ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 47, Number 2, 2017

# MULTIPLE SOLUTIONS FOR KIRCHHOFF-TYPE PROBLEMS WITH CRITICAL GROWTH IN $\mathbb{R}^N$

SIHUA LIANG AND JIHUI ZHANG

ABSTRACT. In this paper, we study the existence of infinitely many solutions for a class of Kirchhoff-type problems with critical growth in $\mathbb{R}^N$. By using a change of variables, the quasilinear equations are reduced to a semilinear one, whose associated functionals are well defined in the usual Sobolev space and satisfy the geometric conditions of the mountain pass theorem for suitable positive parameters $\alpha, \beta$. The proofs are based on variational methods and the concentration-compactness principle.

**1. Introduction.** In this paper, we consider a class of Kirchhoff-type problems involving critical growth of the form

$$-L_p^{a,b}u - a[\Delta_p(u^2)]u = \alpha k(x)|u|^{q-2}u + \beta u^{2(p^*)-2}u, \tag{1.1}$$

where

$$L_p^{a,b}u := \left(a + b\int_{\mathbb{R}^N} \frac{1}{p}|\nabla u|^p dx\right) \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad N \geq 3,$$

$$2 < q < 2p^* = \frac{2Np}{N-p}, \qquad k(x) \in L^r(\mathbb{R}^N),$$

with $r = 2(p^*)/(2(p^*) - q)$, $\alpha, \beta$ real parameters. We believe that $2(p^*)$ is the critical growth for problem (1.1), since when $p = 2$, the critical exponent is $2(2^*)$, see **[34, 35, 36]**.

---

2010 AMS *Mathematics subject classification.* Primary 35J60, Secondary 35J20, 35J25.

*Keywords and phrases.* Kirchhoff-type problems, infinitely many solutions, critical growth, concentration-compactness principle, variational methods.

The first author is supported by the National Natural Science Foundation of China, grant No. 11301038, the Natural Science Foundation of Jilin Province, grant No. 20160101244JC, and the open project program of Key Laboratory of Symbolic Computation and Knowledge Engineering of Ministry of Education, Jilin University, grant No. 93K172013K03. The second author is supported by NSFC, grant No. 11571176.

Received by the editors on October 5, 2014, and in revised form on May 2, 2015.

DOI:10.1216/RMJ-2017-47-2-527 

The acclaimed paper by Brezis and Nirenberg [**4**] generated great interest on problems involving critical exponents; we refer the reader to [**3, 5, 7, 17, 18, 20, 23, 28, 46**] and the references therein for the study of problems with a critical exponent.

On one hand, Kirchhoff-type problems are often referred to as being nonlocal because of the presence of the integral over the entire domain $\Omega$. It is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, namely,

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x,u), \tag{1.2}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $u$ denotes the displacement, $f(x,u)$ is the external force and $b$ is the initial tension while $a$ is related to the intrinsic properties of the string (such as Young's modulus). Equations of this type were first proposed by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string, specifically taking into account the subsequent change in string length caused by oscillations. The solvability of Kirchhoff-type equation (1.2) has been well studied in general dimensions and domains by various authors, see, for example, [**15, 16, 24, 26, 29, 39, 47, 51**] and the references therein.

Nonlocal effect also finds its applications in biological systems. A parabolic version of equation (1.3) can, in theory, be used to describe the growth and movement of a particular species. The movement, modeled by the integral term, is assumed dependent on the "energy" of the entire system with $u$ being its population density. Alternatively, the movement of a particular species may be subject to the total population density within the domain (for instance, the spread of bacteria) which gives rise to equations of the type $u_t - a(\int_{\Omega} u\, dx)\Delta u = f$. Chipot and Lovat [**8**] and Corrêa, et al. [**11**], for example, studied the existence of solutions and their uniqueness for such nonlocal problems as well as their corresponding elliptic problems.

It is well known that the stationary problem of equation (1.2) has the form

$$\begin{cases} -\left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x,u) & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$. Problem (1.3) gained interest only after Lions [**31**] proposed an abstract framework for the problem. Some important and interesting results can be found, see for example, [**1, 2, 12, 13, 21, 30, 37, 41, 43, 44**]. We note that the results dealing with problem (1.3) with critical nonlinearity are relatively scarce.

