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CENTERS FOR GENERALIZED QUINTIC POLYNOMIAL DIFFERENTIAL SYSTEMS

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ABSTRACT. We classify the centers of polynomial differential systems in $\mathbb{R}^2$ of odd degree $d \geq 5$, in complex notation, as $\dot{z} = iz + (z\overline{z})^{(d-5)/2}(Az^5 + Bz^4\overline{z} + Cz^3\overline{z}^2 + Dz^2\overline{z}^3 + Ez\overline{z}^4 + F\overline{z}^5)$, where $A, B, C, D, E, F \in \mathbb{C}$ and either $A = \mathrm{Re}(D) = 0$, $A = \mathrm{Im}(D) = 0$, $\mathrm{Re}(A) = D = 0$ or $\mathrm{Im}(A) = D = 0$.

1. Introduction and statement of the main results. In the qualitative theory of real planar polynomial differential systems one of the main problems is the *center-focus problem*, i.e., the problem of distinguishing between a center and a focus. For singular points whose linear part has a pair of pure imaginary eigenvalues, this problem is equivalent to the existence of an analytic first integral defined in a neighborhood of the singular point, see, for more details, [**2, 12, 13, 24, 25**].

A singular point is a *center* if there exists a neighborhood of it such that all of the orbits in this neighborhood are periodic except the singular point, and a singular point is a *focus* if there is a neighborhood of it such that all of the orbits in this neighborhood spiral either in forward or in backward time to the singular point.

We study the center-focus problem for a class of polynomial differential systems which generalize the class of linear polynomial differential

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systems with homogeneous polynomial nonlinearities of degree 5. The characterization of the centers of polynomial differential systems began with the classes of all quadratic polynomial differential systems and linear polynomial systems with homogeneous polynomial nonlinearities of degree 3, see for instance, [**1, 28, 29, 30, 31**]. Unfortunately, at present, we are very far from obtaining the classification of all of the centers of cubic polynomial differential systems. However, some subclasses of cubic polynomial differential systems with centers have been studied, see for instance, the papers [**32, 33**] and the references cited therein. The centers of linear polynomial differential systems with homogeneous polynomial nonlinearities of degree $k > 3$ are not classified, but there are many partial results for $k = 4, 5, 6, 7, 9$, see [**3, 4, 11, 20, 21, 22, 23**]. In general, the huge number of computations necessary for obtaining complete classification becomes the center problem which is computationally intractable, see for instance, [**16**] and the references cited therein.

In this paper, we work with the real planar polynomial differential systems which have a singular point at the origin with eigenvalues $\pm i$ and which may be written in complex form as

$$(1.1)\quad \dot{z} = iz + (z\overline{z})^{(d-5)/2}(Az^5 + Bz^4\overline{z} + Cz^3\overline{z}^2 + Dz^2\overline{z}^3 + Ez\overline{z}^4 + F\overline{z}^5),$$

where $z = x + iy$, $d \geq 5$ is an arbitrary odd integer and $A, B, C, D, E, F \in \mathbb{C}$ satisfy one of the four conditions:

(c.1) $A = \operatorname{Re}(D) = 0$,
(c.2) $A = \operatorname{Im}(D) = 0$,
(c.3) $\operatorname{Re}(A) = D = 0$,
(c.4) $\operatorname{Im}(A) = D = 0$.

These systems contain as a particular case the results of paper [**21**], where the authors characterize the centers of system (1.1) with $A = D = 0$.

The polynomial differential systems (1.1) when $d = 5$ coincide with the class of quintic polynomial differential systems of the form of a linear center plus homogeneous polynomial nonlinearities of degree 5. Therefore, the polynomial differential systems (1.1) of odd degree $d > 5$ generalize the class of linear polynomial differential systems with quintic homogeneous polynomial nonlinearities.

The main result of this paper is the characterization of centers for the polynomial differential systems (1.1) under the assumptions (c.1)–(c.4). We present the classification of these centers in a different theorem for each of the four classes.

Theorem 1.1. *The polynomial differential systems* (1.1) *satisfying condition* (c.1) *have a center at the origin if one of the following conditions hold.*

(a) $\mathrm{Re}(C) = \mathrm{Im}(D) = \mathrm{Re}(\overline{B}E\overline{F}) = \mathrm{Re}(B^2E) = \mathrm{Im}(BE^2\overline{F}) = \mathrm{Im}(B^2E\overline{F}) = \mathrm{Im}(B^3F) = \mathrm{Re}(E^3\overline{F}^2) = 0$,
(b) $\mathrm{Re}(B) = \mathrm{Re}(C) = F = 3B + \overline{D} = 0$,
(c) $\mathrm{Re}(B) = \mathrm{Re}(C) = \mathrm{Re}(E) = \mathrm{Re}(F) = 0$,
(d) $\mathrm{Re}(C) = E = 2B + \overline{D} = 0$.

The proof of Theorem 1.1 is given in Section 3.

Theorem 1.2. *The polynomial differential systems* (1.1) *satisfying condition* (c.2) *have a center at the origin if one of the following conditions hold.*

(a) $\mathrm{Re}(C) = \mathrm{Im}(D) = \mathrm{Re}(\overline{B}E\overline{F}) = \mathrm{Re}(B^2E) = \mathrm{Im}(BE^2\overline{F}) = \mathrm{Im}(B^2E\overline{F}) = \mathrm{Im}(B^3F) = \mathrm{Re}(E^3\overline{F}^2) = 0$,
(b) $\mathrm{Re}(B) = \mathrm{Re}(C) = F = 3B + \overline{D} = 0$,
(c) $\mathrm{Im}(B) = \mathrm{Re}(C) = \mathrm{Re}(E) = \mathrm{Im}(F) = 0$,
(d) $\mathrm{Re}(C) = E = 2B + \overline{D} = 0$.

We note that the change of variables (2.7) with $\xi = ((a_8/a_7)e^{-i\pi/4})^{1/4}$ transforms condition (c.2) into condition (c.1). Therefore, Theorem 1.2 will not be proved.

Theorem 1.3. *The polynomial differential systems* (1.1) *satisfying condition* (c.3) *have a center at the origin if one of the following conditions hold.*

(a) $\mathrm{Re}(C) = \mathrm{Im}(D) = \mathrm{Re}(\overline{B}E\overline{F}) = \mathrm{Re}(B^2E) = \mathrm{Im}(BE^2\overline{F}) = \mathrm{Im}(B^2E\overline{F}) = \mathrm{Im}(B^3F) = \mathrm{Re}(E^3\overline{F}^2) = 0$,
(b) $\mathrm{Re}(C) = B = 5\overline{A} + E = 0$,
(c) $\mathrm{Re}(C) = A - 3\overline{E} = F = 0$,

(d) $C = F = \mathrm{Re}(E) = \mathrm{Re}(B) - \mathrm{Im}(B) = 7A + E = 49\,\mathrm{Im}(B)^2 - 8\,\mathrm{Im}(E)^2 = 0$ *and* $d = 5$,
(e) $C = F = \mathrm{Re}(E) = \mathrm{Re}(B) + \mathrm{Im}(B) = 7A + E = 49\,\mathrm{Im}(B)^2 - 8\,\mathrm{Im}(E)^2 = 0$ *and* $d = 5$,
(f) $C = F = \mathrm{Re}(E) = 3A + E = 9|B|^2 - 16|E|^2 = 0$ *and* $d = 5$,
(g) $B = C = 3A - 5\overline{E} = 16|E|^2 - 9|F|^2 = 0$, $F = |F|e^{i\psi}$ *with* $\psi = \pi/4 + k\pi$, $k \in \mathbb{Z}$ *and* $d = 5$,
(h) $\mathrm{Re}(B) = \mathrm{Re}(C) = \mathrm{Re}(E) = \mathrm{Re}(F) = 0$,
(i) $\mathrm{Re}(C) = A - C = E = B + \overline{F} = |C|^2 - |F|^2 = 0$ *and* $d = 5$,
(j) $\mathrm{Re}(C) = A + C = E = B - \overline{F} = |C|^2 - |F|^2 = 0$ *and* $d = 5$,
(k) $\mathrm{Re}(C) = \mathrm{Re}(E)$, *conditions* (4.20) *and* $d = 5$,
(l) $C = B + \overline{F} = \mathrm{Re}(E) = A + E = 4|E|^2 - |F|^2 = 0$ *and* $d = 5$,
(m) $C = B - \overline{F} = \mathrm{Re}(E) = A + E = 4|E|^2 - |F|^2 = 0$ *and* $d = 5$,
(n) $\mathrm{Im}(B) = \mathrm{Re}(C) = \mathrm{Im}(E) = \mathrm{Im}(F) = 0$.

