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ZETA FUNCTIONS AND IDEAL CLASSES OF QUATERNION ORDERS

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ABSTRACT. Inspired by Stark's analytic proof of class number finiteness of a ring of integers in an algebraic number field, we give a new proof of the finiteness of the number of classes of ideals in a maximal order of a totally definite quaternion algebra over a totally real number field. Our proof makes use of Epstein zeta function properties. This approach leads to alternative proofs of Eichler's mass formula and even parity of the number of ramified primes in the quaternion algebra.

1. Introduction. In **[10]**, Stark gives an analytic proof of finiteness of the number of ideal classes in the ring of integers of an algebraic number field. Here, we observe that Stark's method can be applied to maximal orders in totally definite quaternion algebras over totally real algebraic number fields, providing a new proof of the standard result that the number of ideal classes is finite in this situation as well. The key is Theorem 5.1, stating that a partial zeta function for a quaternion algebra over a totally real field of degree m may be expressed as a product of gamma factors, exponential factors and a function obtained as the average over all v in an $m-1$-dimensional cube of Epstein zeta functions for quadratic forms parameterized by v. This expression, based on Hecke's method **[7]**, may be of independent interest. As another application, we note that it leads to proofs of Eichler's mass formula and the even parity of the number of ramified primes in the quaternion algebra in Theorems 5.3 and 5.4. These results are standard. Weil provides adelic proofs in **[15]**. Here we provide an alternative approach using some classical analytic methods. Note that

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the Jordan-Zassenhaus theorem provides an algebraic proof of a more general finiteness result, see [**9,** Chapter 26].

The approach used is first to decompose the zeta function of a maximal quaternion order into a sum of partial zeta functions, one for each ideal class. Theorem 5.1 expresses these partial zeta functions in terms of Epstein zeta functions for quadratic forms attached to ideals in the quaternion order. Using formulas for the Epstein zeta functions involving incomplete gamma functions allows showing that their values at real arguments greater than one are bounded away from zero. This yields a positive lower bound on the value of each partial zeta function at such an argument. A determinant computation shows that this lower bound is independent of the ideal class. Then, since the sum over all ideal classes is finite, there must be a finite number of ideal classes. With this background, we are able to equate the zeta function with the finite sum of partial zeta functions at negative values of the argument as well. Eichler's mass formula and the even parity of the number of ramified primes follow from letting the argument approach zero.

2. Preliminaries on orders. In this section, we review the necessary background, see [**9, 13, 14**], for details.

2.1. Orders over rings of algebraic integers. Let F be an algebraic number field of degree m over $\mathbb{Q}$, and denote its ring of integers by $\mathcal{O} = \mathcal{O}_F$. Also, let A be a finite-dimensional, simple F-algebra, and let Λ be an $\mathcal{O}$-order Λ in A. Then, Λ is left (and right) Noetherian as a ring and as an $\mathcal{O}$-module.

We may also view A as a $\mathbb{Q}$-algebra; it is finite-dimensional over $\mathbb{Q}$. Similarly, Λ is a finitely generated, free $\mathbb{Z}$-module. The same applies for any left ideal I of Λ.

The usual norm map $N_{A/F}$ from A to F is multiplicative, and the trace $T_{A/F}$ is additive. For $\gamma \in \Lambda$, we have that γ is integral over $\mathcal{O}$, and hence, $N_{A/F}$ and $T_{A/F}$ restrict to maps from Λ to $\mathcal{O}$.

2.2. Ideals in maximal orders. Hereon, we assume that Λ is a fixed maximal $\mathcal{O}$-order of the simple F-algebra A and that I is a left ideal of Λ with finite index $(\Lambda : I)$. Then, I is a right ideal for a possibly different maximal $\mathcal{O}$-order $\Lambda_R(I)$. The quotient Λ/I is a finite left module over the Dedekind domain $\mathcal{O} = \mathcal{O}_F$. By the structure theorem for such modules, Λ/I is isomorphic to $(\oplus_i \mathcal{O}/\mathfrak{a}_i)$ for a finite set of nonzero ideals $\{\mathfrak{a}_i\}$ of $\mathcal{O}$. We conclude from the multiplicativity of ideal norms in $\mathcal{O}$ that the cardinality

$$|\Lambda/I| = (\Lambda : I) = \prod_i (\mathcal{O} : \mathfrak{a}_i) = \left(\mathcal{O} : \prod_i \mathfrak{a}_i\right).$$

The norm of I to $\mathcal{O}$ is defined as $\mathrm{N}_{A/F}(I) = \prod_i \mathfrak{a}_i$; thus,

$$(\Lambda : I) = (\mathcal{O} : N_{A/F}(I)).$$

The norm $N_{F/\mathbb{Q}}$ is multiplicative from ideals of $\mathcal{O}$ to ideals of $\mathbb{Z}$ and can be interpreted in a similar way. A may also be viewed as a simple $\mathbb{Q}$-algebra; therefore, it is easy to see that Λ is also a maximal $\mathbb{Z}$-order in A. Thus, we have the following.

Proposition 2.1. *With Λ and I as specified at the beginning of this section,*

$$(\Lambda : I) = (\mathbb{Z} : N_{F/\mathbb{Q}}(N_{A/F}(I))) = (\mathbb{Z} : N_{A/\mathbb{Q}}(I)).$$

Two left ideals I and J of Λ are defined to be in the same ideal class if $I = J\gamma$ for some γ in the multiplicative group $A^\times$. The ideal class of J will be denoted $[J]$. The following basic facts will be used later.

Proposition 2.2. *For a left ideal J of finite index in Λ and an element $\gamma \in A$, we have*:

(i) *$J = J\gamma$ if and only if γ is a unit of $\Lambda_R(J)$.*
(ii) *If γ is a unit in A and $J\gamma \subset \Lambda$, then*

$$N_{A/F}(J\gamma) = N_{A/F}(J)\,\mathrm{N}_{A/F}(\gamma).$$

Proof. In order to establish the first statement, suppose that γ is a unit of $\Lambda_R(J)$ with inverse η. Then $J\gamma \subset J = J \cdot 1 = J \cdot \eta\gamma \subset J\gamma$; thus, $J\gamma = J$. Conversely, suppose that $J\gamma = J$. Then, by tensoring over $\mathcal{O}$ with F, we have $A\gamma = A$. Thus, γ is a unit of A with inverse η. Note that it is a two-sided inverse, since A is left Noetherian, see **[9,** Theorem 6.4]. Then, $J = J \cdot 1 = J\gamma\eta = J\eta$, which shows that $\eta \in \Lambda_R(J)$.

For the second statement, see **[9,** Theorems 24.2–24.5]. $\square$

We now consider the inverse of the left ideal I of Λ with finite index $(\Lambda : I) = |\Lambda/I| = c$. Define $I^{-1} = \{a \in A : Ia \subset \Lambda\}$. Note that $c \in I$; thus, $cI^{-1} \subset \Lambda$.

Proposition 2.3.

(i) *I^{-1} is a right Λ-module, and cI^{-1} is a right ideal of Λ of finite index. Both are left $\Lambda_R(I)$-modules. They are free $\mathbb{Z}$-modules of rank equal to the dimension of A as a vector space over $\mathbb{Q}$.*

(ii) *The $\mathcal{O}$-module consisting of all finite sums of elements of the form ab with $a \in I$ and $b \in I^{-1}$ is $I \cdot I^{-1} = \Lambda$.*

(iii) *$(I^{-1} : \Lambda) = (\Lambda : I)$.*

Proof. For the first two statements, see [**9,** Theorem 22.7 and the discussion preceding it]. The third uses [**9,** Theorem 24.5], giving the properties of the extension $N^*_{A/\mathbb{Q}}$ of $N_{A/\mathbb{Q}}$ to $\mathbb{Q}$-multiples of ideals. $\square$

2.3. Reduced traces and norms in quaternion algebras. Hereon, we assume that A is the quaternion algebra over F:

$$A = \left(\frac{a, b}{F}\right),$$

where a and b are elements of F. This means that A is the algebra with center F and F-basis $\{1, i, j, k\}$ in which $i^2 = a$, $j^2 = b$, and $ji = -ij = -k$; thus, $k^2 = -ab$. We can multiply a and b by nonzero squares in F without changing the isomorphism class of A; thus, we will also assume that a and b are elements of $\mathcal{O} = \mathcal{O}_F$. Note that the $\mathbb{Z}$-rank of Λ or any ideal of finite index in Λ now equals $\dim_{\mathbb{Q}}(A) = 4[F : \mathbb{Q}] = 4m$.