In [**16**], by means of a direct variational method, the authors proved the existence and multiplicity of solutions to a class of $p$-Kirchhoff-type problems with Dirichlet boundary data. The existence of infinite solutions to the $p$-Kirchhoff-type quasilinear elliptic equation was shown in [**33**]. In [**9**], the authors studied higher order $p(x)$-Kirchhoff-type problems via variational methods, even in the degenerate case. However, they did not give any further information on the sequence of solutions. Recently, Kajikiya [**25**] established a critical point theorem related to the symmetric mountain pass lemma and applied it to a sublinear elliptic equation. However, there are no such results on Kirchhoff-type problems (1.1).

On the other hand, there are many papers concerned with the quasilinear elliptic equation

$$-\Delta u + V(x)u - [\Delta(u^2)]u = h(x,u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

Such equations arise in various branches of mathematical physics, and they have been the subject of extensive study in recent years. In [**34**], by a change of variables, the quasilinear problem was transformed to a semilinear one, and an Orlicz space framework was used as the working space. They proved the existence of positive solutions of equation (1.4) by the mountain pass theorem. The same method of change of variables was used in [**10**], but the usual Sobolev space $H^1(\mathbb{R}^N)$ framework was used as the working space, and they studied a different class of nonlinearity. In [**35**], the existence of both one sign and nodal ground state-type solutions were established by the Nehari method.

Motivated by the reasons above, the aim of this paper is to show the existence of infinitely many solutions of problem (1.1) and that there exists a sequence of infinitely many arbitrarily small solutions converging to 0 by using a new version of the symmetric mountain pass lemma due to Kajikiya [**25**].

Note that $2(2^*)$ behaves like a critical exponent for the above equations, see [**34**]. For the subcritical case, the existence of solutions for problem (1.4) was studied in [**10, 34, 35, 36**], and it was left open for the critical exponent case, see [**34**]. To the best of our knowledge, the existence of nontrivial radial solutions for equation (1.4) with $h(u) = \mu u^{2(2^*)-1}$ was first studied by Moameni [**38**], using the same Orlicz space as in [**34**]. The existence of multiple solutions for problems (1.1) with $a = 1$ and $b = 0$ using minimax methods and the Krasnoselski genus theory was shown in [**48**]. For other interesting results, see [**22, 42**].

To the best of our knowledge, existence and multiplicity of solutions to problem (1.1) have yet to be studied using variational methods. In this paper, we show that problem (1.1) can be viewed as an elliptic equation coupled with a non-local term. The competing effect of the non-local term with critical nonlinearity and the lack of compactness of the embedding of $H^1(\mathbb{R}^N)$ into the space $L^p(\mathbb{R}^N)$ prevents us from using the variational methods in a standard way. Some new estimates for such a Kirchhoff equation involving Palais-Smale sequences, which are key points for applying this type of theory, need to be re-established. Primarily, we follow the ideas presented in [**19, 25**]. Although this theory has been used for other problems, the adaptation of the procedure to our problem is not at all trivial. The appearance of a non-local term prompts consideration of our problem for suitable spaces; thus, we need more delicate estimates.

The main result of this paper is as follows.

**Theorem 1.1.** *Suppose that $\Omega := \{x \in \mathbb{R}^N : k(x) > 0\}$ is an open subset of $\mathbb{R}^N$ and $0 < |\Omega| < +\infty$, $2 < p < 2q$. Then,*

(i) *for any $\beta > 0$, there exists $\alpha_0 > 0$ such that, if $0 < \alpha < \alpha_0$, then equation (1.1) has a sequence of solutions $\{u_n\}$ with $J(u_n) \le 0$, $J(u_n) \to 0$ and $\{u_n\}$ converges to $0$ as $n \to +\infty$;*

(ii) *for any $\alpha > 0$, there exists $\beta_0 > 0$ such that, if $0 < \beta < \beta_0$, then equation (1.1) has a sequence of solutions $\{u_n\}$ with $J(u_n) \le 0$, $J(u_n) \to 0$ and $\{u_n\}$ converges to $0$ as $n \to +\infty$.*

**Notation 1.2.** In this paper, we use the notation:

- $D^{1,p}(\mathbb{R}^N) : \{u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N)\}$ endowed with the norm

$$\|u\| = \left( \int_{\mathbb{R}^N} |\nabla u|^p \right)^{1/p}.$$

- For $1 \leq r < \infty$, $L^r(\mathbb{R}^N)$ denotes the usual Lebesgue space with norm

$$\|u\|_r = \left( \int_{\mathbb{R}^N} |u|^r \right)^{1/r}.$$

- $c$, $c_1$ and $c_2$ denote positive (possibly different) constants.