The proof of Theorem 1.3 is given in Section 4. Note that Theorem 1.3 (a) coincides with Theorem 1.1 (a), and consequently, it will not be proved.

Theorem 1.4. *Polynomial differential systems* (1.1) *satisfying condition* (c.4) *have a center at the origin if one of the following conditions hold.*

(a) $\mathrm{Re}(C) = \mathrm{Im}(D) = \mathrm{Re}(\overline{B}E\overline{F}) = \mathrm{Re}(B^2E) = \mathrm{Im}(BE^2\overline{F}) = \mathrm{Im}(B^2E\overline{F}) = \mathrm{Im}(B^3F) = \mathrm{Re}(E^3\overline{F}^2) = 0$,
(b) $\mathrm{Re}(C) = B = 5\overline{A} + E = 0$,
(c) $\mathrm{Re}(C) = A - 3\overline{E} = F = 0$,
(d) $C = F = \mathrm{Im}(B) = \mathrm{Im}(E) = 7B + 4E = 7A - E = 0$ *and* $d = 5$,
(e) $C = F = \mathrm{Im}(B) = \mathrm{Im}(E) = 7B - 4E = 7A - E = 0$ *and* $d = 5$,
(f) $C = F = \mathrm{Im}(E) = 3A + E = 9|B|^2 - 16\,\mathrm{Re}(E)^2 = 0$ *and* $d = 5$,
(g) $B = C = 3A - 5\overline{E} = 16|E|^2 - 9|F|^2 = 0$ $F = |F|e^{i\psi}$ *with* $\psi = k\pi/2$, $k \in \mathbb{Z}$ *and* $d = 5$,
(h) $E = \mathrm{Re}(C) = \mathrm{Re}(A) - \mathrm{Im}(C) = B + i\overline{F} = |C|^2 - |F|^2 = 0$ *and* $d = 5$,
(i) $E = \mathrm{Re}(C) = \mathrm{Re}(A) + \mathrm{Im}(C) = |C|^2 - |F|^2 = B - i\overline{F} = 0$ *and* $d = 5$,
(j) $\mathrm{Re}(C) = \mathrm{Im}(E) = \mathrm{Im}(C)^2 - |F|^2 = |B|^2 - 4\,\mathrm{Re}(E)^2 = a_1 + a_9 = a_3a_{11} - a_4a_{12} = 2a_6a_9 - a_4a_{11} - a_3a_{12} = a_4a_6 - 2a_9a_{11} =$

$a_3a_6 - 2a_9a_{12} = a_4^2a_{11} - 4a_9^2a_{11} + a_3a_4a_{12} = 0$ *and* $d = 5$,

(k) $C = \mathrm{Im}(E) = B + i\overline{F} = A - E = 4|E|^2 - |F|^2 = 0$ *and* $d = 5$,

(l) $C = \mathrm{Im}(E) = B - i\overline{F} = A - E = 4|E|^2 - |F|^2 = 0$ *and* $d = 5$,

(m) $\mathrm{Re}(C) = \mathrm{Im}(E) = \mathrm{Re}(F) - \mathrm{Im}(F) = \mathrm{Re}(B) - \mathrm{Im}(B) = 0$,

(n) $\mathrm{Re}(C) = \mathrm{Im}(E) = \mathrm{Re}(F) + \mathrm{Im}(F) = \mathrm{Re}(B) + \mathrm{Im}(B) = 0$.

We note that the change of variables (2.7) with $\xi = ((a_2/a_1)e^{i\pi/2})^{1/4}$ transforms condition (c.4) into condition (c.3). Hence, Theorem 1.4 will not be proved.

2. Preliminary definitions and results. There are very few results about centers for classes of polynomial differential systems of arbitrary degree. The resolution of this problem implies effective computation of the Poincaré-Liapunov constants. Indeed, setting

$$A = a_1 + ia_2, \qquad B = a_3 + ia_4, \qquad C = a_5 + ia_6,$$
$$D = a_7 + ia_8, \qquad E = a_9 + ia_{10}, \qquad F = a_{11} + ia_{12},$$

and writing (1.1) in polar coordinates, i.e., performing a change of variables $r^2 = z\overline{z}$ and $\theta = \arctan(\mathrm{Im}\, z/\,\mathrm{Re}\, z)$, system (1.1) becomes

$$\dot{r} = F(\theta)\, r^d, \qquad \dot{\theta} = 1 + G(\theta)\, r^{d-1}, \tag{2.1}$$

where $F(\theta)$ and $G(\theta)$ are the homogeneous trigonometric polynomials

$$\begin{aligned} F(\theta) &= a_5 + (a_3 + a_7)\cos(2\theta) + (a_8 - a_4)\sin(2\theta) + (a_1 + a_9)\cos(4\theta) \\ &\quad + (a_{10} - a_2)\sin(4\theta) + a_{11}\cos(6\theta) + a_{12}\sin(6\theta), \\ G(\theta) &= a_6 + (a_4 + a_8)\cos(2\theta) + (a_3 - a_7)\sin(2\theta) + (a_{10} + a_2)\cos(4\theta) \\ &\quad + (a_1 - a_9)\sin(4\theta) + a_{12}\cos(6\theta) - a_{11}\sin(6\theta). \end{aligned}$$

In order to determine the necessary conditions for a center, we propose the Poincaré series

$$H(r,\theta) = \sum_{n=2}^{\infty} H_n(\theta) r^n, \tag{2.2}$$

where $H_2(\theta) = 1/2$ and $H_n(\theta)$ are homogeneous trigonometric polynomials with respect to θ of degree n. Imposing this power series as a formal first integral of system (2.1), we obtain

$$\dot{H}(r,\theta) = \sum_{k=2}^{\infty} V_{2k} r^{2k},$$

where V_{2k} are the *Poincaré-Lyapunov constants* that depend upon the parameters of system (1.1). Indeed, it is easy to see by the recursive equations that generate the V_{2k} that these V_{2k} are polynomials in the parameters of system (1.1), see [**8**]. As system (1.1) is polynomial, due to the Hilbert basis theorem, the ideal $J = \langle V_2, V_4, \ldots \rangle$ generated by the Poincaré-Liapunov constants is finitely generated, i.e., there exist $W_1, W_2, \ldots, W_k$ in J such that $J = \langle W_1, W_2, \ldots, W_k \rangle$. Such a set of generators is called a *basis* of J, and the conditions $W_j = 0$ for $j = 1, \ldots, k$ provide a finite set of necessary conditions for a center. The set of coefficients for which all the Poincaré-Liapunov constants V_{2k} vanish is called the *center variety* of the family of polynomial differential systems, and it is also an algebraic set.

