_3y_4)}{x_3^2} + 3\frac{Q_m(1, x_3y_4)}{x_3^3y_4} \\ &\quad - x_3y_4^2(k+3)[P_M(1, x_3y_4) + 3y_4Q_M(1, x_3y_4)]. \end{aligned} \tag{2.7}$$

The singular point $(x_3, y_1) = (0, 0)$ of system (2.4) is blown up to some singular points of system (2.7) lying on $\{x_3 = 0\}$. Since

$$\dot{x}_3|_{x_3=0} = 0,$$
$$\dot{y}_4|_{x_3=0} = 3\gamma y_4,$$

system (2.7) has a singular point $(x_3, y_4) = (0, 0)$ on $\{x_3 = 0\}$, and its differential matrix is

$$\begin{pmatrix} -2\gamma & 0 \\ 0 & 3\gamma \end{pmatrix}.$$

Thus, this singular point is a saddle, and, in this direction, $(x_3, y_4) = (0, 0)$ has been desingularized.

We consider system (2.6) and make the transformation

$$T_8 : x_5 = \frac{x_4}{y_1}, y_1 = y_1.$$

This produces

(2.8)

$$\begin{aligned}
\dot{x}_5 = x_5\Bigg[&(k+4)\frac{P_m(1, y_1)}{y_1^3} - 4\frac{Q_m(1, y_1)}{y_1^4} \\
&+ x_5((k+4)y_1 P_M(1, y_1) - 4Q_M(1, y_1))\Bigg], \\
\dot{y}_1 = \frac{Q_m(1, y_1)}{y_1^3} &- \frac{P_m(1, y_1)}{y_1^2} + x_5 y_1[Q_M(1, y_1) - x_5 y_1^2 P_M(1, y_1)].
\end{aligned}$$

The singular point $(x_4, y_1) = (0, 0)$ of system (2.6) is blown up to some singular points of system (2.8) on $\{y_1 = 0\}$. Since

$$\dot{x}_5|_{y_1=0} = x_5((k+4)\alpha^{(3)} - 4\beta^{(3)} - 4\gamma x_5),$$
$$\dot{y}_1|_{y_1=0} = 0,$$

system (2.8) has two singular points $(x_5, y_1) = (0, 0)$ and

$$(x_5, y_1) = \left(\frac{(k+4)\alpha^{(3)} - 4\beta^{(3)}}{4\gamma}, 0\right)$$

on $\{y_1 = 0\}$.

Note that the region $\{x > 0\}$ for system (1.2) has been transformed into the region $\{x_5 \geq 0\}$ for system (2.8). The dynamics for system (2.8) in the region $\{x_5 < 0\}$ is virtual, that is, it does not appear in coordinates $(x, y)$. It is easy to see that the differential matrix of (2.8) at $(x_5, y_1) = (0, 0)$ is

$$\begin{pmatrix} (k+4)\alpha^{(3)} - 4\beta^{(3)} & 0 \\ 0 & \beta^{(3)} - \alpha^{(3)} \end{pmatrix}.$$

Now consider system (2.6) and make the transformation

$$T_9 : x_4 = x_4, y_5 = \frac{y_1}{x_4}.$$

This yields

$$\begin{aligned} \dot{x}_4 &= (k+3)\frac{P_m(1, x_4y_5)}{x_4^2y_5^2} - 3\frac{Q_m(1, x_4y_5)}{x_4^3y_5^3} \\ &\quad + x_4^2y_5(k+4)P_M(1, x_4y_5) - 3Q_M(1, x_4y_5), \\ \dot{y}_5 &= -(k+4)\frac{P_m(1, x_4y_5)}{x_4^3y_5} + 4\frac{Q_m(1, x_4y_5)}{x_4^4y_5^2} \\ &\quad + x_4y_5^2(k+4)P_M(1, x_4y_5) + 4y_5Q_M(1, x_4y_5). \end{aligned} \tag{2.9}$$