**2. Preliminaries.** The energy functional corresponding to problem (1.1) is defined as:

$$\begin{aligned} I(u) :=& \frac{a}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{b}{2} \left( \int_{\mathbb{R}^N} \frac{1}{p} |\nabla u|^p dx \right)^p \\ &+ \frac{a}{p} \int_{\mathbb{R}^N} 2^{p-1} |u|^p |\nabla u|^p dx \\ &- \frac{\alpha}{q} \int_{\mathbb{R}^N} k(x)|u|^q dx - \frac{\beta}{2(p^*)} \int_{\mathbb{R}^N} |u|^{2(p^*)} dx \\ =& \frac{a}{p} \int_{\mathbb{R}^N} (1 + 2^{p-1}|u|^p) |\nabla u|^p dx + \frac{b}{2} \left( \int_{\mathbb{R}^N} \frac{1}{p} |\nabla u|^p dx \right)^2 \\ &- \frac{\alpha}{q} \int_{\mathbb{R}^N} k(x)|u|^q dx - \frac{\beta}{2(p^*)} \int_{\mathbb{R}^N} |u|^{2(p^*)} dx. \end{aligned}$$

Note that, in general, the functional $I$ is not well defined in general in $D^{1,p}(\mathbb{R}^N)$, for instance. To overcome this difficulty, we employ an argument developed by Colin and Jeanjean **[10]**. We change the variables $v = f^{-1}(u)$, where $f$ is defined by

$$\begin{aligned} f'(t) &= \frac{1}{(1 + 2^{p-1}|f(t)|^p)^{1/p}}, \\ f(0) &= 0 \quad \text{on } [0, +\infty) \end{aligned}$$

and by

$$f(t) = -f(-t) \quad \text{on } (-\infty, 0].$$

The next result is due to Colin and Jeanjean **[10]**.

**Lemma 2.1.** *The following properties are satisfied by* $f$:

($f_0$) $f$ *is uniquely defined* $C^\infty$ *and invertible*;

($f_1$) $|f'(t)| \leq 1$ *for* $t \in \mathbb{R}$;

($f_2$) $|f(t)| \leq |t|$ *for* $t \in \mathbb{R}$;

($f_3$) $f(t)/t \to 1$ *as* $t \to \infty$;

($f_4$) $|f(t)| \leq 2^{1/(2p)}|t|^{1/2}$ *for* $t \in \mathbb{R}$;

($f_5$) $(1/2)f(t) \leq tf'(t) \leq f(t)$ *for all* $t \geq 0$;

($f_6$) $f(t)/\sqrt{t} \to 2^{1/4}$ *as* $t \to \infty$;

($f_7$) *there exists a positive constant* $C$ *such that*

$$|f(t)| \geq \begin{cases} C|t| & |t| \leq 1, \\ C|t|^{1/2} & |t| \geq 1. \end{cases}$$

Thus, after a change of variables, we may write $I(u)$ as

$$\begin{aligned} J(v) := {} & \frac{a}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{b}{2}\bigg(\int_{\mathbb{R}^N} \frac{1}{p}|f'(v)|^p|\nabla v|^p dx\bigg)^2 \\ & - \frac{\alpha}{q}\int_{\mathbb{R}^N} k(x)|f(v)|^q dx - \frac{\beta}{2(p^*)}\int_{\mathbb{R}^N} |f(v)|^{2(p^*)} dx. \end{aligned} \tag{2.1}$$

Then, $J(v)$ is well defined on $D^{1,p}(\mathbb{R}^N)$. Standard arguments **[45, 49]** show that $J(v)$ belongs to $C^1(D^{1,p}(\mathbb{R}^N), \mathbb{R})$.