In practice, we determine a number of Poincaré-Liapunov constants which we believe contains the set of generators of all of the Poincaré-Liapunov constants. From this set, the much more difficult problem is to decompose this algebraic set into its irreducible components. For simple cases, this can be done by hand, see [**3, 4, 15, 18, 19, 21**]. However, for more difficult systems, the use of a computer algebra system is essential. The computational tool which we use is the routine `minAssGTZ` [**9**] of the computer algebra system `Singular` [**17**], which is based on the Gianni-Trager-Zacharias algorithm [**10**]. Since computations are very laborious, they cannot be completed in the field of rational numbers. Therefore, we choose an approach based on the use of modular computations [**27**]. We have chosen the prime $p = 32003$. In order to perform the rational reconstruction, we use `Mathematica` and the algorithm presented in [**27**]. The last step of this algorithm has not been verified because computations cannot be overcome. This step ensures that all of the points of the center variety have been found, that is, we know that all of the encountered points belong to the decomposition of the center variety, but we do not know whether the given decomposition is complete. Nevertheless, it is believed that the given list is complete, see also [**27**]. Therefore, in the following, we provide sufficient conditions for a center, which are necessary from a practical standpoint.

From system (2.1), we obtain the associated equation

$$\frac{dr}{d\theta} = \frac{F(\theta)\, r^d}{1 + G(\theta)\, r^{d-1}}. \tag{2.3}$$

It is clear that equation (2.3) is well defined in a sufficiently small neighborhood of the origin. Hence, if system (2.1) has a center at the origin, then equation (2.3), when $\dot{\theta} > 0$, also has a center at the origin. The transformation $(r, \theta) \to (\rho, \theta)$ introduced by Cherkas [**5**], defined by

$$\rho = \frac{r^{d-1}}{1 + G(\theta) r^{d-1}}, \tag{2.4}$$

whose inverse is

$$r = \frac{\rho^{1/(d-1)}}{(1 - \rho G(\theta))^{1/(d-1)}},$$

is a diffeomorphism from the region $\dot{\theta} > 0$ into its image. If we transform equation (2.3) using transformation (2.4), we obtain the following Abel equation:

$$\begin{aligned} \frac{d\rho}{d\theta} &= -(d-1)G(\theta)F(\theta)\rho^3 + [(d-1)(F(\theta) - G'(\theta)]\rho^2 \\ &= A(\theta)\rho^3 + B(\theta)\rho^2 + C\rho. \end{aligned} \tag{2.5}$$

The solution $\rho(\theta, \rho_0)$ of (2.5) satisfies that $\rho(0, \rho_0) = \rho_0$ can be expanded in a convergent series of $\rho_0 \geq 0$ sufficiently small of the form

$$\rho(\theta, \rho_0) = \rho_1(\theta)\rho_0 + \rho_2(\theta)\rho_0^2 + \rho_3(\theta)\rho_0^3 + \cdots, \tag{2.6}$$

with $\rho_1(\theta) = 1$ and $\rho_k(0) = 0$ for $k \geq 2$. Let $P : [0, \widetilde{\rho}_0] \to \mathbb{R}$ be the Poincaré return map defined by $P(\widetilde{\rho}_0) = \rho(2\pi, \widetilde{\rho}_0)$ for a convenient $\widetilde{\rho}_0$. System (1.1) has a center at the origin if and only if $\rho_k(2\pi) = 0$ for every $k \geq 0$. If we assume that $\rho_2(2\pi) = \cdots = \rho_{m-1}(2\pi) = 0$, we say that $v_m = \rho_m(2\pi)$ is the mth *Poincaré-Liapunov-Abel* constant of system (1.1). Of course, the set of coefficients for which all the Poincaré-Liapunov-Abel constants v_m vanish is the same as that set for which all the Poincaré-Liapunov constants V_{2k} vanish. This set, as previously mentioned, is the center variety of system (1.1).

We note that the space of systems (1.1) with a center at the origin is invariant with respect to the action group $\mathbb{C}^*$ of the change of variables $z \to \xi z$:

(2.7)

$$A \longrightarrow \xi^{(d-7)/2}\overline{\xi}^{(d-5)/2}\xi^5 A, \qquad B \longrightarrow \xi^{(d-7)/2}\overline{\xi}^{(d-5)/2}\xi^4\overline{\xi}B.$$
$$C \longrightarrow \xi^{(d-7)/2}\overline{\xi}^{(d-5)/2}\xi^3\overline{\xi}^2 C, \qquad D \longrightarrow \xi^{(d-7)/2}\overline{\xi}^{(d-5)/2}\xi^2\overline{\xi}^3 D.$$
$$E \longrightarrow \xi^{(d-7)/2}\overline{\xi}^{(d-5)/2}\xi\overline{\xi}^4 E, \qquad F \longrightarrow \xi^{(d-7)/2}\overline{\xi}^{(d-5)/2}\overline{\xi}^5 F;$$

for a proof, see [**18**].

The next result will be used to check when system (1.1) is reversible with respect to a straight line through the origin; it is proven in [**8**]. Indeed, system (1.1) is invariant with respect to a straight line through the origin if it is invariant under the change of variables $w = e^{i\gamma}z$, $\tau = -t$, for some real γ.

Lemma 2.1. *System* (1.1) *is reversible with respect to a straight line if and only if*

$$A = -\overline{A}e^{-4i\gamma}, \qquad B = -\overline{B}e^{-2i\gamma}, \qquad C = -\overline{C},$$
$$D = -\overline{D}e^{2i\gamma}, \qquad E = -\overline{E}e^{4i\gamma}, \qquad F = -\overline{F}e^{6i\gamma}, \tag{2.8}$$

for some $\gamma \in \mathbb{R}$. *Furthermore, in this situation, the origin of system* (1.1) *has a center at the origin.*

Throughout the proof of Theorem 1.3 we will also consider equation (1.1) and its complex conjugated equation, given by

$$\begin{aligned}\dot{\overline{z}} = {} & -i\overline{z} + (z\overline{z})^{(d-5)/2} \\ & \cdot (\overline{A}\overline{z}^5 + \overline{B}\overline{z}^4 z + \overline{C}\overline{z}^3 z^2 + \overline{D}\overline{z}^2 z^3 + \overline{E}\overline{z}z^4 + \overline{F}z^5).\end{aligned} \tag{2.9}$$

In addition, we will also consider the complex system defined by both equations that, after the complex change of time $t \to -it$, is given by

$$\dot{z} = z + P_d(z,\overline{z}), \qquad \dot{\overline{z}} = -\overline{z} + Q_d(z,\overline{z}), \tag{2.10}$$

where P_d and Q_d are homogeneous polynomials of degree d. Since there is no confusion, we will also write it as

$$\dot{x} = x + P_d(x,y), \qquad \dot{y} = -y + Q_d(x,y). \tag{2.11}$$

The next lemma, given in [**14**], will be needed later.

Lemma 2.2. *If system* (2.11) *has a local inverse integrating factor*

$$V = (xy)^\alpha \prod_{i=1}^{m} F_i^{\beta_i},$$

with F_i analytic in x and y, $F_i(0,0) \neq 0$ for $i = 1, \ldots, m$, $\alpha \neq 0$ and α not an integer greater than 1, *then it has an analytic first integral of the form $\Psi = xy + \cdots$.*

In fact, this lemma is a specific case of [**6,** Theorem 4.13 (iii)].

