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ROUGH SINGULAR INTEGRALS ASSOCIATED TO SURFACES OF REVOLUTION ON TRIEBEL-LIZORKIN SPACES

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ABSTRACT. In this paper, we establish the boundedness of rough singular integrals associated to surfaces of revolution generated by two polynomial mappings on the Triebel-Lizorkin spaces and Besov spaces.

1. Introduction. Throughout this paper, let $n \geq 2$ and $\mathcal{A}_n$ denote the class of polynomials of n variables with real coefficients. For $N \geq 1$, let $\mathcal{A}_{n,N}$ be the collection of polynomials in $\mathcal{A}_n$ which have degrees not exceeding N, and let $V_{n,N}$ be the collection of polynomials in $\mathcal{A}_{n,N}$ which are homogeneous of degree N.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. For any nonzero vector $y \in \mathbb{R}^n$, we shall let

$$y' = \frac{y}{|y|} = (y_1', \ldots, y_n').$$

Let d, $m \geq 1$ and $\Gamma_{\Phi,\Psi} = \{(\Phi(y), \Psi(|y|)) : y \in \mathbb{R}^n\}$ be surfaces generated by two suitable mappings

$$\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^d \quad \text{and} \quad \Psi : [0, \infty) \longrightarrow \mathbb{R}^m.$$

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Let K be a kernel of Calderón-Zygmund type on $\mathbb{R}^n$, given by

$$K(y) = \frac{\Omega(y)h(|y|)}{|y|^n}, \tag{1.1}$$

where Ω is homogeneous of degree zero, integrable over S^{n-1} and satisfies

$$\int_{S^{n-1}} \Omega(u)\, d\sigma(u) = 0 \tag{1.2}$$

and $h \in \Delta_1(\mathbb{R}^+)$. Define the singular integral operator $T_{h,\Omega,\Phi,\Psi}$ in $\mathbb{R}^{d+m}$ along $\Gamma_{\Phi,\Psi}$ by

$$T_{h,\Omega,\Phi,\Psi} f(u,v) := \text{p.v.} \int_{\mathbb{R}^n} f(u - \Phi(y), v - \Psi(|y|))K(y)\, dy, \tag{1.3}$$

where $(u, v) \in \mathbb{R}^d \times \mathbb{R}^m = \mathbb{R}^{d+m}$ and $f \in \mathcal{S}(\mathbb{R}^{d+m})$ (the Schwartz class). Here, $\Delta_\gamma(\mathbb{R}^+)$, $\gamma > 0$, denotes the set of all measurable functions h defined on $\mathbb{R}^+ := (0, \infty)$, satisfying

$$\|h\|_{\Delta_\gamma(\mathbb{R}^+)} := \sup_{R>0} \left(R^{-1} \int_0^R |h(t)|^\gamma dt \right)^{1/\gamma} < \infty.$$

Clearly, $L^\infty(\mathbb{R}^+) = \Delta_\infty(\mathbb{R}^+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}^+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}^+)$ for $0 < \gamma_1 < \gamma_2 < \infty$.

In this paper, we aim to establish some new results concerning rough singular integral operators $T_{h,\Omega,\Phi,\Psi}$ associated to certain surfaces of revolution on the Triebel-Lizorkin and Besov spaces. As is well known, the Triebel-Lizorkin and Besov spaces contain many important function spaces, such as Lebesgue, Hardy, Sobolev and Lipschitz spaces.

Here, we recall some definitions. For $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, $p \neq \infty$, the homogeneous Triebel-Lizorkin spaces $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ and homogeneous Besov spaces $\dot{B}_\alpha^{p,q}(\mathbb{R}^n)$ are defined, respectively, by

$$\dot{F}_\alpha^{p,q}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} = \left\| \left(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} |\Psi_i * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty \right\} \tag{1.4}$$

and

$$\dot{B}_{\alpha}^{p,q}(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{B}_{\alpha}^{p,q}(\mathbb{R}^n)} = \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} \|\Psi_i * f\|_{L^p(\mathbb{R}^n)}^q \Big)^{1/q} < \infty \Big\}, \tag{1.5}$$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the tempered distribution class on $\mathbb{R}^n$, $\widehat{\Psi_i}(\xi) = \phi(2^i\xi)$ for $i \in \mathbb{Z}$, and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfies the conditions:

$$0 \le \phi(x) \le 1;$$
$$\operatorname{supp}(\phi) \subset \{x : 1/2 \le |x| \le 2\};$$
$$\phi(x) > c > 0 \quad \text{if } 3/5 \le |x| \le 5/3.$$

Clearly, $\dot{F}_{\alpha}^{p,p}(\mathbb{R}^n) = \dot{B}_{\alpha}^{p,p}(\mathbb{R}^n)$ for any $\alpha \in \mathbb{R}$ and $1 < p < \infty$. Moreover, it is well known that

$$\dot{F}_0^{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \tag{1.6}$$

for any $1 < p < \infty$, see **[16, 17, 28]**, etc., for additional properties of $\dot{F}_{\alpha}^{p,q}(\mathbb{R}^n)$ and $\dot{B}_{\alpha}^{p,q}(\mathbb{R}^n)$.

The nonhomogeneous versions of Triebel-Lizorkin and Besov spaces, denoted by $F_{\alpha}^{p,q}(\mathbb{R}^n)$ and $B_{\alpha}^{p,q}(\mathbb{R}^n)$, respectively, are obtained by adding the term $\|\Theta * f\|_{L^p(\mathbb{R}^n)}$ to the right hand side of (1.4) or (1.5) with $\sum_{i\in\mathbb{Z}}$ replaced by $\sum_{i\ge 1}$, where $\Theta \in \mathcal{S}(\mathbb{R}^n)$ and

$$\operatorname{supp}(\widehat{\Theta}) \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\}, \quad \widehat{\Theta}(x) > c > 0$$

if $|x| \le 5/3$. The following properties are well known, see **[16, 17]**, for example, for any $1 < p,\, q < \infty$:

$$F_{\alpha}^{p,q}(\mathbb{R}^n) \sim \dot{F}_{\alpha}^{p,q}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \tag{1.7}$$

and

$$\|f\|_{F_{\alpha}^{p,q}(\mathbb{R}^n)} \sim \|f\|_{\dot{F}_{\alpha}^{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}, \quad \alpha > 0;$$

$$B_{\alpha}^{p,q}(\mathbb{R}^n) \sim \dot{B}_{\alpha}^{p,q}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \tag{1.8}$$

and

$$\|f\|_{B_{\alpha}^{p,q}(\mathbb{R}^n)} \sim \|f\|_{\dot{B}_{\alpha}^{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}, \quad \alpha > 0.$$

When $\Psi(t) \equiv (0, \dots, 0) \in \mathbb{R}^m$, the operator $T_{h,\Omega,\Phi,\Psi}$ essentially reduces to the lower-dimensional singular integral

$$T_{h,\Omega,\Phi} f(x) := \text{p.v.} \int_{\mathbb{R}^n} f(x - \Phi(y)) K(y)\, dy, \quad x \in \mathbb{R}^d,$$

where $K(\cdot)$ is as in (1.1). When $n = d$, $h \equiv 1$ and $\Phi(y) = y$, the operator $T_{h,\Omega,\Phi}$ reduces to the classical singular integral operator denoted by T_Ω. The boundedness of T_Ω on the Triebel-Lizorkin spaces has been extensively investigated by many authors. For example, see **[8]** for the case $\Omega \in L^r(S^{n-1})$ with some $r > 1$, **[2, 9]** for the case $\Omega \in \mathcal{F}_\beta(S^{n-1})$ with some $\beta > 1$, **[5]** for the case $\Omega \in H^1(S^{n-1})$, **[20]** for the case $\Omega \in L \log^+ L(S^{n-1})$ and **[21]** for the case $\Omega \in L(\log^+ L)^\alpha(S^{n-1})$ with some $0 < \alpha < 1$. Here, $H^1(S^{n-1})$ denotes the Hardy space on the unit sphere S^{n-1} which contains $L \log^+ L(S^{n-1})$ as a proper space, see **[11, 26]**. First introduced by Grafakos and Stefanov **[18]**, $\mathcal{F}_\beta(S^{n-1})$, $\beta > 0$, denotes the set of all $L^1(S^{n-1})$ functions Ω satisfying

$$\sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')| \left(\log \frac{1}{|\xi \cdot y'|} \right)^\beta d\sigma(y') < \infty.$$

Note that

$$\bigcup_{q>1} L^q(S^{n-1}) \subsetneq \bigcap_{\beta>0} \mathcal{F}_\beta(S^{n-1})$$

and

$$\bigcap_{\beta>1} \mathcal{F}_\beta(S^{n-1}) \nsubseteq L \log^+ L(S^{n-1}).$$

Moreover,

$$\bigcap_{\beta>1} \mathcal{F}_\beta(S^{n-1}) \nsubseteq H^1(S^{n-1}) \nsubseteq \bigcup_{\beta>1} \mathcal{F}_\beta(S^{n-1});$$

$$L(\log^+ L)^{\beta_1}(S^{n-1}) \subsetneq L(\log^+ L)^{\beta_2}(S^{n-1}), \quad \text{for all } \beta_1 > \beta_2 > 0;$$

$$L(\log^+ L)^{\beta}(S^{n-1}) \nsubseteq H^1(S^{n-1}) \nsubseteq L(\log^+ L)^{\beta}(S^{n-1}), \quad \text{for all } 0 < \beta < 1.$$

When $n = d$, $m = 1$, $h \equiv 1$ and $\Phi(y) = y$, the operator $T_{h,\Omega,\Phi,\Psi}$ reduces to the classical singular integral operator associated to surfaces of revolution denoted by $T_{\Omega,\Psi}$. The L^p mapping properties of $T_{\Omega,\Psi}$ were first given by Kim, et al. [**19**] under the stronger assumption that Ψ is a convex increasing function with $\Psi(0) = 0$ and $\Omega \in \mathcal{C}^\infty(S^{n-1})$. It should be pointed out that the above result can be extended to the case $\Omega \in L^q(S^{n-1})$ for some $q > 1$ by modifying the proof of [**19,** Theorem 1], see [**27,** pages 372, 373], as well as [**7**]. Later on, the result of [**19**] was extended to the case $\Omega \in L\log^+ L(S^{n-1})$ by Al-Salman and Pan [**3**]. In 2001, Lu, Pan and Yang [**24**] extended the above results to the case $\Omega \in H^1(S^{n-1})$, see [**24,** Theorem 1] for a more general result. Moreover, it follows from [**24,** Theorem 1] that $T_{\Omega,\Psi}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$, provided that $\Omega \in H^1(S^{n-1})$ and $\Psi \in \mathcal{A}_1$. In 2002, Cheng and Pan [**10**] studied the L^p bounds for the operator $T_{\Omega,\Psi}$ with $\Omega \in \mathcal{F}_\beta(S^{n-1})$ and $\Psi \in \mathcal{A}_1$. Recently, Al-Balushi and Al-Salman [**1**] generalized the result of [**10**] and proved the following result.

Theorem A ([**1**]). *Let* $n = d$, $m = 1$, $h \equiv 1$,

$$\Phi(y) = (P_1(|y|)y'_1, \dots, P_n(|y|)y'_n) \\ \text{with } P = (P_1, \dots, P_n) \in (\mathcal{A}_1)^n$$

and $\Psi \in \mathcal{A}_1$. *Suppose that* Ω *satisfies* (1.2) *and* $\Omega \in \mathcal{F}_\beta(S^{n-1})$ *for some* $\beta > 1$. *Then,* $T_{h,\Omega,\Phi,\Psi}$ *is bounded on* $L^p(\mathbb{R}^{n+1})$ *for* $p \in (2\beta/(2\beta - 1), 2\beta)$.

A question which naturally arises is whether the condition $\Omega \in L(\log^+ L)^\alpha(S^{n-1})$ is also sufficient for the L^p boundedness of $T_{h,\Omega,\Phi,\Psi}$ with Φ, Ψ as in Theorem A. An affirmative answer is given by proving a more general result. More precisely, we shall establish the following result.