As in **[10]**, we note that, if $v$ is a nontrivial critical point of $J$, then $v$ is a nontrivial solution to the problem:

$$\begin{aligned} & -a\Delta v - b\int_{\mathbb{R}^N} |f'(v)|^p|\nabla v|^p dx \\ & \cdot (|f'(v)|^{p-2}f'(v)f''(v)|\nabla v|^2 + |f'(v)|^p|\nabla v|^{p-2}\nabla v \Delta v) = g(x,v), \end{aligned} \tag{2.2}$$

where

$$g(x,s) = f'(s)(\alpha k(x)|f(s)|^{q-2}f(s) + \beta|f(s)|^{2(p^*)-2}f(s)).$$

Therefore, let $u = f(v)$, and, since

$$(f^{-1})'(t) = [f'(f^{-1}(t))]^{-1} = (1 + 2^{p-1}|t|^p)^{1/p},$$

we conclude that $u$ is a nontrivial solution to the problem

$$-\left(a + b\int_{\mathbb{R}^N} \frac{1}{p}|\nabla u|^p dx\right) \operatorname{div}(|\nabla u|^{p-2}\nabla u) - a[\Delta_p(u^2)]u$$
$$= \alpha k(x)|u|^{q-2}u + \beta u^{2(p^*)-2}u.$$

The main result of this paper is as follows.

**Theorem 2.2.** *Suppose that $\Omega := \{x \in \mathbb{R}^N : k(x) > 0\}$ is an open subset of $\mathbb{R}^N$ and $0 < |\Omega| < +\infty$, $2 < q < 2p$. Then,*

(i) *for any $\beta > 0$, there exists an $\alpha_0 > 0$ such that, if $0 < \alpha < \alpha_0$, then equation* (2.2) *has a sequence of solutions $\{v_n\}$ with $J(v_n) \leq 0$, $J(v_n) \to 0$ and $\{v_n\}$ converges to $0$ as $n \to +\infty$;*
(ii) *for any $\alpha > 0$, there exists a $\beta_0 > 0$ such that, if $0 < \beta < \beta_0$, then equation* (2.2) *has a sequence of solutions $\{v_n\}$ with $J(v_n) \leq 0$, $J(v_n) \to 0$ and $\{v_n\}$ converges to $0$ as $n \to +\infty$.*

We recall the second concentration-compactness principle of Lions **[32]**.

**Lemma 2.3** ([**32**]). *Let $\{u_n\}$ be a weakly convergent sequence to $u$ in $D^{1,p}(\mathbb{R}^N)$ such that $|u_n|^{p^*} \rightharpoonup \nu$ and $|\nabla u_n| \rightharpoonup \mu$ in the sense of measures. Then, for some at most countable index set $I$,*

(i) $\nu = |u|^{p^*} + \sum_{j\in I} \delta_{x_j}\nu_j$, $\nu_j > 0$;
(ii) $\mu \geq |\nabla u|^p + \sum_{j\in I} \delta_{x_j}\mu_j$, $\mu_j > 0$;
(iii) $\mu_j \geq S\nu_j^{p/p^*}$,

*where $S$ is the best Sobolev constant, i.e.,*

$$S = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^p dx : \int_{\mathbb{R}^N} |u|^{p^*} dx = 1\right\},$$

*$x_j \in \mathbb{R}^N$, $\delta_{x_j}$ are Dirac measures at $x_j$, and $\mu_j$, $\nu_j$ are constants.*

**Lemma 2.4** (**[6, 38]**). *Let $\{u_n\}$ be a weakly convergent sequence to $u$ in $D^{1,p}(\mathbb{R}^N)$, and define*

(i) $\nu_\infty = \lim\limits_{R\to\infty} \limsup\limits_{n\to\infty} \displaystyle\int_{|x|>R} |u_n|^{p^*} dx$;

(ii) $\mu_\infty = \lim\limits_{R\to\infty} \limsup\limits_{n\to\infty} \displaystyle\int_{|x|>R} |\nabla u_n|^p dx$.