Quaternion algebra A has a standard involution sending $\gamma = x + yi + zj + wk$, with x, y, z, w in F to $\overline{\gamma} = x - yi - zj - wk$. The reduced trace of γ is the additive function

$$\operatorname{trd}(\gamma) = \gamma + \overline{\gamma} = 2x \in F,$$

and the reduced norm is the multiplicative function

$$\operatorname{nrd}(\gamma) = \gamma\overline{\gamma} = x^2 - ay^2 - bz^2 + abw^2 \in F.$$

Basic relations are $T_{A/F}(\gamma) = 2\operatorname{trd}(\gamma)$ and $\mathrm{N}_{A/F}(\gamma) = \operatorname{nrd}(\gamma)^2$. If $\gamma \in \Lambda$, then γ is integral over $\mathcal{O}$, and hence, $\operatorname{trd}(\gamma)$ and $\operatorname{nrd}(\gamma)$ lie in

$\mathcal{O} \subset \Lambda$. Thus, $\overline{\gamma} = \operatorname{trd}(\gamma) - \gamma \in \Lambda$. Therefore, the standard involution also maps Λ to itself.

Proposition 2.4. *For an ideal J of Λ of finite index and an element $\gamma \in A^\times$ for which $J\gamma \subset \Lambda$, we have $(\Lambda : J\gamma) = (\Lambda : J) N_{F/\mathbb{Q}}(\operatorname{nrd}(\gamma))^2$.*

Proof. Multiplication by a positive integer allows $\gamma \in \Lambda$. The result can then be seen to follow from an application of Propositions 2.1 and 2.2. □

2.4. Units in maximal orders of totally definite quaternion algebras. Hereon, we assume that F is a totally real field and A is a totally definite quaternion algebra over F. This means that

$$A = \left(\frac{a, b}{F}\right),$$

where a and b are totally negative elements of F. In particular, A is a division algebra. The zeta function of the quaternion algebra is of greatest interest in this case. Now, every non-zero left ideal I of Λ contains a non-zero principal left ideal. Since there are no zero-divisors, such an ideal has the same $\mathbb{Z}$-rank as Λ, namely, $4[F : \mathbb{Q}] = 4m$, and hence, is of finite index. Thus, every non-zero left ideal is of finite index.

Units $\Lambda^\times$ of Λ map to the units of $\mathcal{O} = \mathcal{O}_F$ via the reduced norm map, with the kernel denoted $\Lambda_1^\times$. This leads to the exact sequence

$$1 \longrightarrow \frac{\Lambda_1^\times}{\{\pm 1\}} \longrightarrow \frac{\Lambda^\times}{\mathcal{O}^\times} \longrightarrow \frac{\mathcal{O}^\times}{(\mathcal{O}^\times)^2}.$$

From this and the Dirichlet unit theorem for F, we conclude that $|\Lambda^\times/\mathcal{O}^\times|$ divides $2^m|\Lambda_1^\times/\{\pm 1\}|$, if the latter is finite. This is indeed the case in our totally definite quaternion algebra A, as explained now. Just as $\mathcal{O}$ embeds discretely in $F \otimes_{\mathbb{Q}} \mathbb{R}$, Λ embeds discretely in $A \otimes_{\mathbb{Q}} \mathbb{R} \cong (A \otimes_F \mathbb{R}) \otimes_{\mathbb{Q}} F \cong (A \otimes_F \mathbb{R})^m$. Now, since a and b are totally negative,

$$A \otimes_F \mathbb{R} \cong \left(\frac{a, b}{\mathbb{R}}\right) \cong \mathbb{H},$$

the Hamiltonian quaternions. Consequently, if we denote the Hamiltonian quaternions of reduced norm 1 by $\mathbb{H}_1^\times$, then $\Lambda_1^\times$ embeds discretely in $(\mathbb{H}_1^\times)^m$. Next, $\mathbb{H}_1^\times \cong SU_2(\mathbb{C})$ is a compact group; thus, $\Lambda_1^\times$ is finite. With this in mind, we see that $\Lambda_1^\times/\{\pm 1\}$ is isomorphic to a finite subgroup of $\mathbb{H}_1^\times/\{\pm 1\} \cong SO_3(\mathbb{R})$.

Proposition 2.5. *There exists a constant C_m dependent only upon m such that, if Λ is a maximal $\mathcal{O}_F$-order in the quaternion division algebra*

$$A = \left(\frac{a, b}{F}\right),$$

where a and b are totally negative elements of a totally real field F with $[F : \mathbb{Q}] = m$, then

$$\left|\frac{\Lambda^\times}{\mathcal{O}_F^\times}\right| \leq C_m.$$

Proof. We may in fact take $C_m = 2^{m+5}m^2$. Since $|\Lambda^\times/\mathcal{O}^\times|$ divides $2^m|\Lambda_1^\times/\{\pm 1\}|$, it suffices to show that $|\Lambda_1^\times/\{\pm 1\}| \leq 32m^2$. Now, $\Lambda_1^\times/\{\pm 1\}$ is isomorphic to a finite subgroup of $SO_3(\mathbb{R})$. Finite subgroups of $SO_3(\mathbb{R})$ are either cyclic of order n, dihedral of order $2n$ or isomorphic to the symmetry group of a Platonic solid. Suppose that γ in $\Lambda_1^\times$ has order n. Since $\gamma \in A$, $F[\gamma]$ is a field extension of F of relative degree 1 or 2. Thus, γ is an nth root of unity in a field of degree $2[F : \mathbb{Q}] = 2m$; thus,

$$\begin{aligned} 2m \geq \phi(n) = n \prod_{p|n} \frac{p-1}{p} &\geq \frac{n}{\sqrt{2}} \prod_{p|n} \frac{\sqrt{p}}{p} \\ &= \frac{n}{\sqrt{2}} \prod_{p|n} \frac{1}{\sqrt{p}} \geq \frac{n}{\sqrt{2}} \frac{1}{\sqrt{n}} \\ &= \sqrt{\frac{n}{2}}. \end{aligned}$$

This yields $n \leq 8m^2$. Thus, if $\Lambda_1^\times/\{\pm 1\}$ is cyclic or dihedral, its order is at most $2n$, which is bounded by $16m^2$; otherwise, its order is bounded by $|A_5| = 60$. The larger of these two is $16m^2$ unless $F = \mathbb{Q}$, in which case we cannot have an element of order $n = 5$ by considerations similar to those above. The largest possible group in this case is then S_4 of order 24. Therefore, $\Lambda_1^\times/\{\pm 1\}$ has order at most $24m^2 < 32m^2$ in all cases. $\square$

2.5. Quadratic forms. For our purposes, a key property of the reduced norm is that it defines a quadratic form from which other quadratic forms may be obtained. Our assumption that

$$A = \left(\frac{a, b}{F}\right),$$

with two totally negative elements a and b in the totally real field F implies that the quadratic form is positive-definite. Indeed, set $q(\gamma) = \mathrm{nrd}(\gamma)$ for $\gamma \in A$, and observe that, for $0 \neq \gamma = x + yi + zj + wk$ with x, y, z, w in F, we have $q(\gamma) = x^2 - ay^2 - bz^2 + abw^2$, which is totally positive because x^2, y^2, w^2 and z^2 are all either totally positive or zero, but not all zero. Thus, for each embedding σ_i of F in $\mathbb{R}$, $\sigma_i(q(\gamma))$ is a positive-definite quadratic form on the vector space A over $\mathbb{Q}$, with values in $\mathbb{R}$. Further, a linear combination of quadratic forms is again a quadratic form, and we note that, if $t_1, t_2, \ldots t_n$ are positive real numbers, then

$$\sum_{i=1}^{n} t_i \sigma_i(q(\gamma))$$

is a positive-definite quadratic form on A over $\mathbb{Q}$.