Theorem 1.1. *Let* $n = d$,

$$\Phi(y) = (P_1(|y|)y'_1, \dots, P_n(|y|)y'_n) \quad \text{with } (P_1, \dots, P_d) \in (\mathcal{A}_1)^d$$

and

$$\Psi = (Q_1, \dots, Q_m) \in (\mathcal{A}_1)^m.$$

Suppose that one of the following conditions holds:

(a) $h \in \Delta_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$ *and* $\Omega \in L\log^+ L(S^{n-1})$ *satisfying* (1.2);

(b) $h \in \mathscr{H}_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$ *and* $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$ *satisfying* (1.2), *where* $\mathscr{H}_\gamma(\mathbb{R}^+)$, $\gamma > 0$ *is the set of all measurable functions* $h : \mathbb{R}^+ \to \mathbb{C}$ *satisfying*

$$\|h\|_{\mathscr{H}_\gamma(\mathbb{R}^+)} := \left(\int_0^\infty |h(t)|^\gamma \frac{dt}{t}\right)^{1/\gamma} < \infty.$$

Then,

(i) $T_{h,\Omega,\Phi,\Psi}$ *is bounded on* $\dot{F}_\alpha^{p,q}(\mathbb{R}^{d+m})$ *for* $\alpha \in \mathbb{R}$ *and* $(1/p, 1/q) \in \mathscr{R}_\gamma$, *where* $\mathscr{R}_\gamma$ *is the set of all interiors of the convex hull of three squares*

$$\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{\max\{2,\gamma'\}}\right)^2, \qquad \left(\frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2}\right)^2$$

and

$$\left(\frac{1}{2\gamma}, 1 - \frac{1}{2\gamma}\right)^2.$$

(ii) $T_{h,\Omega,\Phi,\Psi}$ *is bounded on* $\dot{B}_\alpha^{p,q}(\mathbb{R}^{d+m})$ *for* $\alpha \in \mathbb{R}$, $1 < q < \infty$ *and*

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \min\left\{\frac{1}{2}, \frac{1}{\gamma'}\right\}.$$

See Figures 1–3 for $\mathscr{R}_\gamma$.

Here,

$$P_1 = \left(\frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}\right),$$
$$P_2 = \left(\frac{1}{2}, \frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}\right),$$
$$P_3 = \left(\frac{1}{2} + \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2}\right),$$
$$P_4 = \left(\frac{1}{2} + \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2} + \frac{1}{\max\{2,\gamma'\}}\right),$$

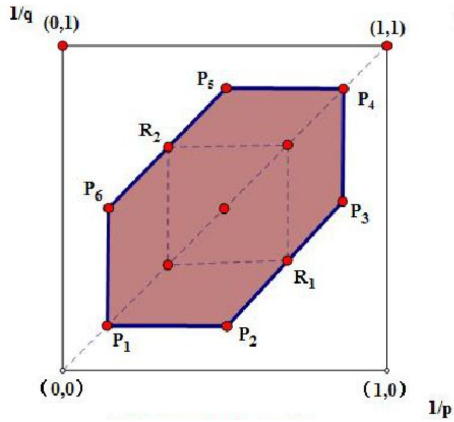

FIGURE 1. $(1 < \gamma \leq 2)$.

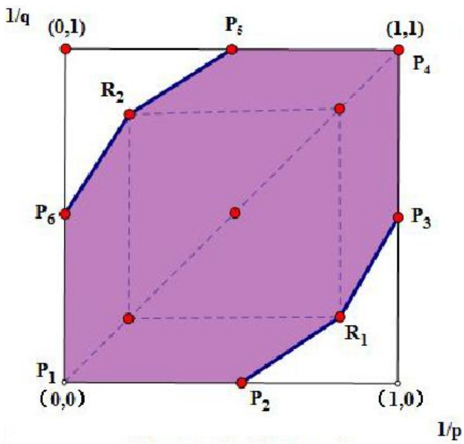

FIGURE 2. $(2 < \gamma < \infty)$.

$$P_5 = \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{\max\{2, \gamma'\}}\right), \qquad P_6 = \left(\frac{1}{2} - \frac{1}{\max\{2, \gamma'\}}, \frac{1}{2}\right),$$
$$Q_1 = (0,0), \qquad Q_2 = (1,0), \qquad Q_3 = (1,1) \quad \text{and} \quad Q_4 = (0,1)$$
$$R_1 = \left(1 - \frac{1}{2\gamma}, \frac{1}{2\gamma}\right), \qquad R_2 = \left(\frac{1}{2\gamma}, 1 - \frac{1}{2\gamma}\right).$$

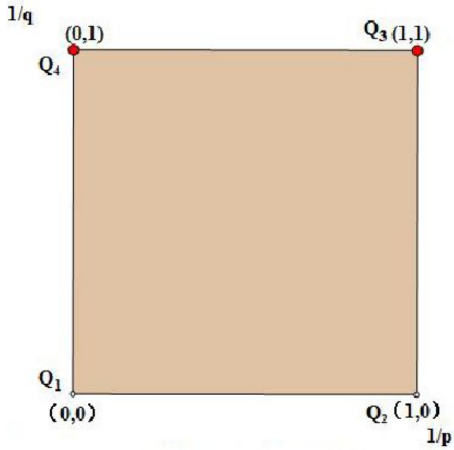

FIGURE 3. ($\gamma = \infty$).

Remark 1.2. Note that $\mathscr{H}_\gamma(\mathbb{R}^+) \subsetneq \Delta_\gamma(\mathbb{R}^+)$ for $0 < \gamma < \infty$ and $\mathscr{H}_\infty(\mathbb{R}^+) = L^\infty(\mathbb{R}^+)$. From Figures 1–3, we see that $\mathscr{R}_{\lambda_1} \subsetneq \mathscr{R}_{\lambda_2}$ for $\lambda_2 > \lambda_1 > 1$ and $\mathscr{R}_\infty = (0,1)^2$.

When $\Psi(t) \equiv (0,\dots,0) \in \mathbb{R}^m$ and $\Phi(y) = \mathcal{P}(y) \in (\mathcal{A}_n)^d$, the operator $T_{h,\Omega,\Phi,\Psi}$ essentially reduces to the class of singular radon transforms $T_{h,\Omega,\mathcal{P}}$, defined by

$$T_{h,\Omega,\mathcal{P}}f(x) := \text{p.v.}\int_{\mathbb{R}^n} f(x-\mathcal{P}(y))K(y)\,dy,$$

where $x \in \mathbb{R}^d$ and $K(\cdot)$ is as in (1.1).

The investigation of boundedness of $T_{h,\Omega,\mathcal{P}}$ on Triebel-Lizorkin spaces has attracted the attention of many authors. For relevant results, one may consult **[6, 22, 23]**.

We now give the results of **[6, 22]** as follows:

Theorem B. *Let* $\mathcal{P} = (P_1,\dots,P_d) \in (\mathcal{A}_n)^d$ *and* Ω *satisfy* (1.2). *Suppose that one of the following conditions holds*:

(i) $h \in \Delta_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$ *and* $\Omega \in H^1(S^{n-1})$ **[6]**;

(ii) $h \in \mathscr{H}_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$ *and* $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$ **[22]**.

Then, $T_{h,\Omega,\mathcal{P}}$ *is bounded on* $\dot{F}^{p,q}_\alpha(\mathbb{R}^d)$ *for* $\alpha \in \mathbb{R}$ *and* $\max\{|1/p - 1/2|, |1/q - 1/2|\} < \min\{1/2, 1/\gamma'\}$.

Remark 1.3. There is a gap in Theorem B, see [**29,** Remark 1]. That proof works in the same region as in our main theorems below.

The remainder of the results of this paper may be stated as follows.

Theorem 1.4. *Let* $\Phi = (P_1, \ldots, P_d) \in (\mathcal{A}_n)^d$ *and* $\Psi = (Q_1, \ldots, Q_m) \in (\mathcal{A}_1)^m$. *Suppose that* $h \in \Delta_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$ *and* $\Omega \in H^1(S^{n-1})$ *satisfying* (1.2). *Then,*

(i) $T_{h,\Omega,\Phi,\Psi}$ *is bounded on* $\dot{F}^{p,q}_\alpha(\mathbb{R}^{d+m})$ *for* $\alpha \in \mathbb{R}$ *and* $(1/p, 1/q) \in \mathscr{R}_\gamma$, *where* $\mathscr{R}_\gamma$ *is as in Theorem* 1.1.

(ii) $T_{h,\Omega,\Phi,\Psi}$ *is bounded on* $\dot{B}^{p,q}_\alpha(\mathbb{R}^{d+m})$ *for* $\alpha \in \mathbb{R}$, $1 < q < \infty$, *and* $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.

Theorem 1.5. *Let* Φ, Ψ *and* $\mathscr{R}_\gamma$ *be as in Theorem* 1.4. *Suppose that* $h \in \mathscr{H}_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$ *and* $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$ *satisfying* (1.2). *Then,*

(i) $T_{h,\Omega,\Phi,\Psi}$ *is bounded on* $\dot{F}^{p,q}_\alpha(\mathbb{R}^{d+m})$ *for* $\alpha \in \mathbb{R}$ *and* $(1/p, 1/q) \in \mathscr{R}_\gamma$.

(ii) $T_{h,\Omega,\Phi,\Psi}$ *is bounded on* $\dot{B}^{p,q}_\alpha(\mathbb{R}^{d+m})$ *for* $\alpha \in \mathbb{R}$, $1 < q < \infty$, *and* $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.

Remark 1.6.

(i) It should be pointed out that the range of (p, q) in our main results was first given by Yabuta [**29**];

(ii) It follows from our main results and (1.6) that, under the assumptions on h, Ω, Φ and Ψ in Theorems 1.1 and 1.4–1.5, the operators $T_{h,\Omega,\Phi,\Psi}$ are bounded on $L^p(\mathbb{R}^{d+m})$ for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, which are new;

(iii) Theorems 1.1 and 1.3 do not hold when replacing $h \in \mathscr{H}_\gamma(\mathbb{R}^+)$ by $h \in \Delta_\gamma(\mathbb{R}^+)$ for $\gamma > 1$ due to $L^\infty(\mathbb{R}^+) \subset \Delta_\gamma(\mathbb{R}^+)$, $L \log^+ L(S^{n-1}) \subsetneq L(\log^+ L)^\alpha(S^{n-1})$ for any $0 < \alpha < 1$, and Calderón and Zygmund's well-known result [**4**];

(iv) Our main results are new, even in the special cases $n = d$, $\Phi(y) = y$ or $m = 1$ and $\Psi(|y|) = |y|$.

Corollary 1.7 immediately follows from (1.6)–(1.8) and Theorems 1.1 and 1.4–1.5.

Corollary 1.7. *Under the same conditions as Theorems* 1.1 *and* 1.4–1.5 *with* $\alpha > 0$*, these operators are bounded on* $F_\alpha^{p,q}(\mathbb{R}^{d+m})$ *and* $B_\alpha^{p,q}(\mathbb{R}^{d+m})$.

The paper is organized as follows. Section 2 is devoted to presenting some auxiliary lemmas. In Section 3, we shall prove Theorems 1.1 and 1.5. The proof of Theorem 1.4 will be given in Section 4. Finally, we shall give some further results in Section 5. We remark that the main method employed in this paper is a combination of ideas and arguments from [**6, 13, 14, 23**] (although we use the standard methods of Fourier transform and Littlewood-Paley theory, they are non-trivial). We would like to point out that our proofs have two main ingredients:

(i) a criterion of boundedness for the operators of convolution type on Triebel-Lizorkin spaces, see Lemma 2.5;
(ii) a refined estimate of vector-valued inequality, see Lemma 2.4, which is a major factor in obtaining our main results.

