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CERTAIN K3 SURFACES PARAMETRIZED BY THE FIBONACCI SEQUENCE VIOLATE THE HASSE PRINCIPLE

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ABSTRACT. For a prime $p \equiv 5 \pmod 8$ satisfying certain conditions, we show that there exist an infinitude of K3 surfaces parameterized by certain solutions to Pell's equation $X^2 - pY^2 = 4$ in the projective 5-space that are counterexamples to the Hasse principle explained by the Brauer-Manin obstruction. Further, these surfaces contain no zero-cycle of odd degree over $\mathbb{Q}$. As an illustration for the main result, we show that the prime $p = 5$ satisfies all of the required conditions in the main theorem, and hence, there exist an infinitude of K3 surfaces parameterized by the Fibonacci sequence that are counterexamples to the Hasse principle explained by the Brauer-Manin obstruction.

1. Introduction. Let $\mathcal{V}$ be a smooth projective geometrically irreducible variety over $\mathbb{Q}$. The variety $\mathcal{V}$ is said to *satisfy the Hasse principle* if the following holds: $\mathcal{V}(\mathbb{Q}) \neq \emptyset$ if and only if $\mathcal{V}(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$, where $\mathbb{A}_{\mathbb{Q}}$ denotes the ring of rational adeles. If $\mathcal{V}$ has points locally everywhere but no rational points over $\mathbb{Q}$, we say that $\mathcal{V}$ is a *counterexample to the Hasse principle.* Although the Hasse principle holds for quadric hypersurfaces of arbitrary dimension, it fails in general. The first counterexamples of genus 1 curves to the Hasse principle were discovered by Lind **[5]** in 1940 and independently shortly thereafter by Reichardt **[10]**.

In 1970, Manin **[6]** introduced a subset of $\mathcal{V}(\mathbb{A}_{\mathbb{Q}})$, say $\mathcal{V}(\mathbb{A}_{\mathbb{Q}})^{\mathbf{Br}}$, such that

$$\mathcal{V}(\mathbb{Q}) \subseteq \mathcal{V}(\mathbb{A}_{\mathbb{Q}})^{\mathbf{Br}} \subseteq \mathcal{V}(\mathbb{A}_{\mathbb{Q}}).$$

If the intermediate set $\mathcal{V}(\mathbb{A}_{\mathbb{Q}})^{\mathbf{Br}}$ explains the failure of the Hasse principle, that is, $\mathcal{V}(\mathbb{A}_{\mathbb{Q}})^{\mathbf{Br}} = \emptyset$, but $\mathcal{V}(\mathbb{A}_{\mathbb{Q}})$ is non-empty, we say that $\mathcal{V}$

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is a *counterexample to the Hasse principle explained by the Brauer-Manin obstruction.*

The main interest of this paper lies in constructing a new family of K3 surfaces of genus 5 for which the following question has an affirmative answer.

Question 1.1. *Is the Brauer-Manin obstruction the only obstruction to the Hasse principle for* K3 *surfaces?*

To the author's knowledge, Question 1.1 is still wide open although there are several explicit examples of K3 surfaces which are counterexamples to the Hasse principle explained by the Brauer-Manin obstruction. We shortly recall some explicit examples of such K3 surfaces. Note that these examples is just a small part of all the well-known examples K3 surfaces that are counterexamples to the Hasse principle.

In [**8**], the author constructs certain quartic del Pezzo surfaces which are counterexamples to the Hasse principle explained by the Brauer-Manin obstruction. The construction of these quartic del Pezzo surfaces generalizes that of a quartic del Pezzo violating the Hasse principle by Birch and Swinnerton-Dyer [**1**]. Using these quartic del Pezzo surfaces, the author [**8**] constructs algebraic families of K3 surfaces of genus 5 which are counterexamples to the Hasse principle explained by the Brauer-Manin obstruction. Thus, Question 1.1 has an affirmative answer for the algebraic families of K3 surfaces of genus 5 in [**8**].

Colliot-Thélène, Coray and Sansuc [**3**] proved that the threefold $\mathcal{Y}_{(5,1,1)}$ in $\mathbb{P}^5$, defined by

$$\mathcal{Y}_{(5,1,1)} : \begin{cases} u_1^2 - 5v_1^2 &= 2xy, \\ u_2^2 - 5v_2^2 &= 2(x+20y)(x+25y), \end{cases}$$

is a counterexample to the Hasse principle explained by the Brauer-Manin obstruction. Coray and Manoil [**4**] constructed *one* K3 surface of genus 5 lying on the threefold $\mathcal{Y}_{(5,1,1)}$ that is a counterexample to the Hasse principle explained by the Brauer-Manin obstruction.

In [**9**], the author generalized the construction of the threefold $\mathcal{Y}_{(5,1,1)}$ in $\mathbb{P}^5$, violating the Hasse principle by Colliot-Thélène, Coray and Sansuc [**3**] to produce arithmetic families of threefolds in $\mathbb{P}^5$ which are counterexamples to the Hasse principle explained by the Brauer-Manin obstruction. In this paper, we will construct arithmetic families of K3 surfaces of genus 5 lying on these same threefolds. *Curiously, there are several* K3 *surfaces of this type that are parametrized by the Fibonacci sequence.*

We now state a special case of our main theorem, see Theorem 3.1, that describes certain K3 surfaces parametrized by the Fibonacci sequence which are counterexamples to the Hasse principle.