3. Proof of Theorem 1.1.

Proof of (a). The conditions of this case expressed in real parameters are $a_5 = a_8 = 0$, i.e., $A = \mathrm{Re}(C) = D = 0$, and

$$\begin{aligned}
p_1 &= a_3a_9a_{11} + a_4a_{10}a_{11} - a_4a_9a_{12} + a_3a_{10}a_{12} = 0,\\
p_2 &= a_3^2a_9 - a_4^2a_9 - 2a_3a_4a_{10} = 0,\\
p_3 &= a_4a_9^2a_{11} - 3a_4a_{10}^2a_{11} - a_3a_9^2a_{12} + 4a_4a_9a_{10}a_{12} - a_3a_{10}^2a_{12} = 0,\\
p_4 &= a_4^2a_9a_{11} + 3a_3a_4a_{10}a_{11} - a_3a_4a_9a_{12} + a_3^2a_{10}a_{12} = 0,\\
p_5 &= 3a_3^2a_4a_{11} - a_4^3a_{11} + a_3^3a_{12} - 3a_3a_4^2a_{12} = 0,\\
p_6 &= a_9^3a_{11}^2 - 3a_9a_{10}^2a_{11}^2 + 6a_9^2a_{10}a_{11}a_{12}\\
&\quad - 2a_{10}^3a_{11}a_{12} - a_9^3a_{12}^2 + 3a_9a_{10}^2a_{12}^2 = 0.
\end{aligned}$$

Each of the conditions p_j, for $j = 1, \ldots, 6$, is now rewritten in terms of the complex parameters of system (1.1). We obtain that

$$p_1 = \mathrm{Re}(\overline{B}E\overline{F}) = 0 \quad \text{and} \quad p_2 = \mathrm{Re}(B^2E) = 0.$$

Using $p_1 = 0$, we get $p_3 = \mathrm{Im}(BE^2\overline{F}) = 0$, and using $p_1 = p_2 = 0$, we get $p_4 = \mathrm{Im}(B^2E\overline{F}) = 0$. Finally, we note that $p_5 = \mathrm{Im}(B^3F) = 0$ and $p_6 = \mathrm{Re}(E^3\overline{F}^2) = 0$. In summary, we have the conditions of statement (1).

From these conditions of (1) we have $A = D = 0$, $\mathrm{Re}(C) = 0$, that is, $C = -\overline{C}$, and

$$\frac{B}{\overline{\overline{B}}} = -\frac{E\overline{F}}{\overline{E}F}, \ \left(\frac{B}{\overline{\overline{B}}}\right)^2 = -\frac{\overline{E}}{E}, \ \frac{B}{\overline{\overline{B}}} = \frac{\overline{E}^2 F}{E^2\overline{F}}, \ \left(\frac{B}{\overline{\overline{B}}}\right)^3 = \frac{\overline{F}}{F}, \ \left(\frac{E}{\overline{\overline{E}}}\right)^3 = -\left(\frac{F}{\overline{\overline{F}}}\right)^2. \tag{3.1}$$

Now, let $\theta_1, \theta_2, \theta_3$ be such that

$$e^{i\theta_1} = -\frac{\overline{B}}{B}, \qquad e^{i\theta_2} = -\frac{\overline{E}}{E}, \qquad e^{i\theta_3} = -\frac{\overline{F}}{F}.$$

From conditions (3.1), we have that

$$\theta_2 = -2\theta_1(\mathrm{mod}(2\pi)), \qquad \theta_3 = -3\theta_1(\mathrm{mod}(2\pi)). \tag{3.2}$$

Now, taking $\gamma = \theta_1/2$ and using (3.2), we have

$$\begin{aligned} e^{2i\gamma} &= e^{i\theta_1} = -\frac{\overline{B}}{B}, \\ e^{-4i\gamma} &= e^{-2i\theta_1} = e^{i\theta_2} = -\frac{\overline{E}}{E}, \\ e^{-6i\gamma} &= e^{-3i\theta_1} = e^{i\theta_3} = -\frac{\overline{F}}{F}. \end{aligned}$$

Hence, by Lemma 2.1, and under the conditions of statement (1), system (1) is reversible and consequently has a center at the origin. □

Proof of (b). The conditions in the real parameters are $a_5 = a_{11} = a_{12} = 3a_4 - a_8 = a_3 = 0$. System (1.1) may be written as:

$$\begin{aligned} \dot{z} &= iz + (z\overline{z})^{(d-5)/2}(Bz^4\overline{z} - 3\overline{B}z^2\overline{z}^3 + Ez\overline{z}^4) \\ &= iz + (z\overline{z})^{(d-3)/2}(Bz^3 - 3\overline{B}z^2 + E\overline{z}^3). \end{aligned} \tag{3.3}$$

If we rescale system (3.3) by $|z|^{d-3}$, we obtain

$$\dot{z} = iz|z|^{3-d} + Bz^3 - 3\overline{B}z^2 + E\overline{z}^3 = i\,\frac{\partial H}{\partial \overline{z}},$$

where, for $d = 5$,

$$H = \log|z|^2 - i(Bz^3\overline{z} - \overline{B}z\overline{z}^3) - \frac{i}{4}(E\overline{z}^4 - \overline{E}z^4),$$

and, for $d \geq 7$ odd, we have

$$H = \frac{2}{5-d}|z|^{5-d} - i(Bz^3\overline{z} - \overline{B}z\overline{z}^3) - \frac{i}{4}(E\overline{z}^4 - \overline{E}z^4).$$

Note that the first integrals $\exp(H)$ for $d = 5$ and H for $d \geq 7$ odd are real functions well defined at the origin. Therefore, the origin is a center. □

Proof of (c). The conditions in the real parameters are $a_3 = a_5 = a_9 = a_{11} = 0$. Note that, in this case, we are under the assumptions of Lemma 2.1 with $\gamma = 0$. Hence, by Lemma 2.1, and under the conditions of statement (3), system (1.1) is reversible and consequently has a center at the origin. □

Proof of (d). The conditions in the real parameters are $a_5 = a_9 = a_{10} = 2a_4 - a_8 = a_3 = 0$. In this case, system (1.1) takes the form:

$$\dot{z} = iz + (z\overline{z})^{(d-5)/2}(Bz^4\overline{z} - 2\overline{B}z^2\overline{z}^3 + F\overline{z}^5). \tag{3.4}$$

Rescaling by $(z\overline{z})^{(d-5)/2} = |z|^{d-5}$, system (3.4) becomes

$$\dot{z} = iz|z|^{5-d} + Bz^4\overline{z} - 2\overline{B}z^2\overline{z}^3 + F\overline{z}^5 = i\frac{\partial H}{\partial \overline{z}}, \tag{3.5}$$

where, for $d \geq 5$ odd with $d \neq 7$, we have

$$H = \frac{2}{7-d}|z|^{7-d} - \frac{i}{2}Bz^4\overline{z}^2 + \frac{i}{2}\overline{B}z^2\overline{z}^4 - \frac{i}{6}F\overline{z}^6 + \frac{i}{6}\overline{F}z^6,$$

and, for $d = 7$, we have

$$H = \log|z|^2 - \frac{i}{2}Bz^4\overline{z}^2 + \frac{i}{2}\overline{B}z^2\overline{z}^4 - \frac{i}{6}F\overline{z}^6 + \frac{i}{6}\overline{F}z^6.$$

Note that the first integrals $\exp(H)$ for $d = 7$ and H for $d \geq 5$ odd with $d \neq 7$ are real functions, well defined at the origin. Therefore, in this case, the origin is a Hamiltonian center. □

4. Proof of Theorem 1.3.

Proof of (b). The conditions in the real parameters are $a_3 = a_4 = a_5 = a_9 = 5a_2 - a_{10} = 0$. System (1.1) can be written as:

$$\dot{z} = iz + (z\overline{z})^{(d-5)/2}(Az^5 + i\,\mathrm{Im}(C)z^3\overline{z}^2 - 5\overline{A}z\overline{z}^4 + F\overline{z}^5). \tag{4.1}$$