2.6. Completions. For each non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}$, let $F_\mathfrak{p}$ denote the completion of F at $\mathfrak{p}$. The completion of the totally definite quaternion algebra A at $\mathfrak{p}$ is $A_\mathfrak{p} = A \otimes_F F_\mathfrak{p}$. The prime $\mathfrak{p}$ is defined as the split in A if $A_\mathfrak{p}$ is isomorphic to the matrix ring $M_2(F_\mathfrak{p})$; otherwise, $\mathfrak{p}$ is ramified in A. Similarly, let $\mathcal{O}_\mathfrak{p}$ denote the completion of $\mathcal{O} = \mathcal{O}_F$ at $\mathfrak{p}$, and let $\Lambda_\mathfrak{p} \cong \Lambda \otimes_\mathcal{O} \mathcal{O}_\mathfrak{p}$ denote the corresponding completion of Λ. It turns out that the prime $\mathfrak{p}$ is ramified in the maximal order Λ if and only if $\Lambda_\mathfrak{p}$ is not isomorphic to the ring $M_2(\mathcal{O}_\mathfrak{p})$ of two-by-two matrices with entries in $\mathcal{O}_\mathfrak{p}$. Equivalently, $\Lambda_\mathfrak{p}$ is the unique maximal $\mathcal{O}_\mathfrak{p}$-order in the unique quaternion division algebra over the complete local field $F_\mathfrak{p}$. There are finitely many ramified primes, and the square of their product D_Λ is the discriminant of Λ. This discriminant does not depend upon the choice of the amaximal $\mathcal{O}$-order in A; thus, it is also called the discriminant of A, see [**9,** Section 25].

3. Zeta functions. We now describe the zeta-functions of quaternion orders, Epstein zeta-functions of quadratic forms and a relation between the two. For the basic theory of zeta-functions of orders, see

[**2, 13, 14**]. For a comprehensive treatment in the case of $F = \mathbb{Q}$, see [**4**]. For a primer on Epstein zeta-functions, see [**12**]. As usual in this setting, s will denote a complex variable and $\Re(s)$ its real part. Recall that we are now assuming $A = (a, b)/F$ for totally negative elements a and b in F, although the definition of the zeta-function of A can be made more generally.

3.1. Zeta functions of maximal orders in quaternion division algebras. For the fixed maximal $\mathcal{O}_F$-order Λ in A, the zeta function of Λ is defined as

$$\zeta_\Lambda(s) = \sum_I (\Lambda : I)^{-s},$$

where I runs through all left Λ-ideals of finite index. This sum converges absolutely and defines ζ_Λ as an analytic function for the real part of s satisfying $\Re(s) > 1$. A proof of the convergence is given in [**2,** Chapter VII], for example, but, because it is essential to our main result, this section includes a sketch of this fact in our case of interest.

Although the ideal theory of Λ requires consideration of other maximal orders of A, consider the reduced norms of ideals of Λ, which lie in the Dedekind domain $\mathcal{O}$. (The reduced norm of an ideal is generated by the reduced norms of its elements.) This allows us to show that ζ_Λ has an Euler product expansion, the product being over non-zero prime ideals $\mathfrak{p}$ of $\mathcal{O}$. The zeta function of $\Lambda_\mathfrak{p}$, $\zeta_{\Lambda_\mathfrak{p}}(s)$, is defined in the same way as for Λ: as the sum of the index of each non-zero left ideal raised to the $-s$ power. For each non-zero ideal $\mathfrak{m}$ of $\mathcal{O}$, let $a_\mathfrak{m}$ denote the number of left ideals of Λ whose reduced norm is $\mathfrak{m}$. Using completions and the fact that left ideals of $\Lambda_\mathfrak{p}$ are principal for each prime $\mathfrak{p}$ allows for $a_{\mathfrak{n}\mathfrak{m}} = a_\mathfrak{n} a_\mathfrak{m}$ whenever $\mathfrak{n}$ and $\mathfrak{m}$ are relatively prime. This is sufficient to give

$$\zeta_\Lambda(s) = \prod_\mathfrak{p} \zeta_{\Lambda_\mathfrak{p}}(s).$$

By direct comparison of Euler factors, a formula involving the Dedekind zeta function $\zeta_F(s)$ of F is obtained:

$$\zeta_\Lambda(s) = \zeta_F(2s)\zeta_F(2s-1) \prod_{\mathfrak{p} | D_\Lambda} (1 - (\mathcal{O} : \mathfrak{p})^{1-2s}). \tag{3.1}$$

By standard comparison with a power of the Riemann zeta function, the Euler product for the right hand side of this equation converges absolutely for $\Re(s) > 1$, and hence, the same applies for $\prod_{\mathfrak{p}} \zeta_{\Lambda_{\mathfrak{p}}}(s)$. This, in turn, implies that the sum $\zeta_\Lambda(s) = \sum_I (\Lambda : I)^{-s}$ converges absolutely to an analytic function for $\Re(s) > 1$, see [**13,** Chapter III] or [**14,** Chapter 17] for details.

3.2. Epstein zeta functions. Let Q be a positive-definite quadratic form on rational k-space $\mathbb{Q}^k$. An element of $\mathbb{Q}^k$ will be represented by a column vector $\mathbf{v}$, and $\mathbf{v}^t$ stands for its transpose. There is a unique positive-definite symmetric matrix P such that $Q(\mathbf{v}) = \mathbf{v}^t P\mathbf{v}$ for all $\mathbf{v} \in \mathbb{Q}^k$. Thus, P is the matrix of Q.

The Epstein zeta function of Q is of the complex variable s, defined as

$$Z_Q(s) = \frac{1}{2} \sum_{0 \neq \mathbf{v} \in \mathbb{Z}^k} Q(\mathbf{v})^{-s}.$$

Epstein introduced these zeta functions as generalizations of the Riemann zeta function [**5, 6**]. They can be used to express Dedekind zeta functions of number fields, and prove some of their properties, as seen in [**12**]. We will show in Theorem 5.1 that they can similarly be used to express zeta functions of quaternion orders.

We list the following properties, of which the first two are of primary interest.

Proposition 3.1.

(i) *The sum* $Z_Q(s) = (1/2)\sum_{0\neq\mathbf{v}\in\mathbb{Z}^k} Q(\mathbf{v})^{-s}$ *converges absolutely to an analytic function for* $\Re(s) > k/2$.

(ii) *Let* $\Gamma(s)$ *denote the gamma function and* $G(s,x)$ *the incomplete gamma function,*

$$G(s,x) = \int_1^\infty y^{s-1} e^{-xy}\, dy,$$

for $x > 0$ *and all complex* s. *This yields the expansion*

$$\Phi_P(s) = \pi^{-s}\Gamma(s) Z_Q(s) = \frac{\det(P)^{-1/2}}{2s-k} - \frac{1}{2s}$$

$$+\frac{1}{2}\sum_{0\neq \mathbf{v}\in\mathbb{Z}^k}\left(G(s,\pi Q(\mathbf{v}))+\frac{G((k/2)-s,\pi\mathbf{v}^tP^{-1}\mathbf{v})}{\det(P)^{1/2}}\right).$$

(iii) $Z_Q(s) > \pi^s[(\det(P)^{-1/2})/(2s-k) - 1/2s]/\Gamma(s)$ *for real* $s > k/2$.
(iv) $Z_Q(s)$ *has a meromorphic continuation to the complex plane.*
(v) $Z_Q(0) = -1/2$, *and the only pole of* Z_Q *is at* $s = k/2$. *It is a simple pole with residue* $(\pi^{k/2}\Gamma(k/2))/2\sqrt{\det(P)}$.
(vi) $Z_Q(s)$ *satisfies the functional equation* $\Phi_P(s) = \det(P)^{-1/2}\Phi_{P^{-1}}$ $(k/2) - s$.