The singular point $(x_4, y_1) = (0, 0)$ of system (2.6) is blown up to some singular points of system (2.9) on $\{x_4 = 0\}$. Since

$$\begin{aligned} \dot{x}_4|_{x_4=0} &= 0, \\ \dot{y}_5|_{x_4=0} &= y_5[(4\beta^{(3)} - (4+k)\alpha^{(3)})y_5 + 4\gamma], \end{aligned}$$

system (2.9) has two singular points $(x_4, y_5) = (0, 0)$ and

$$(x_4, y_5) = \left(0, \frac{4\gamma}{(4+k)\alpha^{(3)} - 4\beta^{(3)}}\right)$$

on $\{x_4 = 0\}$. In particular, when

$$(4+k)\alpha^{(3)} - 4\beta^{(3)} = 0,$$

system (2.9) has a unique singular point $(x_4, y_5) = (0, 0)$. Similarly, the region $\{x > 0\}$ for system (1.2) has been transformed into the region $\{y_5 \geq 0\}$ for system (2.9). Hence, the dynamics for system (2.8) in the region $\{y_5 < 0\}$ are virtual. It is easy to see that the differential matrix of (2.9) at $(x_4, y_5) = (0, 0)$ is

$$\begin{pmatrix} -3\gamma & 0 \\ 0 & 4\gamma \end{pmatrix}.$$

Now we distinguish the following cases.

**Case (a).** $(4+k)\alpha^{(3)} - 4\beta^{(3)} \neq 0$.

**(a1)** If $(4+k)\alpha^{(3)} - 4\beta^{(3)} < 0$, then the singular points of systems (2.8) and (2.9), respectively, satisfy

$$(x_5, y_1) = \left( \frac{(4+k)\alpha^{(3)} - 4\beta^{(3)}}{4\gamma}, 0 \right) \in \{x_5 < 0\}$$

and

$$(x_4, y_5) = \left( 0, \frac{4\gamma}{(4+k)\alpha^{(3)} - 4\beta^{(3)}} \right) \in \{y_5 < 0\}.$$

Therefore, we know that the singular points of systems (2.8) and (2.9) are $(x_5, y_1) = (0,0)$ and $(x_4, y_5) = (0,0)$, respectively. It is easy to see that both are hyperbolic saddles with separatrices in the coordinate axes, and these separatrices do not correspond to characteristic orbits tending to the origin of system (1.2) in the direction $\theta = 0$.

**(a2)** If $(4+k)\alpha^{(3)} - 4\beta^{(3)} > 0$, then the singular point $(x_5, y_1) = (0,0)$ of systems (2.8) is a hyperbolic node, and hence, there are infinite many orbits tending to the origin of system 1.2).

**Case (b).** $\beta^{(3)} - \alpha^{(3)} = 0$. Let $\lambda = (4+k)\alpha^{(3)} - 4\beta^{(3)}$. Then, system (2.8) can be written in the following form:

$$\begin{aligned}
\dot{x}_5 &= x_5[\Phi_1(y_1) + x_5\Phi_2(y_1)],\\
\dot{y}_1 &= \Psi_1(y_1) + x_5\Psi_2(y_1).
\end{aligned}$$

Reparametrizing the above system in order to apply the classification theorem of this type of singular point **[11]**, we obtain

$$\begin{aligned}
\dot{y}_1 &= \frac{1}{\lambda}[\Psi_1(y_1) + x_5\Psi_2(y_1)] := X(y_1, x_5),\\
\dot{x}_5 &= \frac{x_5}{\lambda}[\Phi_1(y_1) + x_5\Phi_2(y_1)] := x_5 + Y(y_1, x_5).
\end{aligned}$$

Note that the unique solution of $x_5 + Y(y_1, x_5) = 0$ passing through $(x_5, y_1) = (0,0)$ is $x_5 = 0$. Since

$$F(y_1) := Q_m(1, y_1) - y_1 P_m(1, y_1) = y_1^3\Psi_1(y_1) \geq 0,$$

the first non-vanishing derivative at $y_1 = 0$ of $F$ is of even order. Without loss of generality, assume that $F^{(2n)}(y_1)|_{y_1=0} > 0$ is the first non-vanishing derivative at $y_1 = 0$ of $F$. It is simple to prove by induction that

$$\begin{aligned} F^{(2n)}(y_1)|_{y_1=0} &= (y_1^3\Psi_1(y_1))^{(2n)}|_{y_1=0} \\ &= \sum_{k=0}^{2n} C_{2n}^k (y_1)^{(k)}(\Psi_1(y_1))^{(2n-k)}|_{y_1=0} \\ &= 6C_{2n}^3\Psi_1^{(2n-3)}(0). \end{aligned}$$

Hence, $\Psi_1^{(2n-3)}(0) > 0$ is the first non-vanishing derivative at $y_1 = 0$ of $\Psi_1(y_1)$. Therefore,

$$X(y_1, 0) = \frac{\Psi_1(y_1)}{\lambda} = \frac{1}{\lambda(2n-3)!}\Psi_1^{(2n-3)}(0)y_1^{2n-3} + \cdots .$$

Applying the classification theorem of this type of singular point **[11]**, we obtain that, if $\lambda < 0$, then the singular point $(x_5, y_1) = (0, 0)$ is a topological saddle; if $\lambda > 0$, then the singular point $(x_5, y_1) = (0, 0)$ is a topological node. This shows that, if $\beta^{(3)} - \alpha^{(3)} \geq 0$, only then $(4 + k)\alpha^{(3)} - 4\beta^{(3)} < 0$ does system (1.2) have no characteristic orbit tending to the origin in the direction $\theta = 0$. Therefore, condition (v) is necessary.