Theorem 1.2. *Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence, that is, $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for every $n \geq 0$. For each integer $n \geq 0$, define*:

$$\begin{cases} \lambda_n = F_{2n}^2/536 - 1099/14472, \\ \mu_n = 55F_{2n}^2/536 - 7669/1608, \\ \nu_n = -325F_{2n}^2/268 + 71219/7236. \end{cases} \tag{1.1}$$

For each integer $n \geq 0$, let $\mathcal{K}_n \subset \mathbb{P}^5$ be the K3 *surface defined by*:

$$\mathcal{K}_n : \begin{cases} u^2 = 2xy + 5z^2, \\ v^2 - 5w^2 = 2(x + 20y)(x + 25y), \\ w^2 = \lambda_n x^2 + \mu_n xy + \nu_n y^2 + z^2. \end{cases} \tag{1.2}$$

Then, for every integer $n \geq 0$, $\mathcal{K}_n$ satisfies the following.

(i) $\mathcal{K}_n$ *is a counterexample to the Hasse principle explained by the Brauer-Manin obstruction*; *and*
(ii) $\mathcal{K}_n$ *contains no zero-cycle of odd degree over* $\mathbb{Q}$.

Theorem 1.2 is a corollary to Theorem 3.1 that is a more general result but requires certain technical conditions. We will prove Theorem 3.1 in subsection 3.1 and Theorem 1.2 in subsection 3.2.

2. Certain threefolds in $\mathbb{P}^5$ violating the Hasse principle. In this section, we recall one of our earlier results in [**9**] that describes the construction of certain threefolds in $\mathbb{P}^5$ violating the Hasse principle. The following theorem plays a key role in constructing certain K3 surfaces which are counterexamples to the Hasse principle explained by the Brauer-Manin obstruction in Section 3.

Theorem 2.1. ([**9,** Theorem 2.7]). *Let p be a prime such that $p \equiv 5 \pmod 8$. Let b and d be integers, and let $q := |pd^2 - 4b^2|$. Assume that the following are true*:

(A1) 3 *is a quadratic non-residue in* $\mathbb{F}_p^{\times}$.
(A2) $b \not\equiv 0 \pmod 3$, $b \not\equiv 0 \pmod p$ *and b and d are odd integers.*
(A3) q *is either* 1 *or an odd prime.*

Let $\mathcal{X}$ be the smooth variety in $\mathbb{A}^5$ defined by

$$\mathcal{X} : \begin{cases} 0 \neq u_1^2 - pv_1^2 = 2x \\ 0 \neq u_2^2 - pv_2^2 = 2(x + 4pb^2)(x + p^2d^2). \end{cases} \tag{2.1}$$

Let $\mathcal{Z}$ be a smooth and proper $\mathbb{Q}$-model of $\mathcal{X}$, and let $\mathcal{Y}$ be the singular variety in $\mathbb{P}^5$ defined by

$$\mathcal{Y} : \begin{cases} u_1^2 - pv_1^2 = 2xy \\ u_2^2 - pv_2^2 = 2(x + 4pb^2y)(x + p^2d^2y). \end{cases} \tag{2.2}$$

Then,

(i) $\mathcal{X}, \mathcal{Y}$ *and* $\mathcal{Z}$ *are counterexamples to the Hasse principle explained by the Brauer-Manin obstruction*; *and*
(ii) $\mathcal{X}, \mathcal{Y}$ *and* $\mathcal{Z}$ *contain no zero-cycle of odd degree over* $\mathbb{Q}$.

Remark 2.2. The author [**9**] showed that there are infinitely many triples (p, b, d) satisfying (A1)–(A3) in Theorem 2.1.

Example 2.3. Let $(p, b, d) = (5, 1, 1)$. It is not difficult to verify that the triple (p, b, d) satisfies (A1)–(A3). Let $\mathcal{Y}_{(5,1,1)}$ be the singular $\mathbb{Q}$-threefold in $\mathbb{P}^5_{\mathbb{Q}}$ defined by

$$\mathcal{Y}_{(5,1,1)} : \begin{cases} u_1^2 - 5v_1^2 = 2xy \\ u_2^2 - 5v_2^2 = 2(x + 20y)(x + 25y). \end{cases}$$

By Theorem 2.1, $\mathcal{Y}_{(5,1,1)}$ is a counterexample to the Hasse principle explained by the Brauer-Manin obstruction and has no zero-cycles of odd degree over $\mathbb{Q}$. The threefold $\mathcal{Y}_{(5,1,1)}$ is the well known Colliot-Thélène-Coray-Sansuc threefold [**3,** Proposition 7.1].

3. Certain K3 surfaces violating the Hasse principle. In this section, we construct certain K3 surfaces of genus 5 which are counterexamples to the Hasse principle explained by the Brauer-Manin obstruction.