Proof. See [**12,** Chapter 1]. The last four statements are direct consequences of the incomplete gamma function expansion. ☐

3.3. Partial zeta functions. The zeta function of a left ideal class $[J]$ in Λ is defined as

$$\zeta_{\Lambda,[J]}(s) = \sum_{I\in[J]}(\Lambda : I)^{-s}. \tag{3.2}$$

By comparison with $\zeta_\Lambda(s)$, this also converges absolutely for $\Re(s) > 1$. Indeed, the absolute convergence of the sum for $\zeta_\Lambda(s)$ implies that

$$\zeta_\Lambda(s) = \sum_{[J]}\zeta_{\Lambda,[J]}(s) \tag{3.3}$$

for $\Re(s) > 1$. Note that this sum is over all non-zero ideal classes $[J]$, and this equality will be used to prove that the number of ideal classes is finite.

Now, for each $I \in [J]$, we have $I = J\gamma$ for some $\gamma \in J^{-1}$. In fact, $J\gamma_1 = J\gamma_2$ if and only if $\gamma_1 = u\gamma_2$ for some $u \in \Lambda_R(J)^\times$, by Proposition 2.2. Let $w_J = |\Lambda_R(J)^\times/\mathcal{O}^\times|$, so $w_J < C_m = 2^{m+5}m^2$, by Proposition 2.5. Then, if we let $J^{-1}/\mathcal{O}^\times$ denote a set of representatives for the nonzero elements of J^{-1} under the equivalence relation for which the equivalence class of a nonzero $\gamma \in J^{-1}$ is $\gamma\mathcal{O}^\times$, there is a w_J-to-one mapping from $J^{-1}/\mathcal{O}^\times$ onto $[J]$ defined by $\gamma \to J\gamma$, as $\Lambda_R(J)^\times\gamma$ is a union of w_J equivalence classes $\gamma\mathcal{O}^\times = \mathcal{O}^\times\gamma$. We find that

$$\zeta_{\Lambda,[J]}(s) = \frac{1}{w_J}\sum_{\gamma\in J^{-1}/\mathcal{O}^\times}(\Lambda : J\gamma)^{-s},$$

again converging absolutely for $\Re(s) > 1$.

Then, using Proposition 2.4, this becomes

$$\zeta_{\Lambda,[J]}(s) = \frac{(\Lambda : J)^{-s}}{w_J} \sum_{\gamma \in J^{-1}/\mathcal{O}^\times} N_{F/\mathbb{Q}}(\mathrm{nrd}(\gamma))^{-2s}, \tag{3.4}$$

converging absolutely for $\Re(s) > 1$.

Now, we let $\{\sigma_1, \ldots, \sigma_m\}$ denote the set of embeddings of F in $\mathbb{R}$. Then, we can express

$$N_{F/\mathbb{Q}}(\mathrm{nrd}(\gamma)) = \prod_{i=1}^{m} \sigma_i(\mathrm{nrd}(\gamma)).$$

Recall that each $\sigma_i(\mathrm{nrd}(\gamma))$ is positive, as seen in subsection 2.5.

Let $\mathbb{R}^+$ denote the positive real numbers and $\mathbb{Z}^+$ the positive integers. For any complex number s with $\Re(s) > 0$, any real number $c \in \mathbb{R}^+$ and any integer $g \in \mathbb{Z}^+$, we have a standard relation

$$\frac{\Gamma(gs)}{c^{gs}} = \int_0^\infty y^{gs} e^{-cy} \frac{dy}{y}, \tag{3.5}$$

in which we set $g = 1$ and $c = \sigma_i(\mathrm{nrd}(\gamma))$. Thus, for each i,

$$\frac{\Gamma(s)}{\sigma_i(\mathrm{nrd}(\gamma))^s} = \int_{\mathbb{R}^+} y_i^s e^{-\sigma_i(\mathrm{nrd}(\gamma)) y_i} \frac{dy_i}{y_i},$$

and hence, upon multiplying these factors together for $i = 1 \ldots m$, and using Tonelli's theorem:

$$\frac{\Gamma(s)^m}{N_{F/Q}(\mathrm{nrd}(\gamma))^s} = \int_{(\mathbb{R}^+)^m} (y_1 \cdots y_m)^s e^{-\sum_{i=1}^m \sigma_i(\mathrm{nrd}(\gamma)) y_i} \frac{dy_1 \cdots dy_m}{y_1 \cdots y_m}.$$

Applying Dirichlet's unit theorem, let $\varepsilon_1, \ldots, \varepsilon_{m-1}$ be fundamental units of $\mathcal{O}$. Following Hecke [**7**], we change to the variables $\mathbf{x} \in \mathbb{R}^{m-1}$ and $u \in \mathbb{R}^+$ where $\mathbf{x}$ has components $x_1, \ldots, x_{m-1}$ and

$$y_i = u \prod_{j=1}^{m-1} |\sigma_i(\varepsilon_j)|^{2x_j} = u\tau_i(\mathbf{x}),$$

for each i. Then

$$\frac{\partial y_i}{\partial x_j} = 2 \log |\sigma_i(\varepsilon_j)| y_i \quad \text{and} \quad \frac{\partial y_i}{\partial u} = \frac{y_i}{u}.$$

Thus, in the Jacobian matrix, we can factor out y_i from the ith row for each i, factor out a 2 from the jth column for $1 \le j \le m-1$ and factor out $1/u$ from the last column. In the remaining matrix, adding all of the rows to the last row produces a new matrix whose (i,j) entry is $\log|\sigma_i(\varepsilon_j)|$ for $1 \le i,\, j \le m-1$, and whose last row has every entry equal to $\log|\pm 1| = 0$ except for a final entry of $1 + \cdots + 1 = m$. Thus, the Jacobian determinant is $(2^{m-1} m \prod_{i=1}^m y_i/u) R_F$, where R_F is the regulator of F.

Since

$$\left| \prod_i \sigma_i(\varepsilon_j) \right| = |N_{F/Q}(\varepsilon_j)| = |\pm 1| = 1,$$

the result of this change of variables is:

$$\frac{\Gamma(s)^m}{N_{F/Q}(\mathrm{nrd}(\gamma))^s} = 2^{m-1} m R_F \cdot \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^+} u^{ms} e^{-\sum_{i=1}^m \sigma_i(\mathrm{nrd}(\gamma))\tau_i(\mathbf{x})u} \frac{du}{u}\, d\mathbf{x}.$$

By equation (3.5) with $g = m$, we can perform the integral over u and arrive at

$$\frac{\Gamma(s)^m}{N_{F/Q}(\mathrm{nrd}(\gamma))^s} = 2^{m-1} m R_F \Gamma(ms) \cdot \int_{\mathbb{R}^{m-1}} \left(\sum_{i=1}^m \sigma_i(\mathrm{nrd}(\gamma))\tau_i(\mathbf{x}) \right)^{-ms} d\mathbf{x}.$$

Replacing s by $2s$ and summing over $\gamma \in J^{-1}/\mathcal{O}^\times$, we see from this and equation (3.4) that

$$\Gamma(2s)^m \zeta_{\Lambda,[J]}(s) = \frac{2^{m-1} m R_F}{w_J} \frac{\Gamma(2ms)}{(\Lambda : J)^s} \cdot \sum_{\gamma \in J^{-1}/\mathcal{O}^\times} \int_{\mathbb{R}^{m-1}} \left(\sum_{i=1}^m \sigma_i(\mathrm{nrd}(\gamma))\tau_i(\mathbf{x}) \right)^{-2ms} d\mathbf{x}, \tag{3.6}$$

the sum converging absolutely for $\Re(s) > 1$.