Throughout the paper, we denote p' by the conjugate index of p, which satisfies $1/p + 1/p' = 1$. The letters C or c, sometimes with certain parameters, will stand for positive constants, not necessarily the same at each occurrence, but independent of the essential variables. We shall use $\delta_{\mathbb{R}^n}$ to denote the Dirac delta function on $\mathbb{R}^n$; J^{-1} denotes the inverse transform of linear transformation J; D^t denotes the transpose of the linear transformation D and π_n^d denotes a projection operator from $\mathbb{R}^d$ to $\mathbb{R}^n$ when $n \le d$. In what follows, we set

$$\sum_{j\in\emptyset} a_j = 0 \quad \text{and} \quad \prod_{j\in\emptyset} a_j = 1.$$

2. Preliminary lemmas. We begin with the following lemma of van der Corput type, which was proven by Ricci and Stein [**25**].

Lemma 2.1. ([**25,** page 186, Corollary]). *Let* $l \in \mathbb{N} \setminus \{0\}$, $\mu_1, \dots, \mu_l \in \mathbb{R}$, $d_1, \dots, d_l$, *be distinct positive real numbers. Let* $\psi \in \mathcal{C}^1([0,1])$. *Then there exists a* $C > 0$ *independent of* $\{\mu_j\}_{j=1}^l$ *such that*

$$\left| \int_\delta^\tau e^{i(\mu_1 t^{d_1} + \ldots + \mu_l t^{d_l})} \psi(t)\, dt \right| \leq C|\mu_1|^{-\epsilon} \left(|\psi(\tau)| + \int_\delta^\tau |\psi'(t)|\, dt \right)$$

holds for $0 \leq \delta < \tau \leq 1$ *and* $\epsilon = \min\{1/d_1, 1/l\}$.

Next, we recall two important vector-valued norm inequalities.

Lemma 2.2. ([**22,** Proposition 2.3]). *Let* $\{a_k\}_{k\in\mathbb{Z}}$ *be a lacunary sequence of positive numbers with the property*

$$\inf_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} \geq a > 1.$$

Let $0 < M \leq N$ *and*

$$H : \mathbb{R}^M \longrightarrow \mathbb{R}^M \quad \textit{and} \quad G : \mathbb{R}^N \longrightarrow \mathbb{R}^N$$

be two nonsingular linear transformations. Let $\Upsilon(\xi) \in \mathcal{S}(\mathbb{R}^M)$ *with* $\widehat{\Upsilon}(0) = 0$ *and* $\Upsilon_k(\xi) = a_k^{-M}\Upsilon(\xi/a_k)$. *Define the transformations* J *and* X_k *by*

$$J(f)(x) = f(G^t(H^t \otimes \mathrm{id}_{\mathbb{R}^{N-M}})x)$$

and

$$X_k(f)(x) = J^{-1}(\Upsilon_k \otimes \delta_{\mathbb{R}^{N-M}}) * J(f))(x).$$

Then, for $1 < p, q < \infty$, $\{g_j\}_{j\in\mathbb{Z}} \in L^p(\mathbb{R}^N, \ell^q)$ *and* $\{g_{k,j}\}_{k,\,j\in\mathbb{Z}} \in L^p(\mathbb{R}^N, \ell^q(\ell^2))$, *there exists a* $C_{M,a} > 0$ *such that*

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |X_k(g_j)|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^N)} \leq C_{M,a} \left\| \left(\sum_{j\in\mathbb{Z}} |g_j|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^N)},$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|X_k(g_{k,j})|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^N)} \leq C_{M,a}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^N)}.$$

Lemma 2.3. ([**6,** Theorem 1.4]). *Let $\mathcal{P}=(P_1,\dots,P_d)\in(\mathcal{A}_n)^d$. Then, for $1<p,\, q<\infty$, the operator $\mathcal{M}_{\mathcal{P}}$ given by*

$$\mathcal{M}_{\mathcal{P}}f(x)=\sup_{r>0}\frac{1}{r^n}\int_{|y|\leq r}|f(x-\mathcal{P}(y))|\,dy$$

satisfies the following $L^p(\mathbb{R}^d,\ell^q)$ inequality

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|\mathcal{M}_{\mathcal{P}}f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\leq C_{p,q}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)},$$

where $C_{p,q}$ is independent of the coefficients of P_j for all $1\leq j\leq d$.

In what follows, for any $\mu\in\mathbb{N}$, we set

$$|||h|||_{\mu,\gamma}=\sup_{k\in\mathbb{Z}}\Big(\int_{2^{(\mu+1)k}}^{2^{(\mu+1)(k+1)}}|h(t)|^\gamma\frac{dt}{t}\Big)^{1/\gamma},\quad \gamma>1. \tag{2.1}$$

For a suitable mapping $\Gamma:\mathbb{R}^n\to\mathbb{R}^d$, define the sequence of measures $\{\sigma_{k,\mu,\Gamma,\Omega}\}_{k\in\mathbb{Z}}$ by

$$\int_{\mathbb{R}^d}f\,d\sigma_{k,\mu,\Gamma,\Omega}=\int_{D_{\mu,k}}f(\Gamma(x))K(x)\,dx, \tag{2.2}$$

where $K(\cdot)$ is as in (1.1) and

$$D_{\mu,k}=\{x\in\mathbb{R}^n:2^{(\mu+1)k}\leq|x|<2^{(\mu+1)(k+1)}\}.$$

Next is a crucial lemma, which will play a key role in the proofs of the main results.

Lemma 2.4. *Let*

$$\Gamma(y) = \left(P_1(|y|)a_1\left(\frac{y}{|y|}\right), \dots, P_d(|y|)a_d\left(\frac{y}{|y|}\right)\right),$$

where $(P_1, \dots, P_d) \in (\mathcal{A}_1)^d$ *and* $a_1, \dots, a_m$ *are arbitrary functions defined on* S^{n-1}. *Suppose that* $\Omega \in L^1(S^{n-1})$ *and* $\||h\||_{\mu,\gamma} < \infty$ *for some* $\mu \in \mathbb{N}$ *and* $\gamma > 1$. *If* $(1/p, 1/q)$ *belongs to the interior of the convex hull of three squares*

$$\left(\frac{1}{2}, \frac{1}{2}+\frac{1}{\max\{2,\gamma'\}}\right)^2, \quad \left(\frac{1}{2}-\frac{1}{\max\{2,\gamma'\}}, \frac{1}{2}\right)^2 \quad \text{and} \quad \left(\frac{1}{2\gamma}, 1-\frac{1}{2\gamma}\right)^2,$$

then, for arbitrary functions $\{g_{k,j}\}_{k,j\in\mathbb{Z}} \in L^p(\mathbb{R}^d, \ell^q(\ell^2))$, *there exists a* $C > 0$ *independent of* μ *and* γ *such that*

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |\sigma_{k,\mu,\Gamma,\Omega} * g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \leq CA_{\mu,\gamma} \left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)}, \tag{2.3}$$

where $A_{\mu,\gamma} = (\mu+1)^{1/\gamma'} \||h\||_{\mu,\gamma} \|\Omega\|_{L^1(S^{n-1})}$.

Proof. We shall prove the lemma by considering the following two cases.

Case 1. $1 < \gamma \leq 2$. We begin by proving (2.3) for $2 < p,\ q < 2\gamma/(2-\gamma)$. By duality, there exists an $\{f_j\}_{j\in\mathbb{Z}}$ satisfying

$$\|\{f_j\}\|_{L^{(p/2)'}(\mathbb{R}^d, \ell^{(q/2)'})} \leq 1$$

such that

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |\sigma_{k,\mu,\Gamma,\Omega} * g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\sigma_{k,\mu,\Gamma,\Omega} * g_{k,j}(x)|^2 f_j(x)\, dx. \tag{2.4}$$

By a similar argument as in **[14,** (7.7)], we have

$$\int_{\mathbb{R}^d} |\sigma_{k,\mu,\Gamma,\Omega} * g_{k,j}(x)|^2 f_j(x)\,dx \leq C\|\Omega\|_{L^1(S^{n-1})}\||h|\|_{\mu,\gamma}^{\gamma} \int_{\mathbb{R}^d} |g_{k,j}(x)|^2 \mathcal{H}_\Gamma(f_j)(x)\,dx, \tag{2.5}$$

where

$$\mathcal{H}_\Gamma(f_j)(x) = \sup_{k\in\mathbb{Z}} \int_{2^{(\mu+1)k}}^{2^{(\mu+1)(k+1)}} \int_{S^{n-1}} |f_j(x+\Gamma(ty'))||\Omega(y')|\,d\sigma(y')|h(t)|^{2-\gamma}\frac{dt}{t}.$$

By Hölder's inequality, we have

$$\begin{aligned}
&\mathcal{H}_\Gamma(f_j)(x) \\
&\leq \||h|\|_{\mu,\gamma}^{2-\gamma} \int_{S^{n-1}} \Bigg(\sup_{k\in\mathbb{Z}} \int_{2^{(\mu+1)k}}^{2^{(\mu+1)(k+1)}} |f_j(x+\Gamma(ty'))|^{\gamma'/2}\frac{dt}{t}\Bigg)^{2/\gamma'} |\Omega(y')|d\sigma(y') \\
&\leq \||h|\|_{\mu,\gamma}^{2-\gamma} \int_{S^{n-1}} \Bigg(\sum_{i=0}^{\mu} \sup_{k\in\mathbb{Z}} \int_{2^{(\mu+1)k+i}}^{2^{(\mu+1)k+i+1}} |f_j(x+\Gamma(ty'))|^{\gamma'/2}\,\frac{dt}{t}\Bigg)^{2/\gamma'} |\Omega(y')|d\sigma(y') \\
&\leq (\mu+1)^{2/\gamma'}\||h|\|_{\mu,\gamma}^{2-\gamma} \int_{S^{n-1}} |\Omega(y')| \\
&\quad \cdot \Bigg(\sup_{r>0}\frac{1}{r}\int_{|t|\leq r} |f_j(x+\Gamma(ty'))|^{\gamma'/2}dt\Bigg)^{2/\gamma'} d\sigma(y').
\end{aligned}$$

Invoking Lemma 2.3 and Minkowski's inequality, we have, for $\gamma'/2 < u, v < \infty$,

$$\begin{aligned}
\Bigg\|\Bigg(\sum_{j\in\mathbb{Z}} |\mathcal{H}_\Gamma(f_j)|^v\Bigg)^{1/v}\Bigg\|_{L^u(\mathbb{R}^d)} &\leq (\mu+1)^{2/\gamma'}\||h|\|_{\mu,\gamma}^{2-\gamma} \\
&\quad \cdot \|\Omega\|_{L^1(S^{n-1})} \Bigg\|\Bigg(\sum_{j\in\mathbb{Z}} |f_j|^v\Bigg)^{1/v}\Bigg\|_{L^u(\mathbb{R}^d)}.
\end{aligned} \tag{2.6}$$

It follows from (2.4)–(2.6) that

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}*g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|^2_{L^p(\mathbb{R}^d)}\\
&\quad\le C\|\Omega\|_{L^1(S^{n-1})}|||h|||^{\gamma}_{\mu,\gamma}\int_{\mathbb{R}^d}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|g_{k,j}(x)|^2\mathcal{H}_\Gamma(f_j)(x)\,dx\\
&\quad\le C\|\Omega\|_{L^1(S^{n-1})}|||h|||^{\gamma}_{\mu,\gamma}\Big\|\Big(\sum_{j\in\mathbb{Z}}|\mathcal{H}_\Gamma(f_j)|^v\Big)^{1/v}\Big\|_{L^u(\mathbb{R}^d)}\\
&\qquad\cdot\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|^2_{L^p(\mathbb{R}^d)}\\
&\quad\le C(\mu+1)^{2/\gamma'}\|\Omega\|^2_{L^1(S^{n-1})}|||h|||^2_{\mu,\gamma}\\
&\qquad\cdot\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|^2_{L^p(\mathbb{R}^d)},
\end{aligned}$$

where we take $u=(p/2)'$ and $v=(q/2)'$. Thus, (2.3) holds for $2<p,\ q<2\gamma/(2-\gamma)$. We also obtain (2.3) for $2\gamma/(3\gamma-2)<p$, $q<2$, by the duality. Interpolating these two cases, we obtain (2.3) for $(1/p,1/q)$ belonging to the interior of the convex hull of two squares $(1/2-1/\gamma',1/2)^2$ and $(1/2,1/2+1/\gamma')^2$. Note that, in this case, the interior of the square $(1/2\gamma,1-1/2\gamma)^2$ is contained in the interior of the convex hull of two squares $(1/2-1/\gamma',1/2)^2$ and $(1/2,1/2+1/\gamma')^2$.