If we rescale system (4.1) by $|z|^{d-5}$, we obtain

$$\dot{z} = iz|z|^{5-d} + Az^5 + i\,\mathrm{Im}(C)z^3\overline{z}^2 - 5\overline{A}z\overline{z}^4 + F\overline{z}^5 = i\frac{\partial H}{\partial \overline{z}},$$

where, for $d \geq 5$ odd with $d \neq 7$, we have

$$H = \frac{2}{7-d}|z|^{7-d} - i(Az^5\overline{z} - \overline{A}z\overline{z}^5) + \frac{\mathrm{Im}(C)}{3}z^3\overline{z}^3 - \frac{i}{6}(F\overline{z}^6 - \overline{F}z^6),$$

and, for $d = 7$, we have

$$H = \log|z|^2 - i(Az^5\overline{z} - \overline{A}z\overline{z}^5) + \frac{\mathrm{Im}(C)}{3}z^3\overline{z}^3 - \frac{i}{6}(F\overline{z}^6 - \overline{F}z^6).$$

Note that the first integrals $\exp(H)$ for $d = 7$ and H for $d \geq 5$ odd, $d \neq 7$, are real functions, well defined at the origin. Therefore, the origin is a center. □

Proof of (c). The conditions in real parameters are $a_{11} = a_{12} = a_9 = a_5 = a_2 + 3a_{10} = 0$. In this case, the associated complex differential system (2.11) is also the Lotka-Volterra case studied in [**14**]. In the real coordinates system (1.1), under the conditions of this case, we have

$$\begin{aligned}(4.2)\quad \dot{x} = -y + (x^2+y^2)^{(d-5)/2}\big(&a_3x^5 + 18a_{10}x^4y - 3a_4x^4y - a_6x^4y \\ &- 2a_3x^3y^2 - 28a_{10}x^2y^3 - 2a_4x^2y^3 - 2a_6x^2y^3 \\ &- 3a_3xy^4 + 2a_{10}y^5 + a_4y^5 - a_6y^5\big), \\ \dot{y} = x - (x^2+y^2)^{(d-5)/2}\big(&2a_{10}x^5 - a_4x^5 - a_6x^5 - 3a_3x^4y \\ &- 28a_{10}x^3y^2 + 2a_4x^3y^2 - 2a_6x^3y^2 - 2a_3x^2y^3 \\ &+ 18a_{10}xy^4 + 3a_4xy^4 - a_6xy^4 + a_3y^5\big).\end{aligned}$$

System (4.2) has the invariant curve $f = x^2 + y^2$ and the inverse integrating factor $V = (x^2+y^2)^{(d+3)/2}$ which, by integration, gives an analytic first integral at the origin. □

Proof of (d). The conditions in real parameters are $a_{11} = a_{12} = a_9 = a_5 = a_6 = a_3 - a_4 = 7a_2 + a_{10} = 49a_4^2 - 8a_{10}^2 = 0$. In this case, the associated complex differential system (2.11) is also the Lotka-Volterra case studied in [**14**]. We take $a_3 = a_4$ and $a_{10} = -7a_2$ and $a_4 = \pm 2\sqrt{2}a_2$. In this case, the complex differential system (2.11) is given by

$$\tag{4.3} \begin{aligned} \dot{x} &= x + a_2x^5 \pm (2-2i)\sqrt{2}a_2x^4y - 7a_2xy^4,\\ \dot{y} &= -y + 7a_2x^4y \mp (2+2i)\sqrt{2}a_2xy^4 - a_2y^5. \end{aligned}$$

System (4.4) has the invariant curve of degree 8 given by

$$\begin{aligned} f(x,y) = 1 &+ 2a_2x^4 + a_2^2x^8 \mp (2-2i)\sqrt{2}a_2x^3y\\ &\mp \left(\frac{10}{3} - \frac{10i}{3}\right)\sqrt{2}a_2^2x^7y - 20ia_2^2x^6y^2 \mp (2+2i)\sqrt{2}a_2xy^3\\ &\pm (18+18i)\sqrt{2}a_2^2x^5y^3 + 2a_2y^4 - \frac{130}{3}a_2^2x^4y^4\\ &\pm (18-18i)\sqrt{2}a_2^2x^3y^5 + 20ia_2^2x^2y^6\\ &\mp \left(\frac{10}{3} + \frac{10i}{3}\right)\sqrt{2}a_2^2xy^7 + a_2^2y^8. \end{aligned}$$

Moreover, system (4.4) has the first integral $H(x,y) = x^ay^bf(x,y)^c$, where

$$\begin{aligned} a &= (-1)^{1/4}(3(-1)^{3/4} - (2-2i)\sqrt{2})/3,\\ b &= i(3i + (2+2i)(-1)^{1/4}\sqrt{2})/3,\\ c &= -i(-3i + (4+4i)(-1)^{1/4}\sqrt{2})/6. \end{aligned}$$ □

Proof of (e). The conditions in real parameters are $a_{11} = a_{12} = a_9 = a_5 = a_6 = a_3 + a_4 = 7a_2 + a_{10} = 49a_4^2 - 8a_{10}^2 = 0$. In this case, the associated complex differential system (2.11) is also the Lotka-Volterra case studied in [**14**]. We take $a_3 = a_4$ and $a_{10} = -7a_2$ and $a_4 = \pm 2\sqrt{2}a_2$. In this case, the complex differential system (2.11) is given by

$$\tag{4.4} \begin{aligned} \dot{x} &= x + a_2x^5 \pm (2+2i)\sqrt{2}a_2x^4y - 7a_2xy^4,\\ \dot{y} &= -y + 7a_2x^4y \mp (2-2i)\sqrt{2}a_2xy^4 - a_2y^5. \end{aligned}$$

System (4.4) has the invariant curve of degree 8 given by

$$\begin{aligned} f(x,y) = 1 &+ 2a_2x^4 + a_2^2x^8 \mp (2+2i)\sqrt{2}a_2x^3y\\ &\mp \left(\frac{10}{3} + \frac{10i}{3}\right)\sqrt{2}a_2^2x^7y - 20ia_2^2x^6y^2 \mp (2-2i)\sqrt{2}a_2xy^3 \end{aligned}$$

$$\pm (18-18i)\sqrt{2}a_2^2x^5y^3 + 2a_2y^4 - \frac{130}{3}a_2^2x^4y^4$$
$$\pm (18+18i)\sqrt{2}a_2^2x^3y^5 + 20ia_2^2x^2y^6$$
$$\mp \left(\frac{10}{3} - \frac{10i}{3}\right)\sqrt{2}a_2^2xy^7 + a_2^2y^8.$$

Moreover, system (4.4) has the first integral $H(x,y) = x^a y^b f(x,y)^c$, where

$$a = (-1)^{3/4}(3(-1)^{1/4} + (2+2i)\sqrt{2})/3,$$
$$b = (-3 + (2+2i)(-1)^{3/4}\sqrt{2})/3,$$
$$c = (-3 - (4+4i)(-1)^{3/4}\sqrt{2})/6. \qquad \square$$

Proof of (f). The conditions in real parameters are $a_{11} = a_{12} = a_9 = a_5 = a_6 = a_{10} + 3a_2 = 0$ and $16a_2^2 - a_3^2 - a_4^2 = 0$. In this case, the associated complex differential system (2.10) is the Lotka-Volterra case studied in [**14**]. Performing the change $\xi = (1/a_2)^{1/4}$, we can take $a_2 = 1$. Now, taking $a_3 = \pm 4\cos\psi$ and $a_4 = \pm 4\sin\psi$ in real coordinates the system takes the form

$$(4.5) \qquad \begin{aligned} \dot{x} &= -y + 4x^4y + 16x^2y^3 - 4y^5 \pm 4x^5\cos\psi \mp 8x^3y^2\cos\psi \\ &\quad \mp 12xy^4\cos\psi \mp 12x^4y\sin\psi \mp 8x^2y^3\sin\psi \pm 4y^5\sin\psi, \\ \dot{y} &= x + 4x^5 - 16x^3y^2 - 4xy^4 \pm 12x^4y\cos\psi \pm 8x^2y^3\cos\psi \\ &\quad \mp 4y^5\cos\psi \pm 4x^5\sin\psi \mp 8x^3y^2\sin\psi \mp 12xy^4\sin\psi. \end{aligned}$$