The next result is the main theorem in this paper.

Theorem 3.1. *We maintain the same notation and assumptions as in Theorem* 2.1. *Assume* (A1)–(A3). *Let* $(\Phi, \Omega) \in \mathbb{Z}^2$ *be a solution to Pell's equation* $X^2 - pY^2 = 4$, *and let* $\lambda, \mu, \nu \in \mathbb{Q}$ *be rational numbers. Assume that the following are true*:

(B1) $\Delta := 2(8b^2 - p^2 - 1)(2pd^2 - p^2 - 1)$ *is a perfect square in* $\mathbb{Z}$.
(B2) *The conic* $\mathcal{Q}_2 \subset \mathbb{P}^2$ *defined by*

$$\mathcal{Q}_2 : U^2 + pV^2 - 2(p - 1 + 8b^2)(p - 1 + 2pd^2)T^2 = 0$$

possesses a point $(\Gamma, \Lambda, \Sigma) \in \mathbb{Z}^3$ *such that* $\gcd(\Gamma, \Lambda, \Sigma) = 1$.
(B3) (λ, μ, ν) *satisfies*

$$\begin{cases} 4\lambda + 2\mu + \nu = -\Omega^2, \\ p^2(p-1)^2\Sigma^2\lambda + 2p(p-1)\Sigma^2\mu + 4\Sigma^2\nu = -p^2\Lambda^2 - 4\Sigma^2, \\ p^2(p^2+1)^2\lambda - 2p(p^2+1)\mu + 4\nu = -4. \end{cases} \tag{3.1}$$

(B4) $\lambda\mu\nu \neq 0$ *and* $\mu^2 - 4\lambda\nu \neq 0$.

Let $\mathcal{K} \subset \mathbb{P}^5$ *be the* K3 *surface defined by*

$$\mathcal{K} : \begin{cases} u^2 = 2xy + pz^2, \\ v^2 - pw^2 = 2(x + 4pb^2y)(x + p^2d^2y), \\ w^2 = \lambda x^2 + \mu xy + \nu y^2 + z^2. \end{cases} \tag{3.2}$$

Then,

(i) $\mathcal{K}$ *is a counterexample to the Hasse principle explained by the Brauer-Manin obstruction*; *and*
(ii) $\mathcal{K}$ *contains no zero-cycle of odd degree over* $\mathbb{Q}$.

Remark 3.2. By the definition of $\mathcal{K}$, it is not obvious that $\mathcal{K}$ is a K3 surface. In Lemma 3.7, we will prove that $\mathcal{K}$ is smooth, and hence, $\mathcal{K}$ is a K3 surface.

Remark 3.3. Note that the conic $\mathcal{Q}_2$ is nonsingular. Indeed, we know that

$$(p-1+8b^2)(p-1+2pd^2) \equiv 1-8b^2 \pmod{p}.$$

Since 2 is a quadratic non-residue in $\mathbb{F}_p^\times$, we see that $1-8b^2 \not\equiv 0 \pmod{p}$. Hence, we deduce that

$$(p-1+8b^2)(p-1+2pd^2) \in \mathbb{Z}_p^\times,$$

and thus, $(p-1+8b^2)(p-1+2pd^2) \neq 0$. Therefore, $\mathcal{Q}_2$ is nonsingular.

Remark 3.4. Suppose that $(\Gamma, \Lambda, \Sigma) \in \mathbb{Z}^3$ with $\gcd(\Gamma, \Lambda, \Sigma) = 1$ belongs to $\mathcal{Q}_2(\mathbb{Q})$. We contend that $\Gamma, \Sigma \not\equiv 0 \pmod{p}$. Indeed, we know that

$$\Gamma^2 + p\Lambda^2 - 2(p-1+8b^2)(p-1+2pd^2)\Sigma^2 = 0. \tag{3.3}$$

Reducing the equation above modulo p, we deduce that $\Gamma^2 \equiv 2(1-8b^2)\Sigma^2 \pmod{p}$. Since 2 is not a square modulo p, we see that $1-8b^2 \not\equiv 0 \pmod{p}$. Hence, $\Gamma \equiv 0 \pmod{p}$ if and only if $\Sigma \equiv 0 \pmod{p}$. Assume that one of these integers is zero modulo p. It then follows that both Γ and Σ are zero modulo p, and hence,

$$\Gamma = p\Gamma_1, \qquad \Sigma = p\Sigma_1,$$

for some integers Γ_1, Σ_1. It follows from (3.3) that

$$p\Gamma_1^2 + \Lambda^2 - 2p(p-1+8b^2)(p-1+2pd^2)\Sigma_1^2 = 0.$$

Taking the above equation modulo p, we deduce that $\Lambda \equiv 0 \pmod{p}$, which is a contradiction to the assumption that $\gcd(\Gamma, \Lambda, \Sigma) = 1$. Therefore, $\Gamma \not\equiv 0 \pmod{p}$ and $\Sigma \not\equiv 0 \pmod{p}$. It also follows from the last two congruences that Γ and Σ are nonzero integers.