Case 2. $\gamma>2$. Since $|||h|||_{\mu,2}\le(\mu+1)^{1/2-1/\gamma}|||h|||_{\mu,\gamma}$ for $\gamma>2$, we can get (2.3) for $(1/p,1/q)$ belonging to the interior of the convex hull of two squares $(0,1/2)^2$ and $(1/2,1)^2$.

Below, we shall prove (2.3) for $(1/p,1/q)$ belonging to the interior of the square $(1/2\gamma,1-(1/2\gamma))^2$. For convenience, we define the measure $|\sigma_{k,\mu,\Gamma,\Omega}|$ in the same way as $\sigma_{k,\mu,\Gamma,\Omega}$, but with Ω and h replaced by $|\Omega|$ and $|h|$, respectively, for any arbitrary functions $\{g_j\}_{j\in\mathbb{Z}}\in L^p(\mathbb{R}^d,\ell^q)$ with $p,\ q>\gamma'$. By a change of variables, and Hölder's inequality,

$$\begin{aligned}
&|\sigma_{k,\mu,\Gamma,\Omega}|*|g_j|(x)\\
&\quad\le\int_{2^{(\mu+1)k}}^{2^{(\mu+1)(k+1)}}\int_{S^{n-1}}|g_j(x-\Gamma(ty'))||\Omega(y')|\,d\sigma(y')|h(t)|\,\frac{dt}{t}
\end{aligned}$$

$$\begin{aligned}
&\le \||h\||_{\mu,\gamma}\Big(\int_{2^{(\mu+1)k}}^{2^{(\mu+1)(k+1)}}\Big|\int_{S^{n-1}}|g_j(x-\Gamma(ty'))||\Omega(y')|\,d\sigma(y')\Big|^{\gamma'}\frac{dt}{t}\Big)^{1/\gamma'}\\
&\le \||h\||_{\mu,\gamma}\|\Omega\|_{L^1(S^{n-1})}^{1/\gamma}\\
&\quad\times\Big(\int_{S^{n-1}}\int_{2^{(\mu+1)k}}^{2^{(\mu+1)(k+1)}}|g_j(x-\Gamma(ty'))|^{\gamma'}\frac{dt}{t}|\Omega(y')|\,d\sigma(y')\Big)^{1/\gamma'}\\
&\le (\mu+1)^{1/\gamma'}\||h\||_{\mu,\gamma}\|\Omega\|_{L^1(S^{n-1})}^{1/\gamma}\\
&\quad\times\Big(\int_{S^{n-1}}\sup_{r>0}\frac{1}{r}\int_{|t|\le r}|g_j(x-\Gamma(ty'))|^{\gamma'}dt|\Omega(y')|\,d\sigma(y')\Big)^{1/\gamma'},
\end{aligned}$$

which, combining Minkowski's inequality with Lemma 2.3, implies that, for any $p, q > \gamma'$,

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sup_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}|*|g_j|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \le C_{p,q}A_{\mu,\gamma}\Big\|\Big(\sum_{j\in\mathbb{Z}}|g_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}. \tag{2.7}$$

It follows that

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sup_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}*g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\quad\le\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sup_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}|*\sup_{k\in\mathbb{Z}}|g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\quad\le C_{p,q}A_{\mu,\gamma}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sup_{k\in\mathbb{Z}}|g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}
\end{aligned}\tag{2.8}$$

for any $p, q > \gamma'$. On the other hand, for any $1 < p, q < \gamma$, then $p', q' > \gamma'$. By the dual argument, there exists $\{h_j\}_{j\in\mathbb{Z}} \in L^{p'}(\mathbb{R}^d, \ell^{q'})$

with $\|\{h_j\}\|_{L^{p'}(\mathbb{R}^d,\ell^{q'})} = 1$ such that

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega} * g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\quad = \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega} * g_{k,j}(x)|h_j(x)\,dx\\
&\quad \le \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}|g_{k,j}(x)||\sigma_{k,\mu,\Gamma,\Omega}| * |\widetilde{h_j}|(-x)\,dx\\
&\quad \le \Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\qquad \times \Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sup_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}| * |\widetilde{h_j}|\Big)^{q'}\Big)^{1/q'}\Big\|_{L^{p'}(\mathbb{R}^d)},
\end{aligned}$$

where $\widetilde{h_j}(x) = h_j(-x)$. Combining this with (2.7) implies that

$$\begin{aligned}
(2.9)\quad &\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega} * g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\qquad \le C_{p,q}A_{\mu,\gamma}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}
\end{aligned}$$

for any $\{g_{k,j}\}_{k,\,j\in\mathbb{Z}} \in L^p(\mathbb{R}^m,\ell^q(\ell^\infty))$ with $1 < p,\, q < \gamma$. Interpolation between (2.8) and (2.9) yields (2.3) for $(1/p, 1/q)$ belonging to the interior of the square $(1/2\gamma, 1-(1/2\gamma))^2$. Again, by interpolation, we obtain (2.3) for the case $\gamma > 2$ and complete the proof of Lemma 2.4. □

A criterion on the bounds of the convolution operators in Triebel-Lizorkin spaces is now given, which is the heart of our proofs.

Lemma 2.5. *Let* Λ, $v \in \mathbb{N}\setminus\{0\}$. *For* $1 \le s \le \Lambda$, *let* $\{a_{k,s,v}\}_{k\in\mathbb{Z}}$ *be a lacunary sequence of positive numbers with the property*:

$$\inf_{k\in\mathbb{Z}}\frac{a_{k+1,s,v}}{a_{k,s,v}} \ge \eta_s^v \quad \textit{for some } \eta_s > 1.$$

For $1 \le s \le \Lambda$, let $\delta_s > 0$, $\ell_s \in \mathbb{N} \setminus \{0\}$ and $L_s : \mathbb{R}^d \to \mathbb{R}^{\ell_s}$ be linear transformations. Let $\{\sigma_{s,k} : 0 \le s \le \Lambda \text{ and } k \in \mathbb{Z}\}$ be a family of measures on $\mathbb{R}^d$ with $\sigma_{0,k} = 0$ for every $k \in \mathbb{Z}$. Suppose that some p_0, $q_0 > 1$ exist satisfying $(p_0, q_0) \ne (2,2)$ and c, $B > 0$ independent of v and $\{L_s\}_{s=1}^{\Lambda}$ such that the following conditions are satisfied for any $1 \le s \le \Lambda$, $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^d$ and $\{g_{k,j}\} \in L^{p_0}(\mathbb{R}^d, \ell^{q_0}(\ell^2))$:

(i) $|\widehat{\sigma_{s,k}}(\xi)| \le cB \min\{1, |a_{k,s,v} L_s(\xi)|^{-\delta_s/v}\}$;
(ii) $|\widehat{\sigma_{s,k}}(\xi) - \widehat{\sigma_{s-1,k}}(\xi)| \le cB |a_{k,s,v} L_s(\xi)|^{1/v}$;
(iii)

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} |\sigma_{s,k} * g_{k,j}|^2 \Big)^{q_0/2} \Big)^{1/q_0} \Big\|_{L^{p_0}(\mathbb{R}^d)} \le cB \Big\| \Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} |g_{k,j}|^2 \Big)^{q_0/2} \Big)^{1/q_0} \Big\|_{L^{p_0}(\mathbb{R}^d)}.$$

Then, for $\alpha \in \mathbb{R}$ and $(1/p, 1/q) \in A_1A_2 \backslash \{(1/p_0, 1/q_0), (1/2, 1/2)\}$, there exists a constant $C > 0$ independent of v and $\{L_s\}_{s=1}^{l}$ such that

$$\Big\| \sum_{k\in\mathbb{Z}} \sigma_{\Lambda,k} * f \Big\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \le CB \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)}, \tag{2.10}$$

where $A_1 = (1/2, 1/2)$, $A_2 = (1/p_0, 1/q_0)$ and A_1A_2 is the line segment from A_1 to A_2.

Proof. For any $1 \le s \le \Lambda$, let $r_s = \operatorname{rank}(L_s)$. By [**14,** Lemma 6.1], there exist two nonsingular linear transformations $H_s : \mathbb{R}^{r_s} \to \mathbb{R}^{r_s}$ and $G_s : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$|H_s \pi_{r_s}^d G_s \xi| \le |L_s(\xi)| \le \ell_s |H_s \pi_{r_s}^d G_s \xi|. \tag{2.11}$$

Let $\zeta \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that $\zeta(t) \equiv 1$ for $|t| \le 1/2$ and $\zeta(t) \equiv 0$ for $|t| \ge 1$. Let $\vartheta(t) = \zeta(t^2)$. For $k \in \mathbb{Z}$ and $1 \le s \le \Lambda$, define the family of measures $\{\mu_{s,k}\}$ by

$$\widehat{\mu_{s,k}}(\xi) = \widehat{\sigma_{s,k}}(\xi) \prod_{j=s+1}^{\Lambda} \vartheta(|a_{k,j,v} H_j \pi_{r_j}^d G_j \xi|) - \widehat{\sigma_{s-1,k}}(\xi) \prod_{j=s}^{\Lambda} \vartheta(|a_{k,j,v} H_j \pi_{r_j}^d G_j \xi|).$$

It can easily be verified that

$$\sigma_{\Lambda,k} = \sum_{s=1}^{\Lambda} \mu_{s,k}, \tag{2.12}$$

$$|\widehat{\mu_{s,k}}(\xi)| \le CB \min\{1, |a_{k,s,v} L_s(\xi)|^{1/v}\}, \tag{2.13}$$

$$|\widehat{\mu_{s,k}}(\xi)| \le CB |a_{k,s,v} L_s(\xi)|^{-\delta_s/v}, \quad \text{if } |a_{k,v,s} H_s \pi_{r_s}^d G_s \xi| \ge 1. \tag{2.14}$$

From (2.12), we can write

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \sigma_{\Lambda,k} * f &= \sum_{k\in\mathbb{Z}} \sum_{s=1}^{\Lambda} \mu_{s,k} * f \\ &= \sum_{s=1}^{\Lambda} \sum_{k\in\mathbb{Z}} \mu_{s,k} * f \\ &=: \sum_{s=1}^{\Lambda} \mathcal{A}_s(f). \end{aligned} \tag{2.15}$$

Thus, to prove Lemma 2.5, it suffices to prove that, for $1 \le s \le \Lambda$, there exists a $C > 0$ independent of v and $\{L_s\}_{s=1}^{\Lambda}$ such that

$$\|\mathcal{A}_s(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \le CB \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \tag{2.16}$$

for $\alpha \in \mathbb{R}$ and p, q satisfying the conditions in Lemma 2.5.