We may write the absolutely convergent integral here as

$$(3.7)\quad \sum_{\lambda\in\mathbb{Z}^{m-1}}\int_{[-1/2,1/2]^{m-1}+\lambda}\left(\sum_{i=1}^{m}\sigma_i(\mathrm{nrd}(\gamma))\tau_i(\mathbf{x})\right)^{-2ms}d\mathbf{x}$$
$$= \sum_{\lambda\in\mathbb{Z}^{m-1}}\int_{[-1/2,1/2]^{m-1}}\left(\sum_{i=1}^{m}\sigma_i(\mathrm{nrd}(\gamma))\tau_i(\mathbf{x}+\lambda)\right)^{-2ms}d\mathbf{x},$$

by changing variables via simple translations.

Since

$$\tau_i(\mathbf{x}) = \prod_{j=1}^{m-1}|\sigma_i(\varepsilon_j)|^{2x_j},$$

clearly $\tau_i(\mathbf{x}+\lambda) = \tau_i(\mathbf{x})\tau_i(\lambda)$ for $\lambda\in\mathbb{Z}^{m-1}$ with components $\lambda_1\cdots\lambda_{m-1}$. Now,

$$\tau_i(\lambda) = \sigma_i\left(\prod_j \varepsilon_j^{\lambda_j}\right)^2 = \sigma_i(\varepsilon_\lambda^2),$$

where $\varepsilon_\lambda = \prod_j \varepsilon_j^{\lambda_j}$ is a unit of $\mathcal{O}$. Since the set $\{\varepsilon_j\}$ consists of fundamental units of $\mathcal{O}$, the elements $\pm\,\varepsilon_\lambda$ run through $\mathcal{O}^\times$ as λ runs through $\mathbb{Z}^{m-1}$. The integral in equation (3.6) thus becomes

$$(3.8)\quad \sum_{\lambda\in\mathbb{Z}^{m-1}}\int_{[-1/2,1/2]^{m-1}}\left(\sum_{i=1}^{m}\sigma_i(\mathrm{nrd}(\gamma)\,\varepsilon_\lambda^2)\tau_i(\mathbf{x})\right)^{-2ms}d\mathbf{x}$$
$$= \frac{1}{2}\sum_{\varepsilon\in\mathcal{O}^\times}\int_{[-1/2,1/2]^{m-1}}\left(\sum_{i=1}^{m}\sigma_i(\mathrm{nrd}(\gamma)\,\varepsilon^2)\tau_i(\mathbf{x})\right)^{-2ms}d\mathbf{x}$$
$$= \frac{1}{2}\sum_{\varepsilon\in\mathcal{O}^\times}\int_{[-1/2,1/2]^{m-1}}\left(\sum_{i=1}^{m}\sigma_i(\mathrm{nrd}(\gamma\,\varepsilon))\tau_i(\mathbf{x})\right)^{-2ms}d\mathbf{x},$$

since $\mathrm{nrd}(\gamma\,\varepsilon) = \mathrm{nrd}(\gamma)\mathrm{nrd}(\varepsilon) = \mathrm{nrd}(\gamma)\,\varepsilon^2$ for $\varepsilon\in\mathcal{O}$.

Replacing the integral in equation (3.6) by this expression results in

$$(3.9)\quad \Gamma(2s)^m\zeta_{\Lambda,[J]}(s) = \frac{2^{m-2}mR_F}{w_J}\frac{\Gamma(2ms)}{(\Lambda:J)^s}.$$

$$\sum_{\substack{\gamma\in J^{-1}/\mathcal{O}^\times\\ \varepsilon\in\mathcal{O}^\times}} \int_{[-1/2,1/2]^{m-1}} \left(\sum_{i=1}^{m} \sigma_i(\mathrm{nrd}(\gamma\,\varepsilon))\tau_i(\mathbf{x})\right)^{-2ms} d\mathbf{x}$$
$$= \frac{2^{m-2}mR_F}{w_J}\frac{\Gamma(2ms)}{(\Lambda:J)^s}\cdot$$
$$\sum_{\gamma\in J^{-1}} \int_{[-1/2,1/2]^{m-1}} \left(\sum_{i=1}^{m} \sigma_i(\mathrm{nrd}(\gamma))\tau_i(\mathbf{x})\right)^{-2ms} d\mathbf{x},$$

again with absolute convergence of the sum for $\Re(s) > 1$.

For fixed $\mathbf{x}$, consider the function of γ that appears in equation (3.9):

$$q_{\mathbf{x}}(\gamma) = \sum_{i=1}^{m} \sigma_i(\mathrm{nrd}(\gamma))\tau_i(\mathbf{x}).$$

By subsection 2.5, $q_{\mathbf{x}}$ is a quadratic form on the $\mathbb{Q}$-space A and is positive-definite since each $t_i = \tau_i(\mathbf{x}) > 0$.

We are interested in the restriction of $q_{\mathbf{x}}$ to certain finitely-generated $\mathbb{Z}$-submodules $\mathcal{L}$ of A of rank $4m$. These are necessarily isomorphic to $\mathbb{Z}^{4m}$ as $\mathbb{Z}$-modules and will be called full *$\mathbb{Z}$-lattices* in A, or simply *lattices*. Important examples of lattices are Λ, J and J^{-1}, by Proposition 2.3. Given a lattice $\mathcal{L}$, we choose a $\mathbb{Z}$-module isomorphism $\phi_{\mathcal{L}} : \mathbb{Z}^{4m} \to \mathcal{L}$. It will be seen that our results do not depend upon the choice of isomorphism.

Extending scalars to $\mathbb{Q}$, we obtain a quadratic form $q_{\mathbf{x},J^{-1}} = q_{\mathbf{x}}\circ\phi_{J^{-1}}$ on $\mathbb{Q}^{4m}$. Rewriting equation (3.9) in light of these observations, we have

$$\Gamma(2s)^m\zeta_{\Lambda,[J]}(s) = \frac{2^{m-2}mR_F}{w_J}\Gamma(2ms)(\Lambda:J)^{-s} \cdot \sum_{\gamma\in J^{-1}} \int_{[-1/2,1/2]^{m-1}} q_{\mathbf{x}}(\gamma)^{-2ms} d\mathbf{x}$$
$$= \frac{2^{m-2}mR_F}{w_J}\Gamma(2ms)(\Lambda:J)^{-s} \cdot \sum_{\mathbf{v}\in\mathbb{Z}^{4m}} \int_{[-1/2,1/2]^{m-1}} q_{\mathbf{x},J^{-1}}(\mathbf{v})^{-2ms} d\mathbf{x}$$
$$= \frac{2^{m-2}mR_F}{w_J}\Gamma(2ms)(\Lambda:J)^{-s} \tag{3.10}$$

$$\cdot \int_{[-1/2,1/2]^{m-1}} \sum_{\mathbf{v}\in\mathbb{Z}^{4m}} q_{\mathbf{x},J^{-1}}(\mathbf{v})^{-2ms} d\mathbf{x},$$

the interchange of limit and summation being justified by the monotone convergence theorem for real s with $s > 1$, and hence, for complex s with $\Re(s) > 1$.