Remark 3.5. Let $(\Phi, \Omega) \in \mathbb{Z}^2$ be a solution to Pell's equation $X^2 - pY^2 = 4$. It is well known that there are infinitely many solutions to the last equation. Let (p, b, d) be a triple of integers satisfying (A1)–(A3) and (B2). It is not difficult to show that there exists $(\lambda, \mu, \nu) \in \mathbb{Q}^3$ that satisfies (3.1), i.e., (B3) is satisfied. Indeed, the matrix M on the right-hand side of (3.1), defined by

$$M = \begin{bmatrix} 4 & 2 & 1 \\ p^2(p-1)^2\Sigma^2 & 2p(p-1)\Sigma^2 & 4\Sigma^2 \\ p^2(p^2+1)^2 & -2p(p^2+1) & 4 \end{bmatrix},$$

is of determinant

$$-2p^2(p+1)(p^2-p-4)(p^3+p+4)\Sigma^2 \neq 0.$$

Upon computing the inverse of M, we deduce that (λ, μ, ν) can be explicitly defined by

(3.4)

$$\begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} \frac{-4}{(p^2-p-4)(p^3+p+4)} & \frac{1}{p^2(p+1)(p^2-p-4)\Sigma^2} & \frac{1}{p^2(p+1)(p^3+p+4)} \\ \frac{-2p(p^2-p+2)}{(p^2-p-4)(p^3+p+4)} & \frac{p^3+p-4}{2p^2(p+1)(p^2-p-4)\Sigma^2} & \frac{-p^2+p-4}{2p^2(p+1)(p^3+p+4)} \\ \frac{p^2(p-1)(p^2+1)}{(p^2-p-4)(p^3+p+4)} & \frac{-p^2-1}{p(p+1)(p^2-p-4)\Sigma^2} & \frac{p-1}{p(p+1)(p^3+p+4)} \end{bmatrix}$$
$$\times \begin{bmatrix} -\Omega^2 \\ -p^2\Lambda^2 - 4\Sigma^2 \\ -4 \end{bmatrix},$$

where "×" denotes the matrix multiplication.

We will prove Theorem 3.1 in subsection 3.1.

We now prove some lemmata that we will need in the proof of Theorem 3.1.

Lemma 3.6. *We maintain the same notation and assumptions as in Theorem* 3.1. *Then,*

$$C_1C_2C_3C_4C_5C_6C_7C_8C_9 \neq 0,$$

where

$$\begin{cases} C_1 := 16p^2b^4\lambda - 4pb^2\mu + \nu + 8b^2, \\ C_2 := p^4d^4\lambda - p^2d^2\mu + \nu + 2pd^2, \\ C_3 := \nu + 8b^2d^2p^2, \\ C_4 := \lambda + \dfrac{2}{p}, \end{cases}$$

$$\begin{cases} C_5 := (p\mu + 8b^2p + 2d^2p^2 - 2)^2 - 4p(\nu + 8b^2d^2p^2)(p\lambda + 2), \\ C_6 := (2 - p\mu)^2 - 4p^2\lambda\nu, \\ C_7 := 16p^2b^4\lambda - 4pb^2\mu + \nu, \\ C_8 := p^4d^4\lambda - p^2d^2\mu + \nu, \\ C_9 := p(\mu + 8b^2 + 2pd^2)^2 - 4(p\lambda + 2)(\nu + 8b^2d^2p^2). \end{cases}$$

Proof. By definition, we see that C_i is a rational number for each $1 \leq i \leq 9$. For each $1 \leq i \leq 9$, denote the numerator of C_i by D_i. By (3.4) and computation, it can be shown that, for each $2 \leq i \leq 9$, there exist positive integers n_i and k_i such that D_i satisfies

$$D_i \equiv 2^{n_i}\Sigma^{k_i} \pmod{p}.$$

Take any integer i with $2 \leq i \leq 9$. If $C_i = 0$, then D_i is zero, and hence,

$$D_i \equiv 0 \pmod{p}.$$

It thus follows that

$$2^{n_i}\Sigma^{k_i} \equiv 0 \pmod{p},$$

which is a contradiction to Remark 3.4. Therefore, $C_i \neq 0$ for each $2 \leq i \leq 9$.

We now prove that $C_1 \neq 0$. By (3.4) and computation, we can show that

$$D_1 \equiv -16\Sigma^2(8b^2 - 1) \pmod{p}.$$

By Remark 3.4, we know that $\Sigma \not\equiv 0 \pmod{p}$. Since 2 is a quadratic non-residue in $\mathbb{F}_p^{\times}$, we deduce that $8b^2 - 1 \not\equiv 0 \pmod{p}$, and thus, $D_1 \not\equiv 0 \pmod{p}$. Therefore, $D_1 \neq 0$, and hence, $C_1 \neq 0$. Hence, our contention follows. $\square$

Lemma 3.7. *We maintain the same notation and assumptions as in Theorem* 3.1. *Then $\mathcal{K}$ is smooth, that is, $\mathcal{K}$ is a* K3 *surface.*

Remark 3.8. We will use the Jacobian criterion to prove Lemma 3.7 in subsection A.1. Since the proof of Lemma 3.7 is rather technical, we postpone the proof of Lemma 3.7 until Appendix A.