Let $\varsigma \in \mathcal{S}(\mathbb{R}^+)$ be such that

$$\varsigma(0) = 0, \quad 0 \le \varsigma(t) \le 1;$$
$$\operatorname{supp}(\varsigma) \subset [\eta_s^{-v\gamma_s}, \eta_s^{v\gamma_s}];$$
$$\sum_{k\in\mathbb{Z}} \varsigma_k^2(t) = 1,$$

where $\varsigma_k(t)=\varsigma(a_{k,s,v}t)$. Define the family of operators $\{S_{k,s}\}_{k\in\mathbb{Z}}$ by

$$\widehat{S_{k,s}f}(\xi)=:\varsigma_k(|H_s\pi^d_{r_s}G_s\xi|)\widehat{f}(\xi). \tag{2.17}$$

We can write

$$\begin{aligned}\mathcal{A}_s(f)&=\sum_{k\in\mathbb{Z}}\mu_{s,k}*\Big(\sum_{j\in\mathbb{Z}}S_{j+k,s}S_{j+k,s}f\Big)\\&=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}S_{j+k,s}(\mu_{s,k}*S_{j+k,s}f)\\&=:\sum_{j\in\mathbb{Z}}\mathcal{A}_{s,j}(f).\end{aligned} \tag{2.18}$$

By the Littlewood-Paley theory, Plancherel's theorem and (2.13)–(2.14),

$$\begin{aligned}&\|\mathcal{A}_{s,j}(f)\|_{L^2(\mathbb{R}^d)}\\&\quad\le CB\Big(\sum_{k\in\mathbb{Z}}\int_{\{\xi\in\mathbb{R}^d:a^{-1}_{k+j,s,v}\eta_s^{-v\gamma_s}\le|H_s\pi^d_{r_s}G_s\xi|\le a^{-1}_{k+j,s,v}\eta_s^{v\gamma_s}\}}|\widehat{\mu_{s,k}}(\xi)|^2|\widehat{f}(\xi)|^2d\xi\Big)^{1/2}\\&\quad\le CB\eta_s^{-c|j|}\|f\|_{L^2(\mathbb{R}^d)},\end{aligned} \tag{2.19}$$

where $c>0$ is independent of v. In view of (1.6) and (2.19), we have

$$\|\mathcal{A}_{s,j}(f)\|_{\dot{F}_0^{2,2}(\mathbb{R}^d)}\le CA\eta_s^{-c|j|}\|f\|_{\dot{F}_0^{2,2}(\mathbb{R}^d)}. \tag{2.20}$$

Now, we estimate $\|\mathcal{A}_{s,j}(f)\|_{\dot{F}_\alpha^{p_0,q_0}(\mathbb{R}^d)}$ for any $\alpha\in\mathbb{R}$. Let $\xi=(\xi^1,\xi^2)$ with $\xi^1=(\xi_1,\dots,\xi_{r_s})$ and $\xi^2=(\xi_{r_s+1},\dots,\xi_d)$. We set

$$\widehat{F_k}(\xi^1)=\widehat{F}(a_{k,s,v}\xi^1)=\zeta_k(|\pi^d_{r_s}\xi|),$$

where ς_k is as in (2.17). It may be easily verified that $F\in\mathcal{S}(\mathbb{R}^{r_s})$ and $\widehat{F}(0)=0$. Define the nonsingular linear transformation J on $\mathbb{R}^d$ by $J=G_s^{-1}(H_s^{-1}\otimes\delta_{\mathbb{R}^{d-r_s}})$. It is easy to see that

$$S_{k,s}(f)(x)=|J|F_k\otimes\delta_{\mathbb{R}^{d-r_s}}*f^J(J^tx), \tag{2.21}$$

where $f^J(x)=|J|^{-1}f((J^t)^{-1}x)$. By a change of variables, (2.21) and Lemma 2.2, we have that, for any $1<p,\ q<\infty$ and $\{g_i\}_{i\in\mathbb{Z}}\in$

$L^p(\mathbb{R}^d, \ell^q)$, there exists a $C > 0$ such that

$$\left\|\left(\sum_{i\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|S_{k,s}(g_i)|^2\right)^{q/2}\right)^{1/q}\right\|_{L^p(\mathbb{R}^d)} \le C\left\|\left(\sum_{i\in\mathbb{Z}}|g_i|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^d)}. \tag{2.22}$$

On the other hand, by assumption (iii), Lemma 2.2 and similar arguments as in [**6,** Proposition 2.3], we can obtain

$$\left\|\left(\sum_{i\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|\mu_{s,k} * g_{i,k}|^2\right)^{q_0/2}\right)^{1/q_0}\right\|_{L^{p_0}(\mathbb{R}^d)} \le CB\left\|\left(\sum_{i\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|g_{i,k}|^2\right)^{q_0/2}\right)^{1/q_0}\right\|_{L^{p_0}(\mathbb{R}^d)}. \tag{2.23}$$

It follows from (2.22)–(2.23) that there exists a $C > 0$ such that

$$\begin{aligned}
\left\|\left(\sum_{i\in\mathbb{Z}}|\mathcal{A}_{s,j}(g_i)|^{q_0}\right)^{1/q_0}\right\|_{L^{p_0}(\mathbb{R}^d)} &= \sup_{\|\{f_i\}\|_{L^{p_0'}(\mathbb{R}^d,\ell^{q_0'})}\le 1} \\
&\quad\times\left|\int_{\mathbb{R}^d}\sum_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}S_{j+k,s}(\mu_{s,k}*S_{j+k,s}(g_i))(x)f_i(x)dx\right| \\
&\le \sup_{\|\{f_i\}\|_{L^{p_0'}(\mathbb{R}^d,\ell^{q_0'})}\le 1}\left\|\left(\sum_{i\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|S^*_{j+k,s}(f_i)|^2\right)^{q_0'/2}\right)^{1/q_0'}\right\|_{L^{p_0'}(\mathbb{R}^d)} \\
&\quad\times\left\|\left(\sum_{i\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|\mu_{s,k} * S_{j+k,s}(g_i)|^2\right)^{q_0/2}\right)^{1/q_0}\right\|_{L^{p_0}(\mathbb{R}^d)} \\
&\le CB\left\|\left(\sum_{i\in\mathbb{Z}}|g_i|^{q_0}\right)^{1/q_0}\right\|_{L^{p_0}(\mathbb{R}^d)},
\end{aligned} \tag{2.24}$$

which leads to

$$\|\mathcal{A}_{s,j}(f)\|_{\dot{F}^{p_0,q_0}_\alpha(\mathbb{R}^d)} = \left\|\left(\sum_{i\in\mathbb{Z}}2^{-i\alpha q_0}|\Psi_i * \mathcal{A}_{s,j}(f)|^{q_0}\right)^{1/q_0}\right\|_{L^{p_0}(\mathbb{R}^d)} \tag{2.25}$$

$$\leq \left\| \left(\sum_{i\in\mathbb{Z}} |\mathcal{A}_{s,j}(2^{-i\alpha}\Psi_i * f)|^{q_0} \right)^{1/q_0} \right\|_{L^{p_0}(\mathbb{R}^d)}$$
$$= CB\|f\|_{\dot{F}_\alpha^{p_0,q_0}(\mathbb{R}^d)}$$

for any $\alpha \in \mathbb{R}$. Then, interpolation **[15, 17]** between (2.20) and (2.25) implies that, for $\alpha \in \mathbb{R}$, p, q satisfying

$$\left(\frac{1}{p}, \frac{1}{q}\right) \in A_1A_2 \setminus \left\{ \left(\frac{1}{p_0}, \frac{1}{q_0}\right), \left(\frac{1}{2}, \frac{1}{2}\right) \right\}$$

and $1 \leq s \leq \Lambda$, there exists an $\epsilon > 0$ such that

$$\|\mathcal{A}_{s,j}(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \leq CB\eta_s^{-c\epsilon|j|}\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)}, \tag{2.26}$$

which, together with (2.18), yields (2.16) and completes the proof of Lemma 2.5. □

3. Proofs of Theorems 1.1 and 1.5. Let $\Omega \in L(\log^+ L)^\alpha(S^{n-1})$ for $\alpha > 0$ satisfy (1.2). Employing the notation in **[3]**, let

$$E_0 = \{y' \in S^{n-1} : |\Omega(y')| < 2\}$$

and

$$E_\mu = \{y' \in S^{n-1} : 2^\mu < |\Omega(y')| \leq 2^{\mu+1}\}$$

for $\mu \in \mathbb{N} \setminus \{0\}$. Let $\Lambda_\Omega = \{\mu \in \mathbb{N} \setminus \{0\} : \sigma(E_\mu) > 2^{-4\mu}\}$ and $\Omega_0 = \Omega - \sum_{\mu\in\Lambda_\Omega} \Omega_\mu$, where

$$\Omega_\mu = \Omega\chi_{E_\mu} - \sigma(S^{n-1})^{-1} \int_{E_\mu} \Omega(y')\, d\sigma(y'), \quad \mu \geq 1.$$

It is easy to verify that

$$\int_{S^{n-1}} \Omega_\mu(y')\, d\sigma(y') = 0, \tag{3.1}$$

$$\|\Omega_\mu\|_{L^1(S^{n-1})} \leq C\|\Omega\|_{L^1(E_\mu)}, \tag{3.2}$$

$$\|\Omega_\mu\|_{L^2(S^{n-1})} \leq C2^{2\mu}\|\Omega\|_{L^1(E_\mu)} \tag{3.3}$$

for $\mu \in \Lambda_\mu \cup \{0\}$, and

$$\sum_{\mu \in \Lambda_\Omega \cup \{0\}} (\mu+1)^\alpha \|\Omega\|_{L^1(E_\mu)} \leq C\|\Omega\|_{L(\log^+ L)^\alpha(S^{n-1})}. \tag{3.4}$$

Clearly,

$$T_{h,\Omega,\Phi,\Psi}(f) = \sum_{\mu \in \Lambda_\Omega \cup \{0\}} T_{h,\Omega_\mu,\Phi,\Psi}(f). \tag{3.5}$$

In what follows, we let $\mathscr{R}_\gamma$ be as in Theorem 1.1 and $\sigma_{k,\mu,\Gamma,\Omega}$ as in (2.2). For $\gamma > 1$, we denote $\widetilde{\gamma} = \max\{2, \gamma'\}$ and $A = (\mu+1)^{1/\gamma'}$ $\|\Omega\|_{L^1(E_\mu)} \||h\||_{\mu,\gamma}$, where $\||h\||_{\mu,\gamma}$ is as in (2.1).

Proof of Theorem 1.1. We first prove part (i). Let $N_1 = \max_{1\leq i\leq n} \deg(P_i)$. There exist $\mathcal{N} \in \mathbb{N}$, some integers $0 = d_0 < d_1 < d_2 < \cdots < d_{\mathcal{N}} = N_1$ and $\{a_{i,j} : 1 \leq i \leq n,\ 0 \leq j \leq \mathcal{N}\}$ such that

$$(a_{1,j}, a_{2,j}, \dots, a_{n,j}) \neq (0, \dots, 0) \quad \text{for } 1 \leq j \leq \mathcal{N},$$

and

$$(P_1(t), P_2(t), \dots, P_n(t)) = \left(\sum_{j=0}^{\mathcal{N}} a_{1,j}t^{d_j}, \sum_{j=0}^{\mathcal{N}} a_{2,j}t^{d_j}, \dots, \sum_{j=0}^{\mathcal{N}} a_{n,j}t^{d_j}\right).$$

For $0 \leq s \leq \mathcal{N}$ and $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, the linear transformation $\mathcal{L}_s : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is defined by

$$\mathcal{L}_s(x,y) = (a_{1,s}x_1, a_{2,s}x_2, \dots, a_{n,s}x_n),$$

where $x = (x_1, \dots, x_n)$. For any $0 \leq s \leq \mathcal{N}$, let

$$\mathcal{P}_s(t,x) = \left(\sum_{j=0}^{s} a_{1,j}t^{d_j}x_1, \sum_{j=0}^{s} a_{2,j}t^{d_j}x_2, \dots, \sum_{j=0}^{s} a_{n,j}t^{d_j}x_n\right).$$

For $k \in \mathbb{Z}$, $\mu \in \Lambda_\Omega \cup \{0\}$ and $0 \leq s \leq \mathcal{N}$, we denote $\sigma_{k,s}^\mu$ by $\sigma_{k,\mu,\Gamma,\Omega}$ with Ω replaced by Ω_μ and $\Gamma(y) = (\mathcal{P}_s(|y|, y'), \Psi(|y|))$. Obviously,

$$\sigma_{k,0}^\mu = 0, \tag{3.6}$$

$$T_{h,\Omega_\mu,\Phi,\Psi}(f) = \sum_{k\in\mathbb{Z}} \sigma_{k,\mathcal{N}}^\mu * (f). \tag{3.7}$$