Now we note the appearance of an Epstein zeta function in equation (3.10). In order to normalize it, again let D_Λ denote the discriminant of Λ so that D_Λ is an ideal in $\mathcal{O}$. Let d_F denote the discriminant of F such that d_F is an integer. Put

$$d_{F,\Lambda} = \frac{d_F^4(\mathcal{O} : D_\Lambda)^{1/4m}}{2},$$

and define the normalized quadratic form

$$Q_{\mathbf{x},J^{-1}}(\mathbf{v}) = \frac{(\Lambda : J)^{1/2m}}{d_{F,\Lambda}}\, q_{\mathbf{x},J^{-1}}(\mathbf{v}). \tag{3.11}$$

We show in the next section (Corollary 4.4) that the symmetric matrix $P_{\mathbf{x},J^{-1}}$ of the positive-definite quadratic form $Q_{\mathbf{x},J^{-1}}$ has determinant 1. This fact is needed to establish a lower bound, independent of the ideal class, for values of partial zeta functions (Corollary 5.2). This independence leads to the main Theorem 5.3 on the finiteness of the number of ideal classes.

4. Computations with determinants and discriminants. In this section, we compute $\det(P_{\mathbf{x},J^{-1}}) = 1$. One may choose to accept this result for now and return later to this section. We begin by considering the matrix $p_{x,\mathcal{R}}$ of $q_{\mathbf{x},\mathcal{R}} = q_x \circ \phi_{\mathcal{R}}$ for a particular order $\mathcal{R} \subset A$ which is most amenable to computation. In general, we let $p_{x,\mathcal{L}}$ denote the matrix representing $q_{\mathbf{x},\mathcal{L}} = q_x \circ \phi_{\mathcal{L}}$.

Let $\mathcal{R} = \mathcal{O} \oplus \mathcal{O}i \oplus \mathcal{O}j \oplus \mathcal{O}k$, and let $\{f_1, \ldots, f_m\}$ be an integral basis of $\mathcal{O}$. Then, $\{f_1, \ldots, f_m, f_1 i, \ldots, f_m i, f_1 j, \ldots, f_m j, f_1 k, \ldots, f_m k\}$ is a free basis for $\mathcal{R}$ as a $\mathbb{Z}$-module. We use this ordered basis to define the $\mathbb{Z}$-module isomorphism $\phi_{\mathcal{R}} : \mathbb{Z}^{4m} \to \mathcal{R}$ in the standard way.

Since each $\gamma = x + yi + zj + wk$, with x, y, z, w in $\mathcal{O}$ has reduced norm

$$\mathrm{nrd}_{A/F}(\gamma) = \gamma\overline{\gamma} = x^2 - ay^2 - bz^2 + abw^2 \in F,$$

we have

(4.1)
$$\begin{aligned} q_{\mathbf{x}}(\gamma) &= \sum_{i=1}^{m} \sigma_i(\mathrm{nrd}(\gamma))\, \tau_i(\mathbf{x}) = \sum_{i=1}^{m} \sigma_i(x^2 - ay^2 - bz^2 + abw^2)\, \tau_i(\mathbf{x}) \\ &= \sum_{i=1}^{m} \sigma_i(x^2)\, \tau_i(\mathbf{x}) + \sum_{i=1}^{m} \sigma_i(-ay^2)\, \tau_i(\mathbf{x}) \\ &\quad + \sum_{i=1}^{m} \sigma_i(-bz^2)\, \tau_i(\mathbf{x}) + \sum_{i=1}^{m} \sigma_i(abw^2)\, \tau_i(\mathbf{x}) \end{aligned}$$

Now, for $\gamma = \phi_{\mathcal{R}}(v_1, \ldots, +v_{4m})$, we have $x = v_1 f_1 + \cdots + v_m f_m$, $y = v_{m+1} f_1 + \cdots + v_{2m} f_m$, $z = v_{2m+1} f_1 + \cdots + v_{3m} f_m$ and $w = v_{3m+1} f_1 + \cdots + v_{4m} f_m$. Thus, we find that $q_{\mathbf{x},\mathcal{R}} = q_{\mathbf{x}} \circ \phi_{\mathcal{R}}$ is represented by a matrix $p_{x,\mathcal{R}}$ consisting of four m-by-m blocks on the diagonal:

$$\left(\sum_{k=1}^{m} \sigma_k(f_i) \sigma_k(f_j) \tau_k(\mathbf{x}) \right)_{i,j},$$

$$\left(\sum_{k=1}^{m} \sigma_k(f_i) \sigma_k(f_j) \sigma_k(-a) \tau_k(\mathbf{x}) \right)_{i,j},$$

$$\left(\sum_{k=1}^{m} \sigma_k(f_i) \sigma_k(f_j) \sigma_k(-b) \tau_k(\mathbf{x}) \right)_{i,j},$$

and

$$\left(\sum_{k=1}^{m} \sigma_k(f_i) \sigma_k(f_j) \sigma_k(ab) \tau_k(\mathbf{x}) \right)_{i,j}.$$

The determinant of $p_{\mathbf{x},\mathcal{R}}$ is the product of these determinants. Each of these matrices may be factored using $M = (\sigma_j(f_i))_{i,j}$ and its transpose M^t, along with diagonal matrices $C(\mathbf{x}, f)$ having diagonal entries $\sigma_k(f)\tau_k(\mathbf{x})$, for certain fixed $f \in F$. The respective factorizations are $MC(\mathbf{x}, 1)M^t$, $MC(\mathbf{x}, -a)M^t$, $MC(\mathbf{x}, -b)M^t$ and $MC(\mathbf{x}, ab)M^t$. The determinant of MM^t is the discriminant d_F of the field F, by definition. The determinant of $C(\mathbf{x}, f)$ is

$$\prod_k \sigma_k(f) \prod_k \tau_k(\mathbf{x}) = \mathrm{N}_{F/\mathbb{Q}}(f)$$

since

$$\tau_k(\mathbf{x}) = \prod_{j=1}^{m-1} |\sigma_k(\varepsilon_j)|^{2x_j}$$

and, for each j, we have

$$\left|\prod_k \sigma_k(\varepsilon_j)\right| = |N_{F/Q}(\varepsilon_j)| = |\pm 1| = 1.$$

Multiplying the factors together yields the next result.

Lemma 4.1. $\det(p_{\mathbf{x},\mathcal{R}}) = d_F^4 N_{F/\mathbb{Q}}(ab)^2$.

Now, we relate this to $\det(p_{\mathbf{x},J^{-1}})$.

Lemma 4.2. *Suppose that $\mathcal{L}_1 \subset \mathcal{L}_2$ are $\mathbb{Z}$-modules of rank m in A. Then,* $\det(p_{\mathbf{x},\mathcal{L}_1}) = (\mathcal{L}_2 : \mathcal{L}_1)^2 \det(p_{x,\mathcal{L}_2})$.

Proof. Each matrix $p_{\mathbf{x},\mathcal{L}_i}$ is defined by choosing a $\mathbb{Z}$-module isomorphism $\phi_{\mathcal{L}_i} : \mathbb{Z}^{4m} \to \mathcal{L}_i$. Let ι be the inclusion map of $\mathcal{L}_1$ in $\mathcal{L}_2$. Then $T = \phi_{\mathbf{x},\mathcal{L}_2}^{-1} \circ \iota \circ \phi_{\mathbf{x},\mathcal{L}_1}$ is a $\mathbb{Z}$-module map from $\mathbb{Z}^{4m}$ to $\mathbb{Z}^{4m}$, represented by a matrix C, and $(\mathcal{L}_2 : \mathcal{L}_1) = |\mathcal{L}_2/\iota(\mathcal{L}_1)| = |\mathbb{Z}^{4m}/T(\mathbb{Z}^{4m})| = |\mathbb{Z}^{4m}/C\mathbb{Z}^{4m}| = |\det(C)|$. Then,

$$\begin{aligned} q_{\mathbf{x},\mathcal{L}_1}(\mathbf{v}) &= q_{\mathbf{x}}(\iota \circ \phi_{\mathcal{L}_1}(\mathbf{v})) = q_{\mathbf{x}}(\phi_{\mathcal{L}_2}(T(\mathbf{v}))) = q_{\mathbf{x}}(\phi_{\mathcal{L}_2}(C\mathbf{v})) \\ &= (C\mathbf{v})^t p_{\mathbf{x},\mathcal{L}_2}(C\mathbf{v}) = \mathbf{v}^t (C^t p_{\mathbf{x},\mathcal{L}_2} C)\mathbf{v}, \end{aligned} \tag{4.2}$$

showing that $p_{\mathbf{x},\mathcal{L}_1} = (C^t p_{\mathbf{x},\mathcal{L}_2} C)$. The result follows upon taking determinants. $\square$

Next, we can compute the determinant of interest. The first expression we give in the next proposition would be sufficient for our purposes, but the second is more natural.