**Case (c).** $(4 + k)\alpha^{(3)} - 4\beta^{(3)} = 0$. Since $(\alpha^{(3)})^2 + (\beta^{(3)})^2 \neq 0$ and $\beta^{(3)} - \alpha^{(3)} \geq 0$,

$$\mu := \beta^{(3)} - \alpha^{(3)} = \frac{k\alpha^{(3)}}{4} > 0.$$

System (2.8) may be rewritten in the following form:

$$\begin{aligned} \dot{x}_5 &= x_5[\Phi_1(y_1) + x_5\Phi_2(y_1)], \\ \dot{y}_1 &= \Psi_1(y_1) + x_5\Psi_2(y_1) = \mu y_1 + \cdots . \end{aligned}$$

In order to apply the classification theorem of this type of singular point **[11]**, we reparametrize the above system and then obtain

$$\begin{aligned} \dot{x}_5 &= \frac{x_5}{\mu}[\Phi_1(y_1) + x_5\Phi_2(y_1)] := X(x_5, y_1), \\ \dot{y}_1 &= \frac{1}{\mu}[\Psi_1(y_1) + x_5\Psi_2(y_1)] := y_1 + Y(x_5, y_1). \end{aligned}$$

Note that the only solution of $y_1 + Y(x_5, y_1) = 0$ passing through $(x_5, y_1) = (0, 0)$ is $y_1 = 0$. Substituting $y_1 = 0$ into $X(x_5, y_1)$ yields

$$X(x_5, 0) = -\frac{4\gamma}{\mu} x_5^2,$$

where $\mu > 0$ and $\gamma > 0$, and then $-(4\gamma/\mu)x_5^2 < 0$. Again applying the classification theorem of this type of singular point **[11]**, we see that the singular point $(x_5, y_1) = (0, 0)$ of system (2.8) is a saddle node, with the nodal sectors on $\{x_5 < 0\}$, the center manifold $W^C$ in the $x_5$-axis, and the $y_1$-axis being the other separatrix. Hence, there is no characteristic orbit of system (1.2) tending to the origin in the direction $\theta = 0$. $\square$

**Lemma 2.5.** *Suppose that $\mathcal{X}$ is a vector field of class $\mathcal{G}_1$. If the system associated with $\mathcal{X}$ satisfies Conditions* (a) *and* (b)*, and, for each characteristic direction $\theta_j$, $j = 1, 2, \ldots, k$,* $\mathrm{sign}_{\vec{0}}(\Theta)((2 + M - m)\alpha_j^{(1)} - 2(\beta_j^{(1)})) \leq 0$ *when* $(\alpha_j^{(1)})^2 + (\beta_j^{(1)})^2 \neq 0$; $\mathrm{sign}_{\vec{0}}(\Theta)((4 + M - m)\alpha_j^{(3)} - 4\beta_j^{(3)}) \leq 0$ *when* $(\alpha_j^{(1)})^2 + (\beta_j^{(1)})^2 = 0$ *but* $(\alpha_j^{(3)})^2 + (\beta_j^{(3)})^2 \neq 0$. *Then, the origin of system* (1.2) *is monodromic.*

*Proof.* For each characteristic direction $\theta_j$, $j = 1, 2, \ldots, k$, after a rotation of angle $\varphi = -\theta_j$, that is, $x = a_j u - b_j v, y = b_j u + a_j v$, where $a_j$ and $b_j$ are as defined above, system (1.2) becomes

(2.10)
$$\begin{aligned}
\dot{u} &= a_j P_m(a_j u - b_j v, b_j u + a_j v) + b_j Q_m(a_j u - b_j v, b_j u + a_j v) \\
&\quad + a_j P_M(a_j u - b_j v, b_j u + a_j v) + b_j Q_M(a_j u - b_j v, b_j u + a_j v), \\
\dot{v} &= -b_j P_m(a_j u - b_j v, b_j u + a_j v) + a_j Q_m(a_j u - b_j v, b_j u - a_j v) \\
&\quad - b_j P_M(a_j u - b_j v, b_j u + a_j v) + a_j Q_M(a_j u - b_j v, b_j u + a_j v).
\end{aligned}$$