3.1. Proof of Theorem 3.1. In this subsection, we will prove Theorem 3.1. Before proceeding, we recall the following lemma which will be needed in the proof of Theorem 3.1.

Lemma 3.9. ([**4,** Lemma 4.8]). *Let k be a number field, and let $\mathcal{V}_1$ and $\mathcal{V}_2$ be (proper) k-varieties. Assume that there is a k-morphism $\alpha : \mathcal{V}_1 \to \mathcal{V}_2$ and that $\mathcal{V}_2(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$. Then, $\mathcal{V}_1(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$.*

Proof of Theorem 3.1. Letting

$$(u, z, v, w) = (u_1, v_1, u_2, v_2),$$

we deduce that $\mathcal{K}$ lies on the threefold $\mathcal{Y}$ in Theorem 2.1. Hence, there exists a $\mathbb{Q}$-morphism

$$\sigma : \mathcal{K} \longrightarrow \mathcal{Y},$$

and it thus follows from Theorem 2.1 (ii) and Lemma 3.9 that $\mathcal{K}$ contains no zero-cycle of odd degree over $\mathbb{Q}$ and $\mathcal{K}(\mathbb{A}_{\mathbb{Q}})^{\mathbf{Br}} = \emptyset$. Therefore, it remains to show that $\mathcal{K}$ is everywhere locally solvable. For any odd prime $l \neq p$, we see that at least one of -1, p, $-p$ is a square in $\mathbb{Q}_l^{\times}$. Hence, it suffices to consider the following cases.

$\star$ *Case* 1. l is an odd prime such that -1 is a square in $\mathbb{Q}_l^{\times}$. Define

$$\begin{aligned} K_1 : &= (x : y : z : u : v : w) \\ &= (p(p-1)\Sigma : 2\Sigma : 2\Sigma : 2p\Sigma : p\Gamma : p\Lambda\sqrt{-1}). \end{aligned}$$

We contend that K_1 lies on $\mathcal{K}$. Indeed, it is obvious that K_1 satisfies the first equation of (3.2). By (B2), we know that

$$\Gamma^2 + p\Lambda^2 = 2(p-1+8b^2)(p-1+2pd^2)\Sigma^2.$$

Hence, multiplying both sides of the above identity by p^2 and writing $p\left(p^2\Lambda^2\right)$ as $-p\left(p\Lambda\sqrt{-1}\right)^2$, we deduce that K_1 satisfies the second equation of (3.2). Furthermore, it follows from the second equation of (3.1) that K_1 satisfies the third equation of (3.2). Since -1 is a square in $\mathbb{Q}_l^{\times}$, we deduce that $K_1 \in \mathcal{K}(\mathbb{Q}_l)$, and therefore, $\mathcal{K}$ is locally solvable at l.

$\star$ *Case* 2. $l = \infty$ or l is an odd prime such that p is a square in $\mathbb{Q}_l^{\times}$. Define

$$K_2 := (x : y : z : u : v : w) = (0 : 0 : 1 : \sqrt{p} : \sqrt{p} : 1).$$

It can easily be verified that $K_2 \in \mathcal{K}(\mathbb{Q}_l)$, and hence, $\mathcal{K}$ is locally solvable at l.

$\star$ *Case* 3. l is an odd prime such that $-p$ is a square in $\mathbb{Q}_l^\times$. Define

$$\begin{aligned} K_3 :&= (x : y : z : u : v : w) \\ &= (-p(p^2+1) : 2 : 2 : 2p\sqrt{-p} : p\sqrt{\Delta} : 0), \end{aligned}$$

where Δ is defined in (B1) of Theorem 3.1. It is obvious that K_3 satisfies the first equation of (3.2). Using (B1), it can easily be checked that K_3 satisfies the second equation of (3.2). Furthermore, it follows from the third equation of (3.1) that the third equation of (3.2) holds for K_3. Since $\sqrt{-p} \in \mathbb{Q}_l^\times$, we deduce that $K_3 \in \mathcal{K}(\mathbb{Q}_l)$. It thus follows that $\mathcal{K}$ is locally solvable at l.

$\star$ *Case* 4. $l = 2$. Define

$$\begin{aligned} K_4 :&= (x : y : z : u : v : w) \\ &= (2 : 1 : \Omega : \Phi : 2\sqrt{(1+2pb^2)(2+p^2d^2)} : 0). \end{aligned}$$

Since $\Phi^2 = 4 + p\Omega^2$, it can easily be verified that K_4 lies on $\mathcal{K}$. Since b and d are odd, we know that $b^2 \equiv d^2 \equiv 1 \pmod 8$. Since $p \equiv 5 \pmod 8$, it may be deduced that

$$(1+2pb^2)(2+p^2d^2) \equiv (1+10)(2+5^2) \equiv 1 \pmod 8.$$

Hence, $\sqrt{(1+2pb^2)(2+p^2d^2)} \in \mathbb{Q}_2^\times$, and thus, $K_4 \in \mathcal{K}(\mathbb{Q}_2)$. Therefore, $\mathcal{K}$ is locally solvable at 2.