*The quantities $\nu_\infty$ and $\mu_\infty$ exist and satisfy*

(iii) $\limsup_{n\to\infty} \displaystyle\int_{\mathbb{R}^N} |u_n|^{p^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty$;

(iv) $\limsup_{n\to\infty} \displaystyle\int_{\mathbb{R}^N} |\nabla u_n|^p dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty$;

(v) $\mu_\infty \geq S\nu_\infty^{p/p^*}$.

**Lemma 2.5.**

(i) *The functional*

$$\mathcal{F}(v) := \int_{\mathbb{R}^N} k(x)|f(v)|^q dx$$

*is well defined and weakly continuous on $D^{1,p}(\mathbb{R}^N)$. Moreover, $\mathcal{F}(v)$ is continuously differentiable; its derivative*

$$\mathcal{F}' : D^{1,p}(\mathbb{R}^N) \longrightarrow (D^{1,p}(\mathbb{R}^N))^*$$

*is given by*

$$\langle \mathcal{F}'(v), h\rangle = p\int_{\mathbb{R}^N} k(x)|f(v)|^{p-2} f(v) f'(v) h\, dx.$$

(ii) *The functional*

$$\mathcal{G}(v) := \int_{\mathbb{R}^N} |f(v)|^{2(p^*)} dx$$

*is well defined. Moreover, $\mathcal{G}(v)$ is continuously differentiable; its derivative*

$$\mathcal{G}' : D^{1,p}(\mathbb{R}^N) \longrightarrow (D^{1,p}(\mathbb{R}^N))^*$$

*is given by*

$$\langle \mathcal{G}'(v), h\rangle = 2(p^*)\int_{\mathbb{R}^N} f(v)^{2(p^*)-1} f'(v) h\, dx.$$

*Proof.* First, by Lemma 2.1 ($f_3$) and ($f_6$), it is clear that $\mathcal{F}(v)$ and $\mathcal{G}(v)$ are well defined on $D^{1,p}(\mathbb{R}^N)$. Next, we prove that $\mathcal{F}(v), \mathcal{G}(v) \in C^1(\mathbb{R}^N)$. It suffices to show that both $\mathcal{F}(v)$ and $\mathcal{G}(v)$ have continuous Gâteaux derivatives on $D^{1,p}(\mathbb{R}^N)$, see [**49**]. We consider only $\mathcal{F}(v)$ since the proof for $\mathcal{G}(v)$ is simpler. Our proof is similar to [**49,** Lemma 3.10]. Let $v, g \in D^{1,p}(\mathbb{R}^N)$. Given $0 < |t| < 1$, by the mean value theorem, there exists a $\lambda \in (0,1)$ such that

$$\begin{aligned}\frac{||f(v+tg)|^q - |f(v)|^q|}{|t|} &= q|f(v+t\lambda g)|^{q-1}|f'(v+t\lambda g)||g| \\ &\le c|v+t\lambda g|^{(q-2)/2}|g| \le c(|v|^{(q-2)/2}|g| + |g|^{q/2}).\end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} k(x)(|v|^{(q-2)/2}|g| + |g|^{q/2})\, dx \\ &\qquad\qquad \le \|k(x)\|_r \|g\|_{p^*}(\|v\|^{(q-2)/2} + \|g\|^{(q-2)/2}).\end{aligned}$$

It follows from the Lebesgue dominated convergence theorem that $\mathcal{F}(v)$ is Gâteaux differentiable and

$$\langle \mathcal{F}'(v), g\rangle = p\int_{\mathbb{R}^N} k(x)|f(v)|^{q-2} f(v) f'(v) g\, dx.$$