By a change of variables and Hölder's inequality,

(3.8)

$$
\begin{aligned}
|\widehat{\sigma^{\mu}_{k,s}}(\xi,\eta)| &= \Big|\int_{2^{k(\mu+1)}}^{2^{(k+1)(\mu+1)}}\int_{S^{n-1}} e^{-2\pi i(\mathcal{P}_s(t,y')\cdot\xi+\Psi(t)\cdot\eta)}\Omega_\mu(y')d\sigma(y')h(t)\frac{dt}{t}\Big| \\
&\le |||h|||_{\mu,\gamma}\Big(\int_{2^{k(\mu+1)}}^{2^{(k+1)(\mu+1)}}\Big|\int_{S^{n-1}} e^{-2\pi i(\mathcal{P}_s(t,y')\cdot\xi+\Psi(t)\cdot\eta)}\Omega_\mu(y')d\sigma(y')\Big|^{\gamma'}\frac{dt}{t}\Big)^{\frac{1}{\gamma'}} \\
&\le |||h|||_{\mu,\gamma}\Big(\sum_{j=0}^{\mu}\int_{2^{k(\mu+1)+j}}^{2^{k(\mu+1)+j+1}}\Big|\int_{S^{n-1}} e^{-2\pi i(\mathcal{P}_s(t,y')\cdot\xi+\Psi(t)\cdot\eta)}\Omega_\mu(y')d\sigma(y')\Big|^{\gamma'}\frac{dt}{t}\Big)^{\frac{1}{\gamma'}} \\
&\le C|||h|||_{\mu,\gamma}\Big(\sum_{j=0}^{\mu}\|\Omega_\mu\|_{L^1(S^{n-1})}^{\max\{0,\gamma'-2\}} \\
&\qquad\cdot\Big(\int_{2^{k(\mu+1)+j}}^{2^{k(\mu+1)+j+1}}\Big|\int_{S^{n-1}} e^{-2\pi i(\mathcal{P}_s(t,y')\cdot\xi+\Psi(t)\cdot\eta)}\Omega_\mu(y')d\sigma(y')\Big|^{2}\frac{dt}{t}\Big)^{\frac{\min\{2,\gamma'\}}{2}}\Big)^{\frac{1}{\gamma'}} \\
&\le C|||h|||_{\mu,\gamma}\|\Omega_\mu\|_{L^1(S^{n-1})}^{\max\{0,1-(2/\gamma')\}}\Big(\sum_{j=0}^{\mu}(\mathcal{I}_{s,j}(\xi,\eta))^{(\min\{2,\gamma'\}/2)}\Big)^{1/\gamma'},
\end{aligned}
$$

where

$$
\mathcal{I}_{s,j}(\xi,\eta) := \int_{2^{k(\mu+1)+j}}^{2^{k(\mu+1)+j+1}}\Big|\int_{S^{n-1}} e^{-2\pi i(\mathcal{P}_s(t,y')\cdot\xi+\Psi(t)\cdot\eta)}\Omega_\mu(y')\,d\sigma(y')\Big|^{2}\frac{dt}{t}.
$$

Invoking Lemma 2.1 and Hölder's inequality, we have

(3.9)

$$
\begin{aligned}
&\mathcal{I}_{s,j}(\xi,\eta) \\
&\quad= \int_{2^{k(\mu+1)+j}}^{2^{k(\mu+1)+j+1}}\iint_{(S^{n-1})^2} e^{-2\pi i(\mathcal{P}_s(t,y')-\mathcal{P}_s(t,\theta))\cdot\xi}\Omega_\mu(y')\overline{\Omega_\mu(\theta)}\,d\sigma(y')d\sigma(\theta)\frac{dt}{t}
\end{aligned}
$$

$$\le \iint_{(S^{n-1})^2} \left| \int_{2^{k(\mu+1)+j}}^{2^{k(\mu+1)+j+1}} e^{-2\pi i(\mathcal{P}_s(t,y')-\mathcal{P}_s(t,\theta))\cdot\xi} \frac{dt}{t} \right|$$
$$\cdot |\Omega_\mu(y')\overline{\Omega_\mu(\theta)}| \, d\sigma(y') \, d\sigma(\theta)$$
$$\le \iint_{(S^{n-1})^2} \min\{1, |2^{(k(\mu+1)+j+1)\, d_s} \mathcal{L}_s(\xi,\eta)\cdot(y'-\theta)|^{-1/d_s}\}$$
$$\cdot |\Omega_\mu(y')\overline{\Omega_\mu(\theta)}| \, d\sigma(y') \, d\sigma(\theta)$$
$$\le \|\Omega_\mu\|^2_{L^2(S^{n-1})} \Big(\iint_{(S^{n-1})^2} \min\{1, |2^{(k(\mu+1)+j+1)\, d_s} \mathcal{L}_s(\xi,\eta)(y'-\theta)|^{-\frac{2}{d_s}}\} d\sigma(y') d\sigma(\theta)\Big)^{\frac{1}{2}}$$
$$\le C\|\Omega_\mu\|^2_{L^2(S^{n-1})} |2^{(k(\mu+1)+j+1)\, d_s} \mathcal{L}_s(\xi,\eta)|^{-1/4d_s},$$

where the last inequality of (3.9) is obtained by the inequality

$$\iint_{(S^{n-1})^2} |y'\cdot(\theta-w)|^{-\alpha} d\sigma(\theta)\, d\sigma(w) < \infty,$$

for $y' \in S^{n-1}$ and $0<\alpha<1$. Then, by (3.3), (3.8) and (3.9), we have

$$|\widehat{\sigma^\mu_{k,s}}(\xi,\eta)| \le C2^{4\mu/\tilde{\gamma}} A |2^{k(\mu+1)\, d_s} \mathcal{L}_s(\xi,\eta)|^{-1/(4d_s\tilde{\gamma})},$$

which, combined with the trivial estimate $|\widehat{\sigma^\mu_{k,s}}(\xi,\eta)| \le CA$, yields

$$|\widehat{\sigma^\mu_{k,s}}(\xi,\eta)| \le CA \min\{1, |2^{kd_s(\mu+1)} \mathcal{L}_s(\xi,\eta)|^{-1/(4d_s\tilde{\gamma}(\mu+1))}\}. \tag{3.10}$$

On the other hand, by a change of variables, (3.2) and Hölder's inequality, we have

(3.11)

$$|\widehat{\sigma^\mu_{k,s}}(\xi,\eta) - \widehat{\sigma^\mu_{k,s-1}}(\xi,\eta)|$$
$$= \int_{2^{k(\mu+1)}\le|y|<2^{(k+1)(\mu+1)}} \left(e^{-2\pi i(\mathcal{P}_s(t,y')\cdot\xi+\Psi(t)\cdot\eta)} - e^{-2\pi i(\mathcal{P}_{s-1}(t,y')\cdot\xi+\Psi(t)\cdot\eta)}\right) \frac{\Omega_\mu(y)h(|y|)}{|y|^n} dy$$

$$\leq C \int_{2^{k(\mu+1)}}^{2^{(k+1)(\mu+1)}} \int_{S^{n-1}} \min\{1, |t^{d_s} \mathcal{L}_s(\xi,\eta)\cdot y'|\} |\Omega_\mu(y')| \, d\sigma(y') |h(t)| \frac{dt}{t}$$
$$\leq CA|2^{d_s(k+1)(\mu+1)} \mathcal{L}_s(\xi,\eta)|^{1/\mu+1}.$$

Invoking Lemma 2.4, we obtain from (3.2) that

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} |\sigma^\mu_{k,s} * g_{k,j}|^2 \Big)^{q/2} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^{n+m})} \leq CA \Big\| \Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} |g_{k,j}|^2 \Big)^{q/2} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^{n+m})}$$

for $\{g_{k,j}\}_{k,j\in\mathbb{Z}} \in L^p(\mathbb{R}^{n+m}, \ell^q(\ell^2))$ with $(1/p, 1/q) \in \mathscr{R}_\gamma$. Then, by (3.6), (3.7), (3.10)–(3.12) and Lemma 2.5, we have

$$\|T_{h,\Omega_\mu,\Phi,\Psi} f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^{n+m})} \leq CA\|f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^{n+m})} \tag{3.12}$$

for any $\alpha \in \mathbb{R}$ and $(1/p, 1/q) \in \mathscr{R}_\gamma$ (the point $(1/2, 1/2)$ may be obtained by interpolation). Theorem 1.1 (i) follows from (3.4), (3.5), (3.13) and the inequality

$$\| |h| \|_{\mu,\gamma} \leq C \min\{(\mu+1)^{1/\gamma} \|h\|_{\Delta_\gamma(\mathbb{R}^+)}, \|h\|_{\mathscr{H}_\gamma(\mathbb{R}^+)}\}. \quad \square$$

Now, we prove Theorem 1.1 (ii). The proof is similar to the arguments in the proof of [**6,** Theorem 1.2]. By Remark 1.3 (ii), we have

$$\begin{aligned} \|T_{h,\Omega,\Phi,\Psi}(f)\|_{\dot{B}^{p,q}_\alpha(\mathbb{R}^{n+m})} &= \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} \|\Psi_i * T_{h,\Omega,\Phi,\Psi}(f)\|^q_{L^p(\mathbb{R}^{n+m})} \Big)^{1/q} \\ &= \Big(\sum_{i\in\mathbb{Z}} \|T_{h,\Omega,\Phi,\Psi}(2^{-i\alpha}\Psi_i * f)\|^q_{L^p(\mathbb{R}^{n+m})} \Big)^{1/q} \\ &\leq C \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} \|\Psi_i * f\|^q_{L^p(\mathbb{R}^{n+m})} \Big)^{1/q} \\ &= C\|f\|_{\dot{B}^{p,q}_\alpha(\mathbb{R}^{n+m})} \end{aligned} \tag{3.13}$$

for $\alpha \in \mathbb{R}$, $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ and $1 < q < \infty$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.5. We only prove part (i), since part (ii) may be obtained from Remark 1.6 (ii) and the same arguments as in the proof of Theorem 1.1 (ii). Following [**14**], we first recall some notation. Let $N_2 = \max_{1\le j\le d}\deg(P_j)$. Then, there are $\mathcal{N}\in\mathbb{N}$, integers $0 < l_1 < l_2 < \cdots < l_{\mathcal{N}} \le N_1$ and polynomials $P_j^s \in V_{n,l_s}$, $R_j \in \mathcal{A}_1$ with $\deg(R_j)\le N_1$ for $1\le s\le\mathcal{N}$, $1\le j\le d$, such that

$$\Phi(x) = \sum_{s=1}^{\mathcal{N}} \mathcal{P}^s(x) + \mathcal{R}(|x|),$$

where $\mathcal{P}^s = (P_1^s,\dots,P_d^s)$ and $\mathcal{R} = (R_1,\dots,R_d)$. For each $s\in\{1,\dots,\mathcal{N}\}$, there is at least one $j\in\{1,\dots,d\}$ such that $P_j^s\neq 0$. For $j=1,\dots,d$ and $1\le s\le\mathcal{N}$, write

$$P_j^s(x) = \sum_{|\beta|=l_s} b_{sj\beta}x^\beta = \sum_{i=1}^{d(s)} b'_{sji}x^{\beta(s,i)},$$

where $d(s)=\dim(V_{n,l_s})$. For $1\le s\le\mathcal{N}$, we define the linear transformations $L_s:\mathbb{R}^d\to\mathbb{R}^{d(s)}$ by

$$L_s(\xi) = \left(\sum_{j=1}^{d} b'_{sj1}\xi_j,\dots,\sum_{j=1}^{d} b'_{sjd(s)}\xi_j\right).$$