$\star$ *Case* 5. $l = p$. Since $p \equiv 5 \pmod 8$, we know that -1 is a square in $\mathbb{Q}_p^\times$. Hence, it follows from Case 1 that $\mathcal{K}$ is locally solvable at p.

Therefore, in each case, $\mathcal{K}$ is everywhere locally solvable, and thus, our contention follows. $\square$

3.2. Proof of Theorem 1.2. In this subsection, we prove Theorem 1.2 using Theorem 3.1.

Proof of Theorem 1.2. Throughout the proof, we maintain the same notation as in Theorem 3.1. Let $(p, b, d) = (5, 1, 1)$. It can easily be verified that the triple $(p, b, d) = (5, 1, 1)$ satisfies Theorem 2.1 (A1)–(A3).

We see that $\Delta = 576 = 24^2$, where Δ is defined as in Theorem 3.1 (B1). Hence, (B1) is true.

We know that the conic $\mathcal{Q}_2$ in $(B1)$ of Theorem 3.1, defined by

$$\mathcal{Q}_2 : U^2 + 5V^2 - 336T^2 = 0,$$

has a point $(\Gamma : \Lambda : \Sigma) = (52 : 8 : 3)$. Thus, (B2) holds.

Let $(L_n)_{n\geq 0}$ be the Lucas sequence, that is, $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for every $n \geq 0$. It is well known [**7,** page 61] that

$$L_{2n}^2 - 5F_{2n}^2 = 4 \tag{3.5}$$

for every nonnegative integer $n \geq 0$. Thus, the pair $(X, Y) := (L_{2n}, F_{2n})$ satisfies Pell's equation $X^2 - 5Y^2 = 4$ for every $n \geq 0$.

Let n be an arbitrary nonnegative integer. We see that the triple $(\lambda_n, \mu_n, \nu_n)$ defined by (1.1) satisfies system (3.1) in Theorem 3.1 (B3), where p, Ω, Λ, Σ are taken to be 5, F_{2n}, 8, 3, respectively. Hence, $(\lambda_n, \mu_n, \nu_n)$ satisfies (B3).

It is clear from (1.1) that λ_n, μ_n, ν_n are nonzero. Furthermore, we see that

$$\mu_n^2 - 4\lambda_n\nu_n = \frac{5625}{287296}\left(F_{2n}^2 - \frac{16}{75}F_{2n} - \frac{73417}{2025}\right)\left(F_{2n}^2 + \frac{16}{75}F_{2n} - \frac{73417}{2025}\right).$$

Since F_{2n} is a positive integer and the polynomial defined by

$$\left(X^2 - \frac{16}{75}X - \frac{73417}{2025}\right)\left(X^2 + \frac{16}{75}X - \frac{73417}{2025}\right)$$

has no zeros in $\mathbb{Q}$, we deduce that $\mu_n^2 - 4\lambda_n\nu_n$ is non-zero. Thus, (B4) is true.

Let $\mathcal{K}_n$ be the K3 surface defined by (1.2). It is easily seen that $\mathcal{K}_n$ is precisely the K3 surface defined by (3.2) with 5, 1, 1, λ_n, μ_n, ν_n in the roles of p, b, d, λ, μ, ν, respectively. It, thus, follows from Theorem 3.1 that $\mathcal{K}_n$ satisfies Theorem 1.2 (i), (ii). Therefore, our contention follows. $\square$

APPENDIX

A. Proof of Lemma 3.7.

A.1. Proof of Lemma 3.7. In this subsection, we prove Lemma 3.7. We prove that $\mathcal{K}$ is smooth if and only if

$$\prod_{i=1}^{9} C_i \neq 0,$$

where the C_i are as defined in Lemma 3.6. We use the Jacobian criterion to prove that $\mathcal{K}$ is smooth. Assume the contrary, that is, $\mathcal{K}$ is singular at some point $P := (x : y : z : u : v : w)$. Then, the Jacobian matrix of $\mathcal{K}$, defined by

(A.1)
$$\mathcal{J}_{\mathcal{K}} = \begin{pmatrix} 2y & 2x & 2pz & -2u & 0 & 0 \\ 4x + 2(p^2d^2 + 4pb^2)y & 16b^2d^2p^3y + 2(p^2d^2 + 4pb^2)x & 0 & 0 & -2v & 2pw \\ 2\lambda x + \mu y & 2\nu y + \mu x & 2z & 0 & 0 & -2w \end{pmatrix},$$

is of rank less than 3. Consider the following cases.