This system has a characteristic direction in $v = 0$. Applying Lemmas 2.3 and 2.4 to system (2.10), we see that it contains no orbits tending to the origin in the direction $v = 0$. This direction corresponds to the direction $\theta = \theta_j$ for system (1.2). Hence, the origin is a monodromic singular point. $\square$

*Proof of Theorem* 1.3. Follows straightforwardly from Theorem 1.1 and Lemma 2.5. $\square$

**3. An example.** Consider the following system

$$\dot{x} = y^3 - x^2 y^3, \qquad \dot{y} = -3x^5 - y^5. \tag{3.1}$$

It is not difficult to see that (3.1) can be written in polar coordinates $(r, \theta)$ as

$$\begin{aligned} \dot{r} &= \cos\theta \sin^3\theta - r(\cos^3\theta \sin^3\theta + 3\sin\theta\cos^5\theta + \sin^6\theta), \\ \dot{\theta} &= -\sin^4\theta - r^2(3\cos^6\theta + \cos\theta\sin^5\theta - \sin^4\theta\cos\theta). \end{aligned} \tag{3.2}$$

Its origin has two directions given by $\{\theta_1 = 0\}$ and $\{\theta_2 = \pi\}$. Since $(\dot{r}(r, \theta + \pi), \dot{\theta}(r, \theta + \pi)) = (\dot{r}(r, \theta), \dot{\theta}(r, \theta))$, we only need verify the monodromic conditions for $\theta_1 = 0$. Condition (a) is satisfied since $\Theta(x, y) := xQ(x, y) - yP(x, y) = -3x^6 - y^4(x^2 + x + 1) < 0$, except at the origin. Also, Condition (b) is trivially satisfied since $P_3(\theta_i) = Q_3(\theta_i) = 0$ for $i = 1, 2$,

$$\alpha^{(1)} = \beta^{(1)} = 0, \alpha^{(2)} = \beta^{(2)} = 0, (\alpha^{(3)})^2 + (\beta^{(3)})^2 = 1 \neq 0$$

and

$$(4 + M - m)\alpha^{(3)} - 4\beta^{(3)} = 2\alpha^{(3)} = 2 > 0.$$

From Theorem 1.3, we see that the origin of system (3.1) is a monodromic singular point.

The numerical solution of system (3.1) by `Maple 6.0` is given in Figure 1. It shows that the origin is actually a monodromic singular point which, in fact, is a stable fine focus.

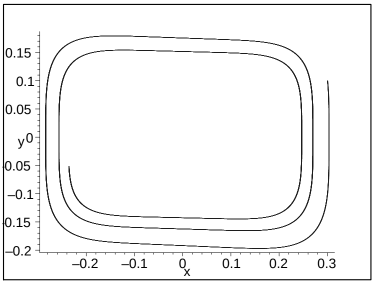


FIGURE 1. Numerical solution of system (3.1). The initial condition is chosen to be the point (0.3, 0.1), close to the origin.

**4. Conclusion.** Since many mathematical models which are proposed from the physical, chemical and biological communities are polynomial systems, it is important to analyze qualitative properties of polynomial systems. There are very few results on the monodromy problem for the polynomial system which has a zero linear portion at its singular point. The monodromy problem of a type of degenerate system defined by the sum of two homogeneous vector fields with a zero linear portion was studied, and a monodromy criterion of such a degenerate polynomial system satisfying special conditions, i.e., $(\alpha_j^{(1)})^2 + (\beta_j^{(1)})^2 \neq 0$, for all $j = 1, 2, \ldots, k$, is given in **[13]**. In this paper, we obtained a monodromy criterion of a type of degenerate polynomial system with weaker conditions, that is,

$$(\alpha_j^{(1)})^2 + (\beta_j^{(1)})^2 = 0;$$

however,

$$(\alpha_j^{(3)})^2 + (\beta_j^{(3)})^2 \neq 0$$

for some $j$, $j = 1, 2, \ldots, k$, by means of a blow-up technique and a classification theorem. An example was presented to show that our result generalizes the corresponding result in **[13]**. The monodromy criterion of such a degenerate system defined by the sum of two homogeneous vector fields with no additional conditions will be considered in our future work.

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