In this case, the complex differential system (2.11) is given by

$$(4.6) \qquad \begin{aligned} \dot{x} &= x + x^5 + 3xy^4 \mp 4ix^4y\cos\psi \pm 4x^4y\sin\psi, \\ \dot{y} &= -y - 3x^4y - y^5 \mp 4ixy^4\cos\psi \pm 4xy^4\sin\psi. \end{aligned}$$

System (4.6) is Lotka-Volterra; consequently, it has invariant curves $x = 0$ and $y = 0$. Moreover, it has the invariant curve of degree 12 given by $f = 0$, where f is

$$f = 1 + 24x^4y^4(1 + 4x^4 + 4y^4)$$
$$+ 4xy\Big[\pm i(x-y)(x+y)(-3 + 4x^2y^2(3 + 2(x^2-y^2)^2))\cos\psi$$

$$- xy(9y^4 + x^4(9 + 16y^4)) \cos 2\psi - 9ixy(x^4 - y^4) \sin 2\psi \\ \pm (x^2 + y^2)(3 + 4x^2y^2(3 + 2(x^2 + y^2)^2)) \sin \psi \Big].$$

Moreover, an inverse integrating factor of system (4.6) is given by $V = x^{-1}y^{-1}f^{5/6}$. This inverse integrating factor is not well defined at the origin. However, applying Lemma 2.2, system (4.6) has an analytic first integral at the origin, and consequently, so does system (4.5). □

Proof of (g). The four conditions of statement (g) in real parameters are $a_9 = a_5 = a_6 = a_3 = a_4 = 3a_2+5a_{10} = 16a_{10}^2-9(a_{11}^2+a_{12}^2) = 0$. Performing the change of variables $\xi = (1/a_{10})^{1/4}$ where the last condition is $|F| = 4|a_{10}|/3$, we obtain $F = 4/3|a_{10}|e^{i\psi}$ with $\psi \in (0, 2\pi]$. Then, we get

$$\dot{z} = iz - i\frac{5}{3}z^5 + iz\overline{z}^4 \pm \frac{4}{3}e^{i\psi}\overline{z}^5. \tag{4.7}$$

In real coordinates, system (4.7) becomes

$$\begin{aligned} \dot{x} &= -y + \frac{34}{3}x^4y - \frac{44}{3}x^2y^3 + \frac{2}{3}y^5 \pm \frac{4}{3}x^5 \cos\psi \mp \frac{40}{3}x^3y^2 \cos\psi \\ &\quad \pm \frac{20}{3}xy^4 \cos\psi \pm \frac{20}{3}x^4y \sin\psi \mp \frac{40}{3}x^2y^3 \sin\psi \pm \frac{4}{3}y^5 \sin\psi, \\ \dot{y} &= x - \frac{2}{3}x^5 + \frac{44}{3}x^3y^2 - \frac{34}{3}xy^4 \mp \frac{20}{3}x^4y \cos\psi \pm \frac{40}{3}x^2y^3 \cos\psi \\ &\quad \mp \frac{4}{3}y^5 \cos\psi \pm \frac{4}{3}x^5 \sin\psi \mp \frac{40}{3}x^3y^2 \sin\psi \pm \frac{20}{3}xy^4 \sin\psi. \end{aligned} \tag{4.8}$$

In this case, the complex differential system (2.11) is given by

$$\begin{aligned} \dot{x} &= x - \frac{5}{3}x^5 + xy^4 \mp \frac{4}{3}iy^5 \cos\psi \pm \frac{4}{3}y^5 \sin\psi, \\ \dot{y} &= -y - x^4y + \frac{5}{3}y^5 \mp \frac{4}{3}ix^5 \cos\psi \mp \frac{4}{3}x^5 \sin\psi. \end{aligned} \tag{4.9}$$

In fact, if we compute Poincaré-Liapunov constants for system (4.9), we obtain that the first 12 are zero, but the next is nonzero, and its value is $V_{13} = \pi \sin 4\psi$. Therefore, we have that this constant vanishes only for $\psi = k\pi/4$ with $k \in \mathbb{Z}$. Hence, $\psi = 0 + k\pi$, $\psi = \pi/2 + k\pi$, $\psi = \pi/4 + k\pi$ and $\psi = 3\pi/4 + k\pi$ with $k \in \mathbb{Z} \setminus \{0\}$. The first two cases give time-reversible systems. For the third and fourth cases, system (4.9) takes the form

$$\begin{aligned} \dot{x} &= x - \frac{5}{3}x^5 + xy^4 \pm \frac{2\sqrt{2}}{3}(1-i)y^5, \\ \dot{y} &= -y - x^4y + \frac{5}{3}y^5 \mp \frac{2\sqrt{2}}{3}(1+i)x^5. \end{aligned} \tag{4.10}$$

System (4.10) has no invariant algebraic curves of degree ≤ 16 except the curve of fourth degree $f_1 = 1-x^4 \pm (1-i)\sqrt{2}\, x^3 y \pm (1+i)\sqrt{2}\, xy^3 - y^4$. From now on, we work only with system (4.10) with upper signs to simplify the computations. For the other determination, we can obtain similar results. We write f_1 as

$$f_1 = 1 - ((-1-i)x + \sqrt{2}y)^3((1+i)x + \sqrt{2}y)/4.$$

This factorization suggests the following change of coordinates:

$$X = (1+i)x + \sqrt{2}y \quad \text{and} \quad Y = (-1-i)x + \sqrt{2}y,$$

whose inverse change is

$$x = \frac{1}{4}(1-i)(X-Y), \qquad Y = \frac{1}{2\sqrt{2}}\,(X+Y).$$

With these new coordinates, system (4.10) with the upper signs becomes

$$\begin{aligned} \dot{X} &= -Y + \frac{X^5}{16} + \frac{X^3Y^2}{2} + \frac{3XY^4}{16}, \\ \dot{Y} &= -X - \frac{X^4Y}{48} + \frac{X^2Y^3}{12} + \frac{Y^5}{48}, \end{aligned} \tag{4.11}$$

and the invariant curve has the form $\widetilde{f}_1 = 1 - XY^3/4$. Now, we have the transformation

$$U = \frac{1 - G/12}{(1-G/4)^{1/3}} - 1, \qquad V = -\frac{3G^2 + Y^8}{144(1-G/4)^{2/3}},$$

where $G = XY^3$ and system (4.11) takes the form

$$\dot{U} = V, \qquad \dot{V} = -7(U+1)V - 4(3U + 3U^2 + U^3). \tag{4.12}$$

Finally, we have the rotation $U = u + v$, $V = -4u - 3v$, obtaining the system

$$\begin{aligned} \dot{u} &= -4u - 16u^2 + 4u^3 - 25uv + 12u^2v - 9v^2 + 12uv^2 + 4v^3, \\ \dot{v} &= -3v + 16u^2 - 4u^3 + 25uv - 12u^2v + 9v^2 - 12uv^2 - 4v^3. \end{aligned} \tag{4.13}$$

System (4.13) has a node at the origin whose eigenvalues are 3 and 4 and, consequently, is a linearizable node, see [**7**]. Moreover, it is easy to check that, going back through all the changes of coordinates, pulls the first meromorphic integral back (or the linearizing change of coordinates) to a first integral of the original system (4.10). Thus, for this case, we have a center. □