⋆ *Case* 1. $uv \neq 0$. We know that the matrix defined by

$$\begin{pmatrix} -2u & 0 & 2pz \\ 0 & -2v & 0 \\ 0 & 0 & 2z \end{pmatrix}$$

is of determinant $8zuv$. Since $uv \neq 0$, we deduce that $z = 0$, and it thus follows from the first equation of (3.2) that $2xy = u^2 \neq 0$. Hence, $x \neq 0$ and $y \neq 0$. Since $\text{rank}(\mathcal{J}_{\mathcal{K}}) < 3$, we know that both of the matrices defined by

$$\begin{pmatrix} -2u & 0 & 2y \\ 0 & -2v & 4x + 2(p^2d^2 + 4pb^2)y \\ 0 & 0 & 2\lambda x + \mu y \end{pmatrix}$$

and

$$\begin{pmatrix} -2u & 0 & 2x \\ 0 & -2v & 16b^2d^2p^3y + 2(p^2d^2 + 4pb^2)x \\ 0 & 0 & 2\nu y + \mu x \end{pmatrix}$$

are of determinant 0. Therefore, it follows that $2\lambda x+\mu y = 2\nu y+\mu x = 0$. Eliminating x and y in the last two equations, we deduce that

$$\mu^2 - 4\lambda\nu = 0,$$

which is a contradiction to (B4).

$\star$ *Case* 2. $u = 0$ and $v = 0$. In this case, we consider the following subcases.

• *Subcase* 1. $w = 0$. It follows from the second equation of (3.2) that $x = -4pb^2y$ or $x = -p^2d^2y$. If $x = -4pb^2y$, then it follows from the first equation of (3.2) that $pz^2 = -2xy = 8pb^2y^2$. We contend that $y \neq 0$; otherwise, $x = y = z = u = v = w = 0$, which is a contradiction. Thus, $y \neq 0$, and, with no loss of generality, we can assume that $y = 1$. Therefore, $x = -4pb^2$ and $z^2 = 8b^2$. Substituting the last two identities into the third equation of (3.2), we deduce that

$$C_1 = 16p^2b^4\lambda - 4pb^2\mu + \nu + 8b^2 = 0,$$

which is a contradiction to Lemma 3.6.

If $x = -p^2d^2y$, then, repeating the same arguments as above, we deduce that

$$C_2 = p^4d^4\lambda - p^2d^2\mu + \nu + 2pd^2 = 0,$$

which is a contradiction to Lemma 3.6.

• *Subcase* 2. $w \neq 0$. First assume that $z = 0$. By the first equation of (3.2), we see that $x = 0$ or $y = 0$. If $x = 0$, then it follows from (3.2) that $w^2 = -8b^2d^2p^2y^2$ and $w^2 = \nu y^2$. Using the last two identities, it can be shown that $yw \neq 0$. Eliminating y and w in the last two identities, we deduce that $C_3 = \nu + 8b^2d^2p^2 = 0$, which is a contradiction to Lemma 3.6. If $y = 0$, then, repeating the same arguments as above, we deduce that $C_4 = \lambda + (2/p) = 0$, which is a contradiction to Lemma 3.6.

Now suppose that $z \neq 0$. By (3.2), we know that $xy = -pz^2/2 \neq 0$.

The matrices defined by

$$\begin{pmatrix} 2pz & 0 & 2y \\ 0 & 2pw & 4x + 2(p^2d^2 + 4pb^2)y \\ 2z & -2w & 2\lambda x + \mu y \end{pmatrix}$$

and

$$\begin{pmatrix} 2pz & 0 & 2x \\ 0 & 2pw & 16b^2d^2p^3y + 2(p^2d^2 + 4pb^2)x \\ 2z & -2w & 2\nu y + \mu x \end{pmatrix}$$

are of determinants

$$4pzw(y(p\mu + 8b^2p + 2d^2p^2 - 2) + 2x(p\lambda + 2))$$

and

$$4pzw(x(p\mu + 8b^2p + 2d^2p^2 - 2) + 2py(8b^2d^2p^2 + \nu)),$$

respectively. Since P is a singular point of $\mathcal{K}$ and $zw \neq 0$, it follows that

$$\begin{aligned} y(p\mu + 8b^2p + 2d^2p^2 - 2) + 2x(p\lambda + 2) &= x(p\mu + 8b^2p + 2d^2p^2 - 2) \\ &\quad + 2py(8b^2d^2p^2 + \nu) = 0. \end{aligned}$$

Eliminating x and y in the last two equations, we deduce that

$$C_5 = (p\mu + 8b^2p + 2d^2p^2 - 2)^2 - 4p(\nu + 8b^2d^2p^2)(p\lambda + 2) = 0,$$

which is a contradiction to Lemma 3.6.

$\star$ *Case* 3. $u = 0$ and $v \neq 0$. First assume that $w \neq 0$. We know that the matrices defined by

$$\begin{pmatrix} 2x & 0 & 0 \\ 16b^2d^2p^3y + 2(p^2d^2 + 4pb^2)x & -2v & 2pw \\ 2\nu y + \mu x & 0 & -2w \end{pmatrix}$$

and

$$\begin{pmatrix} 2y & 0 & 0 \\ 4x + 2(p^2d^2 + 4pb^2)y & -2v & 2pw \\ 2\lambda x + \mu y & 0 & -2w \end{pmatrix}$$

are of determinant 0. Since $vw \neq 0$, it follows that $x = y = 0$, and it thus follows from (3.2) that $z = w = 0$, a contradiction.