For $0\le s\le\mathcal{N}$, we define $\mathcal{P}_s$ by

$$\mathcal{P}_s(x) = \mathcal{R}(|x|) + \sum_{u=1}^{s}\mathcal{P}^u(x).$$

For $k\in\mathbb{Z}$, $\mu\in\Lambda_\Omega\cup\{0\}$ and $0\le s\le\mathcal{N}$, we denote $\sigma_{k,s}^{\mu}$ by $\sigma_{k,\mu,\Gamma,\Omega}$ with Ω replaced by Ω_μ and $\Gamma(y)=(\mathcal{P}_s(y),\Psi(|y|))$. Obviously,

$$\sigma_{k,0}^{\mu} = 0; \tag{3.14}$$

$$T_{h,\Omega_\mu,\Phi,\Psi}(f) = \sum_{k\in\mathbb{Z}} f*\sigma_{k,\mathcal{N}}^{\mu}. \tag{3.15}$$

By a change of variables, Hölder's inequality and (3.2),

(3.16)

$$
\begin{aligned}
&|\widehat{\sigma^{\mu}_{k,s}}(\xi,\eta)-\widehat{\sigma^{\mu}_{k,s-1}}(\xi,\eta)|\\
&\quad=\Big|\int_{2^{(\mu+1)k}}^{2^{(\mu+1)(k+1)}}\int_{S^{n-1}}\Omega_\mu(y')(e^{-2\pi i(\mathcal{P}_s(ty')\cdot\xi+\Psi(t)\cdot\eta)}\\
&\qquad\qquad\qquad -e^{-2\pi i(\mathcal{P}_{s-1}(ty')\cdot\xi+\Psi(t)\cdot\eta)})\,d\sigma(y')h(t)\,\frac{dt}{t}\Big|\\
&\quad\le C|2^{(\mu+1)(k+1)l_s}L_s(\xi)|\|\Omega_\mu\|_{L^1(S^{n-1})}\int_{2^{(\mu+1)k}}^{2^{(\mu+1)(k+1)}}|h(t)|\,\frac{dt}{t}\\
&\quad\le CA|2^{(\mu+1)(k+1)l_s}L_s(\xi)|.
\end{aligned}
$$

It is easily verified that

$$
|\widehat{\sigma^{\mu}_{k,s}}(\xi,\eta)|\le CA. \tag{3.17}
$$

Combining this with (3.16) yields

$$
|\widehat{\sigma^{\mu}_{k,s}}(\xi,\eta)-\widehat{\sigma^{\mu}_{k,s-1}}(\xi,\eta)|\le CA|2^{(\mu+1)kl_s}L_s(\xi)|^{1/(\mu+1)}. \tag{3.18}
$$

On the other hand, by a change of variables, (3.2) and Hölder's inequality,

(3.19)

$$
\begin{aligned}
&|\widehat{\sigma^{\mu}_{k,s}}(\xi,\eta)|\\
&\ =\Big|\int_{2^{(\mu+1)k}}^{2^{(\mu+1)(k+1)}}\int_{S^{n-1}}\Omega_\mu(y')e^{-2\pi i(\mathcal{P}_s(ty')\cdot\xi+\Psi(t)\cdot\eta)}d\sigma(y')h(t)\frac{dt}{t}\Big|\\
&\le |||h|||_{\mu,\gamma}\Big(\int_{2^{(\mu+1)k}}^{2^{(\mu+1)(k+1)}}\Big|\int_{S^{n-1}}\Omega_\mu(y')e^{-2\pi i(\mathcal{P}_s(ty')\cdot\xi+\Psi(t)\cdot\eta)}d\sigma(y')\Big|^{\gamma'}\frac{dt}{t}\Big)^{1/\gamma'}\\
&\le |||h|||_{\mu,\gamma}\Big(\sum_{j=0}^{\mu}\int_{2^{(\mu+1)k+j}}^{2^{(\mu+1)k+j+1}}\Big|\int_{S^{n-1}}\Omega_\mu(y')e^{-2\pi i(\mathcal{P}_s(ty')\cdot\xi+\Psi(t)\cdot\eta)}d\sigma(y')\Big|^{\gamma'}\frac{dt}{t}\Big)^{\frac{1}{\gamma'}}
\end{aligned}
$$

$$\le \||h\||_{\mu,\gamma}\Big(\sum_{j=0}^{\mu}\|\Omega\|_{L^1(E_\mu)}^{\max\{\gamma'-2,0\}}$$
$$\times\Big(\int_{2^{(\mu+1)k+j}}^{2^{(\mu+1)k+j+1}}\Big|\int_{S^{n-1}}\Omega_\mu(y')e^{-2\pi i(\mathcal{P}_s(ty')\cdot\xi+\Psi(t)\cdot\eta)}d\sigma(y')\Big|^2\frac{dt}{t}\Big)^{\gamma'/\tilde{\gamma}}\Big)^{\frac{1}{\gamma'}}.$$

Applying [**14,** Corollary 4.3] with $\epsilon = 1/(8l_s)$ and $p = 2$, we have that, for any $r > 0$,

$$\Big(\int_r^{2r}\Big|\int_{S^{n-1}}\Omega_\mu(y')e^{-2\pi i(\mathcal{P}_s(ty')\cdot\xi+\Psi(t)\cdot\eta)}d\sigma(y')\Big|^2\frac{dt}{t}\Big)^{1/2} \le C\|\Omega_\mu\|_{L^2(S^{n-1})}|r^{l_s}L_s(\xi)|^{-1/(8l_s)}. \tag{3.20}$$

Combining (3.3) with (3.19)–(3.20) implies

$$|\widehat{\sigma_{k,s}^{\mu}}(\xi,\eta)| \le C2^{4\mu/\tilde{\gamma}}A|2^{(\mu+1)kl_s}L_s(\xi)|^{-1/(4l_s\tilde{\gamma})}. \tag{3.21}$$

It follows from (3.17) and (3.21) that

$$|\widehat{\sigma_{k,s}^{\mu}}(\xi,\eta)| \le CA\min\{1,|2^{(\mu+1)kl_s}L_s(\xi)|\}^{-1/(4(\mu+1)l_s\tilde{\gamma})}. \tag{3.22}$$

Invoking Lemma 2.5, we have

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,s}^{\mu}*g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^{d+m})} \le CA\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^{d+m})} \tag{3.23}$$

for $(1/p,1/q)\in\mathscr{R}_\gamma$.

For $1\le s\le N_1$, define the linear transformation $\mathcal{L}_s:\mathbb{R}^{d+m}\to\mathbb{R}^{d(s)}$ by $\mathcal{L}_s(\xi,\eta) = L_s(\xi)$ for any $(\xi,\eta)\in\mathbb{R}^{d+m}$. It follows from (3.14)–(3.16), (3.22), (3.23) and Lemma 2.5 that

$$\|T_{h,\Omega_\mu,\Phi,\Psi}(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^{d+m})} \le CA\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^{d+m})} \tag{3.24}$$

for any $\alpha\in\mathbb{R}$ and $(1/p,1/q)\in\mathscr{R}_\gamma$ (the point $(1/2,1/2)$ is obtained by interpolation). Inequality (3.24), together with equations (3.4), (3.5) and the fact that

$$|||h|||_{\mu,\gamma} \leq \|h\|_{\mathscr{H}_\gamma(\mathbb{R}^+)} \quad \text{for any } \gamma > 0,$$

yields Theorem 1.5 (i). $\square$

4. Proof of Theorem 1.4. Recall the Hardy space on S^{n-1} and its atomic decomposition. The Hardy space $H^1(S^{n-1})$ is the set of all $L^1(S^{n-1})$ functions Ω satisfying $\|\Omega\|_{H^1(S^{n-1})} < \infty$, where

$$\|\Omega\|_{H^1(S^{n-1})} = \int_{S^{n-1}} \sup_{0\leq r<1} \left| \int_{S^{n-1}} \Omega(\theta) \frac{1-r^2}{|rw-\theta|^n} \, d\sigma(\theta) \right| d\sigma(w).$$

We say that a function $a(\cdot)$ on S^{n-1} is a *regular atom* if there exist $\varepsilon \in S^{n-1}$ and $\varrho \in (0,2]$ such that

$$\operatorname{supp}(a) \subset S^{n-1} \cap B(\varepsilon, \varrho), \tag{4.1}$$

$$\text{where } B(\varepsilon,\varrho) = \{y \in \mathbb{R}^n : |y-\varepsilon| < \varrho\};$$

$$\|a\|_{L^\infty(S^{n-1})} \leq \varrho^{-n+1}; \tag{4.2}$$

$$\int_{S^{n-1}} a(y)\, d\sigma(y) = 0. \tag{4.3}$$

Below is the well-known atomic decomposition of the Hardy space.

Lemma 4.1. ([**11, 12**]). *If $\Omega \in H^1(S^{n-1})$ satisfies* (1.2), *then there exist $\{c_j\} \subset \mathbb{C}$ and H^1 regular atoms $\{\Omega_j\}$ such that*

$$\Omega = \sum_j c_j \Omega_j \quad \text{and} \quad \sum_j |c_j| \approx \|\Omega\|_{H^1(S^{n-1})}.$$

Proof of Theorem 1.4. We only prove part (i), since part (ii) may be obtained from Remark 1.6 (ii) and similar arguments as in Theorem 1.1 (ii). Without loss of generality, we may assume that Ω is a regular atom satisfying (4.1)–(4.3) with $0 < \varrho < 1/4$ and $\varepsilon = \mathbf{e} = (0,\dots,0,1)$.

Next, we give some notation, which is identical to that in [**14**]. In what follows, we denote $x = (\widetilde{x}, x_n)$ with $\widetilde{x} = (x_1,\dots,x_{n-1})$. Let $N_3 = \max_{1\leq j\leq d} \deg(P_j)$. Then there are $\mathcal{N} \in \mathbb{N}$, integers $0 < l_1 < l_2 < \cdots < l_{\mathcal{N}} \leq N_1$ and polynomials $P_j^u \in V_{n,l_u}$, $R_j \in \mathcal{A}_1$ with

$\deg(R_j) \le N_1$ for $1 \le u \le \mathcal{N}$, $1 \le j \le d$, such that

$$\mathcal{P}(x) = \sum_{u=1}^{\mathcal{N}} \mathcal{P}^u(x) + \mathcal{R}(|x|),$$

where $\mathcal{P}^u = (P_1^u, P_2^u, \dots, P_d^u)$ and $\mathcal{R} = (R_1, R_2, \dots, R_d)$. For $j = 1, \dots, d$, denote

$$P_j^u(x) = \sum_{|\beta|=l_u} b_{uj\beta} x^\beta.$$

For $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| = l$, we choose $\eta_{l,\alpha}(\cdot) \in \mathcal{A}_{n-1}$ such that

$$|x^\alpha - \eta_{l,\alpha}(\widetilde{x})| \le C\varrho^{4(n-1)} \quad \text{for } x \in S^{n-1} \cap B(\mathbf{e}, \varrho).$$

For each $u \in \{1, \dots, \mathcal{N}\}$, $j \in \{1, \dots, d\}$, we define $q_j^u \in \mathcal{A}_{n-1}$ by

$$q_j^u(\widetilde{x}) = \sum_{|\beta|=l_u} b_{uj\beta} \eta_{l_u,\beta}(\widetilde{x}),$$

and set $q^u(\widetilde{x}) = (q_1^u(\widetilde{x}), q_2^u(\widetilde{x}), \dots, q_d^u(\widetilde{x}))$. Fixing each $u \in \{1, \dots, \mathcal{N}\}$, there are positive integers $v(u)$, $0 < h_{u,1} < \cdots < h_{u,v(u)}$ and polynomials

$$\{W_{j\eta}^u : j = 1, \dots, d;\ \eta = 1, \dots, v(u)\} \subset \mathcal{A}_{n-1}$$

such that

(a) for $j \in \{1, \dots, d\}$, $\eta \in \{1, \dots, v(u)\}$, $W_{j\eta}^u(\cdot)$ is homogeneous of degree $h_{u,\eta}$;
(b) for each $\eta \in \{1, \dots, v(u)\}$, there exists at least one $j \in \{1, \dots, d\}$ such that $W_{j\eta}^u \ne 0$;
(c) for each $j \in \{1, \dots, d\}$, there is a $v_j^u \in \mathbb{R}$ such that $q_j^u(\widetilde{x}) = \sum_{\eta=1}^{v(u)} W_{j\eta}^u(\widetilde{x}) + v_j^u$.