Proof of (h). The conditions in real parameters are $a_3 = a_5 = a_9 = a_{11} = 0$. Note that, in this case, we are under the assumptions of Lemma 2.1 with $\gamma = 0$. Hence, by Lemma 2.1, under the conditions of statement (8), system (1) is reversible and consequently has a center at the origin. □

Proof of (i). The conditions in real parameters are $a_5 = a_9 = a_{10} = a_4 - a_{12} = a_3 + a_{11} = a_2 - a_6 = a_6^2 - (a_{11}^2 + a_{12}^2)$. Making the change of variables $\xi = (1/a_6)^{1/4}$ and since the last condition is $|F| = |C|$, we get that $F = |a_6|e^{i\psi}$ with $\psi \in (0, 2\pi]$. Moreover, we have $B = -\overline{F}$, that is, $B = -|a_6|e^{-i\psi}$. Then, we obtain

$$\dot{z} = iz + iz^5 \mp e^{-i\psi} z^4\overline{z} + iz^3\overline{z}^2 \pm e^{i\psi}\overline{z}^5. \tag{4.14}$$

In real coordinates, system (4.14) becomes

$$\begin{aligned} \dot{x} &= -y - 6x^4y + 8x^2y^3 - 2y^5 \mp 8x^3y^2\cos\psi \pm 8xy^4\cos\psi \\ &\quad \pm 2x^4y\sin\psi \mp 12x^2y^3\sin\psi + 2y^5\sin\psi, \\ \dot{y} &= x + 2x^5 - 8x^3y^2 + 6xy^4 \mp x^4y\cos\psi \pm 8x^2y^3\cos\psi \\ &\quad \pm 2x^5\sin\psi \mp 12x^3y^2\sin\psi \pm 2xy^4\sin\psi. \end{aligned} \tag{4.15}$$

In this case, the complex differential system (2.11) is given by

$$\begin{aligned} \dot{x} &= x + x^5 + x^3y^2 \pm i(xy^4 - y^5)\cos\psi \pm (x^4y + y^5)\sin\psi, \\ \dot{y} &= -y - x^2y^3 - y^5 \mp i(x^5 - xy^4)\cos\psi \mp (x^5 + xy^4)\sin\psi. \end{aligned} \tag{4.16}$$

System (4.16) has the invariant curve $f_1 = 1 + (x^2 + y^2)^2$ and the invariant curve of degree 8

$$f_2 = \frac{1}{4}\Big(4 + 2(x^2+y^2)^2(2+3x^2y^2) + 4ix(x-y)y(x+y)(2+(x^2+y^2)^2)\cos\psi - (x^2+y^2)^2(x^4+y^4)\cos 2\psi + 4xy(x^2+y^2)(2+(x^2+y^2)^2)\sin\psi + i(x-y)(x+y)(x^2+y^2)^3\sin(2\psi)\Big).$$

Moreover, system (4.16) has an inverse integrating factor of the form $V = f_1^{1/4} f_2$, well defined at the origin. □

Proof of (j). The conditions in real parameters are $a_5 = a_9 = a_{10} = a_4 + a_{12} = a_3 - a_{11} = a_2 + a_6 = a_6^2 - (a_{11}^2 + a_{12}^2)$. Making the change of variables $\xi = (1/a_6)^{1/4}$ and, since the last condition is $|F| = |C|$, we get that $F = |a_6|e^{i\psi}$ with $\psi \in (0, 2\pi]$. Moreover, we have $B = \overline{F}$, that is, $B = |a_6|e^{-i\psi}$. Then, we obtain

$$\dot{z} = iz - iz^5 \pm e^{-i\psi} z^4\overline{z} + iz^3\overline{z}^2 \pm e^{i\psi}\overline{z}^5. \tag{4.17}$$

In real coordinates, system (4.17) becomes

$$\begin{aligned}\dot{x} &= -y + 4x^4y - 12x^2y^3 \pm 2x^5\cos\psi \mp 12x^3y^2\cos\psi \\ &\quad \pm 2xy^4\cos\psi \pm 8x^4y\sin\psi \mp 8x^2y^3\sin\psi, \\ \dot{y} &= x + 12x^3y^2 - 4xy^4 \mp 2x^4y\cos\psi \pm 12x^2y^3\cos\psi \\ &\quad \mp 2y^5\cos\psi \mp 8x^3y^2\sin\psi \pm 8xy^4\sin\psi.\end{aligned} \tag{4.18}$$

In this case, the complex differential system (2.11) is given by

$$\begin{aligned}\dot{x} &= x - x^5 + x^3y^2 \mp i(xy^4 + y^5)\cos\psi \mp (x^4y - y^5)\sin\psi, \\ \dot{y} &= -y - x^2y^3 + y^5 \mp i(x^5 + xy^4)\cos\psi \mp (x^5 - xy^4)\sin\psi.\end{aligned} \tag{4.19}$$

System (4.19) has the invariant curve $f = 1 - (x^2 + y^2)^2$ and the invariant curve of degree 8

$$f_2 = \frac{1}{4}\Big(4 - 2(x^2-y^2)^2(2+3x^2y^2) + 4ix(x-y)y(x+y)(-2+(x^2-y^2)^2)\cos\psi$$

$$+ (x^2 - y^2)^2(x^4 + y^4)\cos 2\psi - i(x^2 - y^2)^3 \sin 2\psi$$
$$+ (x^2 + y^2)(4xy(-2 + (x^2 - y^2)^2)) \sin\psi\Big).$$

Moreover, system (4.19) has an inverse integrating factor of the form $V = f_1^{1/4} f_2$, well defined at the origin. □

Proof of (k). The conditions in real parameters are $a_5 = a_9 = 0$ and

$$
\begin{aligned}
p_1 &= a_2 - a_{10} = 0\\
p_2 &= a_4a_{11} + a_3a_{12} = 0,\\
p_3 &= 2a_6a_{10} + a_3a_{11} - a_4a_{12} = 0,\\
p_4 &= a_6^2 - a_{11}^2 - a_{12}^2 = 0,\\
p_5 &= a_4a_6 - 2a_{10}a_{12} = 0,\\
p_6 &= a_3a_6 + 2a_{10}a_{11} = 0,\\
p_7 &= a_3^2 + a_4^2 - 4a_{10}^2 = 0.
\end{aligned}
\tag{4.20}
$$

We can take $a_6 = 1$ by making the change $\xi = (1/a_6)^{1/4}$. From $p_1 = 0$, we get $a_2 = a_{10}$. Furthermore, condition $p_4 = 0$ implies $|F| = |a_6|$, and thus, $F = |a_6|e^{i\psi} = \pm e^{i\psi}$, i.e., $a_{11} = \pm\sin\psi$, $a_{12} = \pm\cos\psi$. From $p_5 = 0$, we get $a_4 = 2a_{10}a_{12}$, and, from $p_6 = 0$, we get $a_3 = -2a_{10}a_{11}$. With these parameters, we obtain that $p_j = 0$ for $j = 1, \dots, 7$.