Now suppose that $w = 0$. If $z = 0$, then it follows from (3.2) that $x = 0$ or $y = 0$. By the third equation of (3.2), we deduce that $x = \nu y^2 = 0$ or $y = \lambda x^2 = 0$. Therefore, $x = y = 0$, and hence, $v = 0$, a contradiction. Thus, z is nonzero, and hence, $xy \neq 0$.

The matrices defined by

$$\begin{pmatrix} 2y & 2pz & 0 \\ 4x + 2(p^2d^2 + 4pb^2)y & 0 & -2v \\ 2\lambda x + \mu y & 2z & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 2x & 2pz & 0 \\ 16b^2d^2p^3y + 2(p^2d^2 + 4pb^2)x & 0 & -2v \\ 2\nu y + \mu x & 2z & 0 \end{pmatrix}$$

are of determinants

$$4vz((2 - p\mu)y - 2p\lambda x)$$

and

$$4vz((2 - p\mu)x - 2p\nu y),$$

respectively. Since $vz \neq 0$, it follows that

$$(2 - p\mu)y - 2p\lambda x = (2 - p\mu)x - 2p\nu y = 0.$$

Eliminating x and y in the above equations, we deduce that

$$C_6 = (2 - p\mu)^2 - 4p^2\lambda\nu = 0,$$

a contradiction to Lemma 3.6.

$\star$ *Case* 4. $u \neq 0$ and $v = 0$. We know that the matrix defined by

$$\begin{pmatrix} -2u & 2pz & 0 \\ 0 & 0 & 2pw \\ 0 & 2z & -2w \end{pmatrix}$$

is of determinant $8pzuw$. Hence, $z = 0$ or $w = 0$. We consider the following subcases.

• *Subcase* 1. $z = 0$. By the first equation of (3.2), we deduce that $xy = u^2/2 \neq 0$. We contend that $w \neq 0$; otherwise, it follows from the second equation of (3.2) that $x = -4pb^2y$ or $x = -p^2d^2y$. If $x = -4pb^2y$, it then follows from the third equation of (3.2) that

$$C_7 = 16p^2b^4\lambda - 4pb^2\mu + \nu = 0,$$

a contradiction to Lemma 3.6.

If $x = -p^2d^2y$, then, using the same arguments as above, we deduce that

$$C_8 = p^4d^4\lambda - p^2d^2\mu + \nu = 0,$$

a contradiction to Lemma 3.6. Hence, $w \neq 0$. We know that the matrices defined by

$$\begin{pmatrix} -2u & 0 & 2y \\ 0 & 2pw & 4x + 2(p^2d^2 + 4pb^2)y \\ 0 & -2w & 2\lambda x + \mu y \end{pmatrix}$$

and

$$\begin{pmatrix} -2u & 0 & 2x \\ 0 & 2pw & 16b^2d^2p^3y + 2(p^2d^2 + 4pb^2)x \\ 0 & -2w & 2\nu y + \mu x \end{pmatrix}$$

are of determinants

$$-4uw\left((2p\lambda + 4)x + (p\mu + 8pb^2 + 2p^2d^2)y\right)$$

and

$$-4uw\left((p\mu + 8pb^2 + 2p^2d^2)x + (2p\nu + 16b^2d^2p^3)y\right),$$

respectively. Since $uw \neq 0$, we deduce that

$$\begin{cases} (2p\lambda + 4)x + (p\mu + 8pb^2 + 2p^2d^2)y = 0 \\ (p\mu + 8pb^2 + 2p^2d^2)x + (2p\nu + 16b^2d^2p^3)y = 0. \end{cases}$$

Eliminating x and y in the equations above, it is easily seen that

$$C_9 = p(\mu + 8b^2 + 2pd^2)^2 - 4(p\lambda + 2)(\nu + 8b^2d^2p^2) = 0,$$

a contradiction to Lemma 3.6.

• *Subcase* 2. $w = 0$. By subcase 1, it may be assumed that $z \neq 0$. We know that the matrix defined by

$$\begin{pmatrix} -2u & 2pz & 2y \\ 0 & 0 & 4x + 2(p^2d^2 + 4pb^2)y \\ 0 & 2z & 2\lambda x + \mu y \end{pmatrix}$$

is of determinant

$$8zu(2x + (p^2d^2 + 4pb^2)y).$$

It thus follows that

$$2x + (p^2d^2 + 4pb^2)y = 0. \tag{A.2}$$

On the other hand, it follows from the second equation of (3.2) that $x = -4pb^2y$ or $x = -p^2d^2y$. We contend that $xy \neq 0$; otherwise, $x = y = 0$, and hence, it follows from the third equation of (3.2) that $z = 0$, a contradiction. If $x = -4pb^2y$, then it follows from (A.2) that

$$p^2d^2 - 4pb^2 = 0.$$

Hence, it follows that

$$pq = \pm p(pd^2 - 4b^2) = \pm(p^2d^2 - 4pb^2) = 0,$$

a contradiction to (A3). If $x = -p^2d^2y$, then the same arguments as above show that $pq = 0$, a contradiction to (A3).

From these arguments, we deduce that $\mathcal{K}$ is smooth.

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