Proposition 4.3.

$$\det(p_{\mathbf{x},J^{-1}}) = \frac{d_F^4 N_{F/\mathbb{Q}}(ab)^2}{(\Lambda : J)^2 (\Lambda : \mathcal{R})^2} = \frac{d_F^4 (\mathcal{O} : D_\Lambda)}{(\Lambda : J)^2 2^{4m}}.$$

Proof. From Lemmas 4.1 and 4.2, with $\mathcal{L}_1 = \mathcal{R} \subset \Lambda$ and $\mathcal{L}_2 = J^{-1} \supset \Lambda$, we get

$$\det(p_{\mathbf{x},J^{-1}}) = \frac{d_F^4 N_{F/\mathbb{Q}}(ab)^2}{(J^{-1}:\mathcal{R})^2} = \frac{d_F^4 N_{F/\mathbb{Q}}(ab)^2}{(J^{-1}:\Lambda)^2(\Lambda:\mathcal{R})^2}.$$

Then, Proposition 2.3 gives $(J^{-1}:\Lambda) = (\Lambda : J)$, establishing the first equality.

For the second, consider the discriminants $D_{\mathcal{R}}$ and D_Λ of $\mathcal{R}$ and Λ, respectively. These are ideals of $\mathcal{O}$. We have from [**9,** Chapter 4] or [**14,** Chapter 12],

$$(16a^2b^2) = D_{\mathcal{R}} = [\Lambda:\mathcal{R}]^2 D_\Lambda, \tag{4.3}$$

where $\Lambda/\mathcal{R} \cong \oplus_i \mathcal{O}/\mathfrak{a}_i$ as an $\mathcal{O}$-module and $[\Lambda:\mathcal{R}] = \prod_i \mathfrak{a}_i$. Hence,

$$(\Lambda:\mathcal{R}) = |\Lambda/\mathcal{R}| = \left|\bigoplus_i \frac{\mathcal{O}}{\mathfrak{a}_i}\right| = \left|\frac{\mathcal{O}}{\prod_i \mathfrak{a}_i}\right| = \left|\frac{\mathcal{O}}{[\Lambda:\mathcal{R}]}\right| = (\mathcal{O}:[\Lambda:\mathcal{R}]).$$

This equality and equation (4.3) lead to

$$\begin{aligned} 16^m N_{F/\mathbb{Q}}(ab)^2 &= N_{F/\mathbb{Q}}(16a^2b^2) = (\mathcal{O}:(16a^2b^2)) = (\mathcal{O}:[\Lambda:\mathcal{R}]^2 D_\Lambda) \\ &= (\mathcal{O}:[\Lambda:\mathcal{R}])^2(\mathcal{O}:D_\Lambda) = (\Lambda:\mathcal{R})^2(\mathcal{O}:D_\Lambda). \end{aligned} \tag{4.4}$$

Setting $N_{F/\mathbb{Q}}(ab)^2 = (\Lambda:\mathcal{R})^2(\mathcal{O}:D_\Lambda)/2^{4m}$ in the first expression for $\det(p_{\mathbf{x},J^{-1}})$ yields the second. □

Corollary 4.4. $\det(P_{\mathbf{x},J^{-1}}) = 1$.

Proof. Since

$$Q_{\mathbf{x},J^{-1}} = \frac{(\Lambda:J)^{1/2m}}{d_{F,\Lambda}}\, q_{\mathbf{x},J^{-1}},$$

we have that $Q_{\mathbf{x},J^{-1}}$ is represented by the $4m \times 4m$ matrix

$$P_{\mathbf{x},J^{-1}} = \frac{(\Lambda:J)^{1/2m}}{d_{F,\Lambda}}\, p_{\mathbf{x},J^{-1}}.$$

It follows that $d_{F,\Lambda} = (d_F^4(\mathcal{O} : D_\Lambda))^{1/4m}/2$, and
(4.5)

$$\begin{aligned}\det(P_{\mathbf{x},J^{-1}}) &= \det\left(\frac{(\Lambda : J)^{1/2m}}{d_{F,\Lambda}}\, p_{\mathbf{x},J^{-1}}\right) = \left(\frac{(\Lambda : J)^{1/2m}}{d_{F,\Lambda}}\right)^{4m} \det(p_{\mathbf{x},J^{-1}}) \\ &= \frac{(\Lambda : J)^2}{d_{F,\Lambda}^{4m}} \det(p_{\mathbf{x},J^{-1}}) = \frac{(\Lambda : J)^2 2^{4m}}{d_F^4(\mathcal{O} : D_\Lambda)} \det(p_{\mathbf{x},J^{-1}}) = 1.\end{aligned}$$

The last equality holds by Proposition 4.3. $\square$

5. Main results. We now arrive at the following key result, directly relating a partial zeta function to Epstein zeta functions.

Theorem 5.1.

(i) *For* $\Re(s) > 1$,

$$\begin{aligned}\Gamma(2s)^m \zeta_{\Lambda,[J]}(s) &= \frac{2^{m-2} m R_F}{w_J} \frac{\Gamma(2ms)}{d_{F,\Lambda}^{2ms}} \\ &\quad \cdot \int_{[-1/2,1/2]^{m-1}} \sum_{\mathbf{v} \in \mathbb{Z}^{4m}} Q_{\mathbf{x},J^{-1}}(\mathbf{v})^{-2ms} d\mathbf{x} \\ &= \frac{2^{m-1} m R_F}{w_J} \frac{\Gamma(2ms)}{d_{F,\Lambda}^{2ms}} \\ &\quad \cdot \int_{[-1/2,1/2]^{m-1}} Z_{Q_{\mathbf{x},J^{-1}}}(2ms)\, d\mathbf{x}.\end{aligned} \tag{5.1}$$

(ii) *The function* $\zeta_{\Lambda,[J]}(s)$ *extends to a meromorphic function on the complex plane with its only poles at* $s = 1$ *and at negative integer multiples of* $1/2m$.

(iii) *The extended function* $\zeta_{\Lambda,[J]}(s)$ *has a zero of order* $m-1$ *at* $s = 0$ *and*

$$\lim_{s \to 0} \frac{\zeta_{\Lambda,[J]}(s)}{s^{m-1}} = \frac{-2^{2m-3} R_F}{w_J}.$$

Proof.

(i) These equalities directly follow from (3.10), together with the definition of $Q_{\mathbf{x},J^{-1}}$ in (3.11), and the definition of $Z_{Q_{\mathbf{x},J^{-1}}}$.

(ii) From part (i),

$$\zeta_{\Lambda,[J]}(s) = \frac{2^{m-1}mR_F}{w_J}\frac{\Gamma(2ms)}{d_{F,\Lambda}^{2ms}\Gamma(2s)^m}\int_{[-1/2,1/2]^{m-1}} Z_{Q_{\mathbf{x},J^{-1}}}(2ms)\,d\mathbf{x}, \tag{5.2}$$

and $Z_{Q_{\mathbf{x},J^{-1}}}(2ms)$ is meromorphic on the complex plane with a single pole at $s=1$, by Proposition 3.1. For each $s \neq 1$, it is a continuous function of $\mathbf{x}$, so when integrated with respect to $\mathbf{x}$ over the compact region, it yields a meromorphic function with a single pole at $s=1$. Since $\Gamma(s)$ is also meromorphic on the complex plane with no zeroes and poles only at $s=0$ and negative integers, the expression for $\zeta_{\Lambda,[J]}(s)$ represents it as a meromorphic function with the stated properties.