In real coordinates, we get

$$
\begin{aligned}
\dot{x} = {}& -y - x^4y - 2a_{10}x^4y - 2x^2y^3 + 12a_{10}x^2y^3 - y^5 - 2a_{10}y^5\\
&\pm 5x^4y\cos\psi \mp 6a_{10}x^4y\cos\psi \mp 10x^2y^3\cos\psi \mp 4a_{10}x^2y^3\cos\psi\\
&\pm y^5\cos\psi \pm 2a_{10}y^5\cos\psi \pm x^5\sin\psi \mp 2a_{10}x^5\sin\psi\\
&\mp 10x^3y^2\sin\psi \pm 4a_{10}x^2y^3\sin\psi \pm 5xy^4\sin\psi \pm 6a_{10}xy^4\sin\psi,\\
\dot{y} = {}& x + x^5 + 2a_{10}x^5 + 2x^3y^2 - 12a_{10}x^3y^2 + xy^4 + 2a_{10}xy^4\\
&\pm x^5\cos\psi \pm 2a_{10}x^5\cos\psi \mp 10x^3y^2\cos\psi \mp 4a_{10}x^3y^2\cos\psi\\
&\pm 5xy^4\cos\psi \mp 6a_{10}xy^4\cos\psi \mp 5x^4y\sin\psi \mp 6a_{10}x^4y\sin\psi\\
&\pm 10x^2y^3\sin\psi \mp 4a_{10}x^2y^3\sin\psi \mp y^5\sin\psi \pm 2a_{10}y^5\sin\psi.
\end{aligned}
$$

We integrate this system into the complex saddle form (2.11) as

$$
\begin{aligned}
\dot{x} = {}& x + a_{10}x^5 + x^3y^2 + a_{10}xy^4\\
&\mp iy^5(i\cos\psi + \sin\psi) \mp ia_{10}x^4y(2i\cos\psi - 2\sin\psi),
\end{aligned}
\tag{4.21}
$$

$$\dot{y} = -y - a_{10}x^4y - x^2y^3 - a_{10}y^5 \pm x^5(-\cos\psi - i\sin\psi) \pm 2a_{10}xy^4(-\cos\psi + i\sin\psi).$$

System (4.21) is Darboux integrable because it has three invariant algebraic curves of degree 4 of the form $f_i(0,0) \neq 0$ and, with these three curves, it is possible to construct an integrating factor of system (4.21) of the form $V = f_1f_2f_3$. Consequently, it has a complex center at the origin. □

Proof of (l). The conditions in real parameters are $a_5 = a_9 = a_6 = a_4 - a_{12} = a_3 + a_{11} = a_2 + a_{10} = 4a_{10}^2 - (a_{11}^2 + a_{12}^2) = 0$.

Making the change of variables $\xi = (1/a_{10})^{1/4}$, and, since the last condition is $|F| = 2|E|$, we get that $F = 2|a_{10}|e^{i\psi}$ with $\psi \in (0, 2\pi]$. Moreover, we have $B = -\overline{F}$, that is, $B = -2|a_{10}|e^{-i\psi}$. Then, we get

$$\dot{z} = iz - iz^5 \pm 2e^{-i\psi}z^4\overline{z} + iz\overline{z}^4 \pm 2e^{i\psi}\overline{z}^5. \tag{4.22}$$

In real coordinates, system (4.22) becomes

$$\begin{aligned} \dot{x} &= -y + 8x^4y - 8x^2y^3 \\ &\quad \mp 16x^3y^2\cos\psi \pm 16xy^4\cos\psi \\ &\quad \pm 4x^4y\sin\psi \mp 24x^2y^3\sin\psi \pm 4y^5\sin\psi, \\ \dot{y} &= x + 8x^3y^2 - 8xy^4 \\ &\quad \mp 16x^4y\cos\psi \pm 16x^2y^3\cos\psi \\ &\quad \pm 4x^5\sin\psi \mp 24x^3y^2\sin\psi \pm 4xy^4\sin\psi. \end{aligned} \tag{4.23}$$

We integrate this system into the complex saddle form (2.11) as

$$\begin{aligned} \dot{x} &= x - x^5 + xy^4 \pm 2i(x^4y - y^5)\cos\psi \pm 2(x^4y + y^5)\sin\psi, \\ \dot{y} &= -y - x^4y + y^5 \mp 2i(x^5 - xy^4)\cos\psi \mp 2(x^5 + xy^4)\sin\psi. \end{aligned} \tag{4.24}$$

System (4.24) is Darboux integrable because it has three invariant algebraic curves of degree 4 given by $f_1 = 1 - (x^2 + y^2)^2$ and two other curves that we do not write here due to their extension. In order to prove their existence, we take polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ in the real system (4.23) and, following **[11]**, we impose the existence of an invariant algebraic curve of degree a that, in polar coordinates, takes the form $f = 1 + U_1(\theta)r^4$, i.e., f must satisfy the equation

$$\frac{\partial f}{\partial r}\dot{r} + \frac{\partial f}{\partial \psi}\dot{\psi} - (U'(\psi)r^4)f \equiv 0, \tag{4.25}$$

Now, substituting $U_1(\theta)$ by an arbitrary homogeneous polynomial of degree 4, i.e., taking $U_1(\theta) = B_1 \cos 4\theta + B_2 \sin 4\theta + B_3 \cos 2\theta + B_4 \sin 2\theta + B_5$, it is easy to prove that equation (4.25) has three solutions f_1, f_2 and f_3, where f_1 has been previously given. Moreover, $V = f_1^{-1/2} f_2 f_3$ is an inverse integrating factor of system (4.24). □

Proof of (m). The conditions in real parameters are $a_5 = a_9 = a_6 = a_4 + a_{12} = a_3 - a_{11} = a_2 + a_{10} = 4a_{10}^2 - (a_{11}^2 + a_{12}^2) = 0$. Making the change of variables $\xi = (1/a_6)^{1/4}$ and, since the last condition is $|F| = 2|E|$, we get that $F = 2|a_{10}|e^{i\psi}$ with $\psi \in (0, 2\pi]$. Moreover, we have $B = \overline{F}$, that is, $B = 2|a_{10}|e^{-i\psi}$. Then, we obtain

$$\dot{z} = iz - iz^5 \mp 2e^{-i\psi} z^4 \overline{z} + iz\overline{z}^4 \pm 2e^{i\psi} \overline{z}^5. \tag{4.26}$$

In real coordinates, system (4.26) becomes

$$\begin{aligned} \dot{x} &= -y + 8x^4 y - 8x^2 y^3 \pm 4x^5 \cos\psi \mp 24x^3 y^2 \cos\psi \\ &\quad \pm 4xy^4 \cos\psi \pm 16x^4 y \sin\psi \mp 16x^2 y^3 \sin\psi, \\ \dot{y} &= x + 8x^3 y^2 - 8xy^4 \mp 4x^4 y \cos\psi \pm 24x^2 y^3 \cos\psi \\ &\quad \mp 4y^5 \cos\psi \mp 16x^3 y^2 \sin\psi \pm 16xy^4 \sin\psi. \end{aligned} \tag{4.27}$$

We can integrate this system into the complex saddle form (2.11) as

$$\begin{aligned} \dot{x} &= x - x^5 + xy^4 \mp 2i(x^4 y + y^5)\cos\psi \mp 2(x^4 y - y^5)\sin\psi, \\ \dot{y} &= -y - x^4 y + y^5 \mp 2i(x^5 + xy^4)\cos\psi \mp 2(x^5 - xy^4)\sin\psi. \end{aligned} \tag{4.28}$$

System (4.28) is Darboux integrable because it has three invariant algebraic curves of degree 4 given by $f_1 = 1 - (x^2 - y^2)^2$ and two other curves that we do not write here due to their extension. However, as in the previous case, we can prove their existence. Moreover, this case also has an inverse integrating factor of the form $V = f_1^{-1/2} f_2 f_3$. □

Proof of (n). The conditions in real parameters are $a_4 = a_5 = a_9 = a_{12} = 0$. Note that, in this case, we are under the assumptions of Lemma 2.1 with $\gamma = \pi/2$. Hence, by Lemma 2.1, under the conditions of statement (14), system (1.1) is reversible and, consequently, has a center at the origin. □

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