Now, we prove continuity of the Gâteaux derivative. Assume that $v_n \to v$ in $D^{1,p}(\mathbb{R}^N)$; then $f^2(v_n) \to f^2(v)$ in $D^{1,p}(\mathbb{R}^N)$. By the continuity of the embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, $f^2(v_n) \to f^2(v)$ in $L^{p^*}(\mathbb{R}^N)$. Let us define $\mathcal{K}(v) := pk(x)|f(v)|^{q-2}f(v)f'(v)$. Then $\mathcal{K} \in (L^{p^*}(\mathbb{R}^N), C(L^{p^*}(\mathbb{R}^N))')$. The proof is analogous to [**27**] and [**49,** Theorem A.4]. It follows that $\mathcal{K}(v_n) \to \mathcal{K}(v)$ in $(L^{p^*}(\mathbb{R}^N))'$. Using the Hőlder and Sobolev inequalities, we have

$$\begin{aligned}\langle \mathcal{F}'(v_n) - \mathcal{F}'(v), g\rangle &\le \|\mathcal{K}(v_n) - \mathcal{K}(v)\|_{(p^*)'}\|g\|_{p^*} \\ &\le c\|\mathcal{K}(v_n) - \mathcal{K}(v)\|_{(p^*)'}\|g\|.\end{aligned}$$

Hence, $\|\mathcal{F}'(v_n) - \mathcal{F}'(v)\| \to 0$ and $\mathcal{F} \in C^1$. □

Recall that a $C^1$ functional $J$ on Banach space $X$ is said to satisfy the Palais-Smale condition at level $c$, denoted $(PS)_c$, if every sequence $\{u_n\} \subset X$ satisfying $\lim\limits_{n\to\infty} J(u_n) = c$ and $\lim\limits_{n\to\infty} \|J(u_n)\|_{X^*} = 0$ has a convergent subsequence.

**Lemma 2.6.** *Assume that* $2 < q < 2p$. *Then, any* $(PS)_c$ *sequence* $\{v_n\}$ *is bounded in* $D^{1,p}(\mathbb{R}^N)$.

*Proof.* Let $\{v_n\}$ be a $(PS)_c$ sequence, that is,

$$\begin{aligned} c + o(1) &= J(v_n) \\ &= \frac{a}{p}\int_{\mathbb{R}^N} |\nabla v_n|^p dx \\ &\quad + \frac{b}{2}\biggl(\int_{\mathbb{R}^N} \frac{1}{p}|f'(v_n)|^p|\nabla v_n|^p dx\biggr)^2 \\ &\quad - \frac{\alpha}{q}\int_{\mathbb{R}^N} k(x)|f(v_n)|^q dx - \frac{\beta}{2(p^*)}\int_{\mathbb{R}^N} |f(v_n)|^{2(p^*)} dx, \end{aligned} \tag{2.3}$$

and, for any $w \in D^{1,p}(\mathbb{R}^N)$,

$$\begin{aligned} o(1)\|v_n\| &= \langle J'(v_n), w\rangle \\ &= a\int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \nabla w\, dx \\ &\quad - \alpha\int_{\mathbb{R}^N} k(x)|f(v_n)|^{q-2} f(v_n) f'(v_n) w\, dx \\ &\quad - \beta\int_{\mathbb{R}^N} f^{2(p^*)-1}(v_n) f'(v_n) w\, dx \\ &\quad + b\biggl(\int_{\mathbb{R}^N} \frac{1}{p}\frac{|\nabla v_n|^p}{1 + 2^{p-1}|f(v_n)|^p}\, dx\biggr)\Upsilon, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \Upsilon = \biggl(\int_{\mathbb{R}^N} &\frac{|\nabla v_n|^{p-2}\nabla v_n \nabla w (1 + 2^{p-1}|f(v_n)|^p)}{[1 + 2^{p-1}|f(v_n)|^p]^2} \\ &- \frac{2^{p-1}|\nabla v_n|^p |f(v_n)|^{p-2} f(v_n) f'(v_n) w}{[1 + 2^{p-1}|f(v_n)|^p]^2}\, dx\biggr). \end{aligned}$$

Choose

$$w = w_n = \sqrt[p]{1 + 2^{p-1}|f(v_n)|^p}\, f(v_n).$$

We have $w_n \in D^{1,p}(\mathbb{R}^N)$. From $(f_4)$, and, since

$$|\nabla w_n| = \biggl(1 + \frac{2^{p-1}|f(v_n)|^p}{1 + 2^{p-1}|f(v_n)|^p}\biggr)|\nabla v_n|,$$

we deduce that $\|w_n\| \leq c\|v_n\|$. In particular, noting that $\{v_n\}$ is a $(PS)_c$ consequence,

$$
\begin{aligned}
o(1)\|v_n\| &= \langle J'(v_n), w_n\rangle \\
&= a\int_{\mathbb{R}^N}\left(1+\frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p}\right)|\nabla v_n|^p dx \\
&\quad + \frac{b}{p}\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p}{1+2^{p-1}|f(v_n)|^p}\,dx\right)^2 \\
&\quad - \alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^q dx - \beta\int_{\mathbb{R}^N}f^{2(p^*)}(v_n)\,dx.
\end{aligned}
\tag{2.5}
$$