(iii) Using the functional equation for the Gamma function in equation (5.2) gives

$$\begin{aligned}\frac{\zeta_{\Lambda,[J]}(s)}{s^{m-1}} &= \frac{2^{m-1}mR_F}{w_J s^{m-1}}\frac{\Gamma(2ms+1)/(2ms)}{d_{F,\Lambda}^{2ms}(\Gamma(2s+1)/2s)^m}\\ &\quad\cdot\int_{[-1/2,1/2]^{m-1}} Z_{Q_{\mathbf{x},J^{-1}}}(2ms)\,d\mathbf{x}\\ &= \frac{2^{m-1}mR_F}{w_J}\frac{\Gamma(2ms+1)}{d_{F,\Lambda}^{2ms}\Gamma(2s+1)^m}\frac{2^{m-1}}{m}\\ &\quad\cdot\int_{[-1/2,1/2]^{m-1}} Z_{Q_{\mathbf{x},J^{-1}}}(2ms)\,d\mathbf{x}.\end{aligned} \tag{5.3}$$

Now, letting s approach zero and using $Z_{Q_{\mathbf{x},J^{-1}}}(0) = -1/2$ from Proposition 3.1 gives the result. $\square$

Our main result is obtained by taking s to be a real value greater than 1. We will use $s=2$ to be explicit.

Corollary 5.2. $\zeta_{\Lambda,[J]}(2) > (R_F/2^9m^2)(\pi^4/6d_{F,\Lambda}^4)^m$.

Proof. Setting $s = 2$ in Theorem 5.1 and dividing by $\Gamma(4)^m = 6^m$ yield

$$\zeta_{\Lambda,[J]}(2) = \frac{2^{m-1} m R_F}{w_J (6 d_{F,\Lambda}^4)^m} \Gamma(4m) \int_{[-1/2,1/2]^{m-1}} Z_{Q_{\mathbf{x},J^{-1}}}(4m)\, d\mathbf{x}, \tag{5.4}$$

Since $\det(P_{\mathbf{x},J^{-1}}) = 1$ as shown in Corollary 4.4, Proposition 3.1 (iii) with $s = k = 4m$ gives

$$Z_{Q_{\mathbf{x},J^{-1}}}(4m) > \frac{\pi^{4m}}{\Gamma(4m)} \left(\frac{1}{8m - 4m} - \frac{1}{8m} \right) = \frac{\pi^{4m}}{\Gamma(4m)} \frac{1}{8m} . \tag{5.5}$$

Using this in equation (5.4), along with $w_J < 2^{m+5} m^2$ from Proposition 2.5, gives the result. □

We are now positioned to give our proof of the finiteness of the number of ideal classes.

Theorem 5.3. *In a maximal $\mathcal{O}_F$-order Λ of a quaternion algebra A that is ramified at all infinite places over a totally real field F, the number of left ideal classes $[J]$ is finite.*

Proof. From equation (3.3) with $s = 2$, we have that

$$\zeta_\Lambda(2) = \sum_{[J]} \zeta_{\Lambda,[J]}(2), \tag{5.6}$$

a sum over all classes of non-zero left ideals of Λ. Now, Corollary 5.2 gives

$$\zeta_{\Lambda,[J]}(2) \geq C(F, \Lambda),$$

with a constant $C(F, \Lambda) > 0$, independent of $[J]$. Since $\zeta_\Lambda(2)$ is finite, the sum has a finite number of terms, giving the desired result. □

By considering the equation

$$\zeta_\Lambda(s) = \sum_{[J]} \zeta_{\Lambda,[J]}(s)$$

as s approaches zero, we can now prove the Eichler mass formula (**[3]**) and the even parity of the number of ramified primes in A over F. The latter is typically obtained as a consequence of Hilbert reciprocity for number fields, proved in **[11,** Section 10, Theorem B]. Analytic proofs of these results appear in the literature; however, we wish to emphasize the use of s approaching zero rather than s approaching 1, and the ability to determine the behavior at $s = 0$ directly from explicit formulas rather than using functional equations.

Theorem 5.4.

(i) **Eichler's mass formula.**

$$\sum_{[J]} \frac{1}{w_J} = \frac{h_F|\zeta_F(-1)|}{2^{m-1}} \prod_{\mathfrak{p}|D_\Lambda} ((\mathcal{O}:\mathfrak{p}) - 1).$$

In particular, $\zeta_F(-1)$ is rational.

(ii) **A special case of Hilbert reciprocity.** *The number of primes of F that ramify in A is even.*

Proof. Equation (3.3) states that

$$\zeta_\Lambda(s) = \sum_{[J]} \zeta_{\Lambda,[J]}(s) \tag{5.7}$$

for $\Re(s) > 1$. Now that the sum on the right is finite, we can conclude that this equation holds for the meromorphic continuation of these functions as well. Dividing by s^{m-1} and taking the limit as s approaches zero on both sides of this equation with the use of equation (3.1) and Theorem 5.1, we obtain

$$\begin{aligned} \left(\lim_{s\to 0} \frac{\zeta_F(2s)}{s^{m-1}}\right)\zeta_F(-1) \prod_{\mathfrak{p}|D_\Lambda} (1 - (\mathcal{O}:\mathfrak{p})) &= \lim_{s\to 0} \frac{\zeta_\Lambda(s)}{s^{m-1}} \\ = \sum_{[J]} \lim_{s\to 0} \frac{\zeta_{\Lambda,[J]}(s)}{s^{m-1}} &= -2^{2m-3} R_F \sum_{[J]} \frac{1}{w_J}. \end{aligned} \tag{5.8}$$

The first limit is given by the general "analytic class number formula" in terms of the class number h_F and number of roots of unity w_F in F:

$$\lim_{s\to 0} \frac{\zeta_F(2s)}{s^{m-1}} = 2^{m-1} \lim_{s\to 0} \frac{\zeta_F(2s)}{(2s)^{m-1}} = 2^{m-1} \frac{-h_F R_F}{w_F}.$$

For the totally real field F, we have $w_F = 2$. After substituting and canceling in equation (5.8), this yields

$$h_F \zeta_F(-1) \prod_{\mathfrak{p} \mid D_\Lambda} (1 - (\mathcal{O} : \mathfrak{p})) = 2^{m-1} \sum_{[J]} \frac{1}{w_J}. \tag{5.9}$$

Taking absolute values now yields (i).

For (ii), we simply consider signs. In equation (5.9), the number of negative terms in the product equals the number of finite primes dividing D_Λ, which is the number of finite primes of F that ramify in A. Also, it follows from the analytic continuation formula for ζ_F that the sign of $\zeta_F(-1)$ is $(-1)^m$, a product of m negative terms. Now, m is the number of infinite primes of F that ramify in A, since F is totally real of degree m and all infinite primes ramify by assumption. Since the right side of equation (5.9) is positive, we see that the total number of finite and infinite ramified primes is even. $\square$

Remark 5.5. The Birch-Tate conjecture specifies the exact value of $\zeta_F(-1)$ for totally real F in terms of the algebraic K-group $K_2(\mathcal{O}_F)$ and a quantity $w_2(F)$ which can be most simply described as the maximum number of roots of unity in an abelian Galois extension of F whose Galois group over F has exponent 2. The conjecture that

$$\zeta_F(-1) = (-1)^m \frac{|K_2(\mathcal{O}_F)|}{w_2(F)}$$

holds for F abelian over $\mathbb{Q}$, and holds up to powers of 2 as consequences of [**8, 16**].

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