For $u \in \{1, \dots, \mathcal{N}\}$ and $\eta \in \{1, \dots, v(u)\}$, we define $\mathcal{R}^u(x)$ and $\mathcal{W}^{u,\eta}(\widetilde{x})$ by

$$\mathcal{R}^u(x) = \mathcal{R}(|x|) + \sum_{u \le k \le \mathcal{N}} |x|^{l_k}(v_1^k, \dots, v_d^k) + \sum_{1 \le k \le u-1} \mathcal{Q}^k(x),$$

and

$$\mathcal{W}^{u,\eta}(\widetilde{x}) = (W_{1\eta}^u(\widetilde{x}), \dots, W_{d\eta}^u(\widetilde{x})).$$

Let $M(0)=0$,

$$M(u)=\sum_{k=1}^{u}[v(k)+1] \quad \text{for } 1\le u\le \mathcal{N},$$

and define $\Gamma_0,\ \Gamma_1,\ \ldots,\Gamma_{M(\mathcal{N})}$ by

$$\Gamma_{M(u-1)+\theta}(x)=\mathcal{R}^u(x)+|x|^{l_u}\sum_{1\le k\le\theta}\mathcal{W}^{u,k}\Big(\frac{\widetilde{x}}{|x|}\Big)$$

for $1\le u\le\mathcal{N},\ 0\le\theta\le M(u)-M(u-1)$ and $\Gamma_{M(m)}(x)=\Phi(x)$. Let $d(u)=\dim(V_{n,l_u})$. For each $u\in\{1,\ldots,\mathcal{N}\}$, write

$$\{\beta\in\mathbb{N}^n:|\beta|=l_u\}:=\{\beta(u,1),\ldots,\beta(u,d(u))\}.$$

Hence, we can write

$$P_j^u(x)=\sum_{s=1}^{d(u)}b'_{ujs}x^{\beta(u,s)},$$

where $b'_{ujs}=b_{uj\beta(u,s)}$. Denote by $d(u,\eta)$ the number of distinct elements in $\{\varpi\in\mathbb{N}^{n-1}:\ |\varpi|=h_{u,\eta}\}$. For $1\le u\le\mathcal{N}$, $1\le\eta\le v(u)$ and $1\le j\le d$, write

$$\{\varpi:|\varpi|=h_{u,\eta}\}=\{\varpi(u,\eta,1),\ldots,\varpi(u,\eta,d(u,\eta))\}$$

and

$$W_{j\eta}^u(\widetilde{x})=\sum_{s=1}^{d(u,\eta)}w_{u,j,\eta,s}\widetilde{x}^{\varpi(u,\eta,s)}.$$

For $1\le u\le\mathcal{N}$, we define $\Lambda_1,\ldots,\Lambda_{M(\mathcal{N})}\in\mathbb{N}$ by

$$\Lambda_{M(u-1)+\theta}=\begin{cases}d(u,\theta) & \text{if } 1\le\theta<M(u)-M(u-1);\\ d(u) & \text{if } \theta=M(u)-M(u-1).\end{cases}$$

Also, we define linear transformations $L_i:\mathbb{R}^d\to\mathbb{R}^{\Lambda_i}$ for $1\le i\le M(\mathcal{N})$ by

$$L_{M(u-1)+\theta}(\xi)=\begin{cases}\big(\sum_{j=1}^d w_{u,j,\theta,s}\xi_j,\ldots,\sum_{j=1}^d w_{u,j,\theta,d(u,\theta)}\xi_j\big)\\ \qquad\text{if } 1\le\theta<M(u)-M(u-1);\\ \big(\sum_{j=1}^d b'_{uj1}\xi_j,\ldots,\sum_{j=1}^d b'_{ujd(u)}\xi_j\big)\\ \qquad\text{if } \theta=M(u)-M(u-1).\end{cases}$$

For $s = 1, \dots, M(\mathcal{N})$, we set

$$\begin{cases} l(s) = l_u, \quad \delta(s) = h_{u,\theta}, \quad \gamma(s) = 1/(4h_{u,\theta} l_u \gamma') \\ \qquad\qquad \text{if } \eta \in [M(u-1), M(u)); \\ l(s) = l_u, \quad \delta(s) = 4l_u(n-1), \quad \gamma(s) = 1/(8l_u\gamma') \\ \qquad\qquad \text{if } \eta = M(u). \end{cases}$$

For each $k \in \mathbb{Z}$ and $0 \le s \le M(\mathcal{N})$, we denote $\sigma_{k,s}$ by $\sigma_{k,\mu,\Gamma,\Omega}$ with $\Gamma(y) = (\Gamma_s(y), \Psi(|y|))$ and $\mu = 0$. Obviously,

$$T_{h,\Omega,\Phi,\Psi}(f) = \sum_{k\in\mathbb{Z}} \sigma_{k,M(\mathcal{N})} * f. \tag{4.4}$$

It is easily verified that

$$\sigma_{k,0} = 0, \qquad |\widehat{\sigma_{k,s}}(\xi,\eta)| \le C. \tag{4.5}$$

By a change of variables and Hölder's inequality,

(4.6)

$$\begin{aligned} &|\widehat{\sigma_{k,s}}(\xi,\eta) - \widehat{\sigma_{k,s-1}}(\xi,\eta)| \\ &\quad \le \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} \left|e^{-2\pi i\xi\cdot\Gamma_s(ty')} - e^{-2\pi i \xi\cdot\Gamma_{s-1}(ty')}\right| |\Omega(y')|\, d\sigma(y') |h(t)| \,\frac{dt}{t} \\ &\quad \le C 2^{kl(s)} \varrho^{\delta(s)} |L_s(\xi)|. \end{aligned}$$

On the other hand, by [**14,** Propositions 5.1, 5.3, Remark 5.2], we get

$$|\widehat{\sigma_{k,s}}(\xi,\eta)| \le C(2^{kl(s)}\varrho^{\delta(s)}|L_s(\xi)|)^{-\gamma(s)}. \tag{4.7}$$

Invoking Lemma 2.4 with $\mu = 0$, we obtain

$$\begin{aligned} &\left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |\sigma_{k,s} * g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^{d+m})} \\ &\qquad\qquad \le C \left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^{d+m})} \end{aligned} \tag{4.8}$$

for $s \in \{1, \dots, M(\mathcal{N})\}$ and $(1/p, 1/q) \in \mathscr{R}_\gamma$. For $1 \le s \le M(\mathcal{N})$, define the linear transformation $\mathcal{L}_s : \mathbb{R}^{d+m} \to \mathbb{R}^{\Lambda_s}$ by $\mathcal{L}_s(\xi, \eta) = \varrho^{\delta(s)} L_s(\xi)$. It follows from (4.4)–(4.8) and Lemma 2.5 that

$$\|T_{h,\Omega,\Phi,\Psi}(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^{d+m})} \le C\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^{d+m})}$$

for any $\alpha \in \mathbb{R}$ and $(1/p, 1/q) \in \mathscr{R}_\gamma$ (the point $(1/2, 1/2)$ may be obtained by interpolation). Thus, we can prove Theorem 1.4 (i) for Ω an H^1 atom on S^{n-1} satisfying (4.1)–(4.3). The above result, together with Lemma 4.1, yields Theorem 1.4 (i). □

5. Additional results. In this section, we highlight more general results than those previously shown. Let $\mathcal{G}$ be the set of all nonnegative (or non-positive) and monotonic $\mathcal{C}^1(\mathbb{R}^+)$ functions ϕ such that $\Upsilon_\phi(t) := \phi(t)/(t\phi'(t))$ with $|\Upsilon_\phi(t)| \le C_\phi$, where C_ϕ is a positive constant which depends only upon ϕ. Let Φ, Ψ and K be as in (1.3), and let $\varphi \in \mathcal{G}$. Define the singular integral operators $T_{h,\Omega,\Phi,\Psi,\varphi}$ by

(5.1)
$$T_{h,\Omega,\Phi,\Psi,\varphi}(f)(x) := \text{p.v.} \int_{\mathbb{R}^n} (f)(u - \Phi(\varphi(|y|)y'), v - \Psi(\varphi(|y|))K(y)\, dy,$$
$$(u, v) \in \mathbb{R}^d \times \mathbb{R}^m.$$

In what follows, let $\mathscr{R}_\gamma$ be as in Theorem 1.1. We have the following general results.

Theorem 5.1. *Let $n = d$, $\Phi(y) = (P_1(|y|)y'_1, \dots, P_n(|y|)y'_n)$ with $(P_1, \dots, P_d) \in (\mathcal{A}_1)^d$, $\Psi = (Q_1, \dots, Q_m) \in (\mathcal{A}_1)^m$ and $\varphi \in \mathcal{G}$. Suppose that one of the following conditions holds*:

(a) *$h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\Omega \in L\log^+ L(S^{n-1})$ satisfying* (1.2);
(b) *$h \in \mathscr{H}_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$ satisfying* (1.2).

Then,

(i) *$T_{h,\Omega,\Phi,\Psi,\varphi}$ is bounded on $\dot{F}_\alpha^{p,q}(\mathbb{R}^{d+m})$ for $\alpha \in \mathbb{R}$ and $(1/p, 1/q) \in \mathscr{R}_\gamma$.*

(ii) $T_{h,\Omega,\Phi,\Psi,\varphi}$ *is bounded on* $\dot{B}^{p,q}_{\alpha}(\mathbb{R}^{d+m})$ *for* $\alpha \in \mathbb{R}$ *and* p, q *satisfying* $1 < q < \infty$ *and* $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.

Theorem 5.2. *Let* $\Phi = (P_1, \ldots, P_d) \in (\mathcal{A}_n)^d$, $\Psi = (Q_1, \ldots, Q_m) \in (\mathcal{A}_1)^m$ *and* $\varphi \in \mathcal{G}$. *Suppose that one of the following conditions holds*:

(a) $h \in \Delta_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$ *and* $\Omega \in H^1(S^{n-1})$ *satisfying* (1.2)*;*
(b) $h \in \mathscr{H}_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$ *and* $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$ *satisfying* (1.2)*.*

Then,

(i) $T_{h,\Omega,\Phi,\Psi,\varphi}$ *is bounded on* $\dot{F}^{p,q}_{\alpha}(\mathbb{R}^{d+m})$ *for* $\alpha \in \mathbb{R}$ *and* $(1/p, 1/q) \in \mathscr{R}_\gamma$.
(ii) $T_{h,\Omega,\Phi,\Psi,\varphi}$ *is bounded on* $\dot{B}^{p,q}_{\alpha}(\mathbb{R}^{d+m})$ *for* $\alpha \in \mathbb{R}$ *and* p, q *satisfying* $1 < q < \infty$ *and* $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.

The proofs of Theorems 5.1 and 5.2 are based on combining similar arguments as in the proofs of [**13,** Theorem 1.1] and [**22,** Theorem 1.4] with Theorems 1.1 and 1.4–1.5. We omit the details.

It follows immediately from (1.6)–(1.8) and Theorems 5.1 and 5.2 that

Corollary 5.3. *Under the same conditions as Theorems* 5.1 *and* 5.2 *with* $\alpha > 0$, *the operator* $T_{h,\Omega,\Phi,\Psi,\varphi}$ *is also bounded on* $F^{p,q}_{\alpha}(\mathbb{R}^{d+m})$ *and* $B^{p,q}_{\alpha}(\mathbb{R}^{d+m})$, *respectively.*

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