By equations (2.3) and (2.5), we have

$$
\begin{aligned}
c+o(1)\|v_n\| &= J(v_n) - \frac{1}{2(p^*)}\langle J'(v_n), w_n\rangle \\
&= a\int_{\mathbb{R}^N}\left[\frac{1}{p}-\frac{1}{2(p^*)}\left(1+\frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p}\right)\right]|\nabla v_n|^p dx \\
&\quad + \left(\frac{1}{p}-\frac{1}{p^*}\right)\frac{b}{2p}\left(\int_{\mathbb{R}^N}|f'(v_n)|^p|\nabla v_n|^p dx\right)^2 \\
&\quad + \left(\frac{1}{2(p^*)}-\frac{1}{q}\right)\alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^p dx \\
&\geq \frac{a}{N}\int_{\mathbb{R}^N}|\nabla v_n|^p dx + \frac{b}{2Np}\left(\int_{\mathbb{R}^N}|f'(v_n)|^p|\nabla v_n|^p dx\right)^2 \\
&\quad - \frac{\alpha}{r}\int_{\mathbb{R}^N}k(x)|f(v_n)|^q dx \geq \frac{a}{N}\int_{\mathbb{R}^N}|\nabla v_n|^p dx \\
&\quad - \frac{\alpha}{r}\int_{\mathbb{R}^N}k(x)|f(v_n)|^p dx \geq \frac{a}{N}\int_{\mathbb{R}^N}|\nabla v_n|^p dx \\
&\quad - \frac{\alpha}{r}\left(\int_{\mathbb{R}^N}|k(x)|^r dx\right)^{1/r}\left(\int_{\mathbb{R}^N}|f(v_n)|^{2(p^*)}dx\right)^{q/2(p^*)} \\
&\geq \frac{a}{N}\int_{\mathbb{R}^N}|\nabla v_n|^p dx - \frac{\alpha}{pr}\left(\int_{\mathbb{R}^N}|k(x)|^r dx\right)^{1/r} \\
&\quad \cdot\left(\int_{\mathbb{R}^N}|\nabla f^2(v_n)|^p dx\right)^{q/2p} \geq \frac{a}{N}\|v_n\|^p - c\|v_n\|^{q/2},
\end{aligned}
\tag{2.6}
$$

which implies that $\{v_n\}$ is bounded since $q < 2p$. $\square$

The next result from **[40]** will be useful in the sequel.

**Lemma 2.7** (**[50]**)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ a sequence such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $p \geq 2$. Then,*

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^p dx \geq \lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^p dx + \lim_{n\to\infty} \int_\Omega |\nabla u|^p dx.$$

**Lemma 2.8.** *Let $c < 0$ and $2 < q < 2p$.*

(i) *For any $\beta > 0$, there exists an $\alpha_0 > 0$ such that, if $0 < \alpha < \alpha_0$, then $J$ satisfies $(PS)_c$.*
(ii) *For any $\alpha > 0$, there exists a $\beta_0 > 0$ such that, if $0 < \beta < \beta_0$, then $J$ satisfies $(PS)_c$.*

*Proof.* Let $\{v_n\}$ be a $(PS)_c$ sequence. By Lemma 2.6, $\{v_n\}$ is bounded in $D^{1,p}(\mathbb{R}^N)$. It is easy to check that $\{f(v_n)\}$ is also bounded in $D^{1,p}(\mathbb{R}^N)$. Therefore, we can assume that $v_n \rightharpoonup v$ in $D^{1,p}(\mathbb{R}^N)$, $v_n \to v$ almost everywhere in $\mathbb{R}^N$, since if $f \in C^\infty$, then $|f(v_n)|^2 \to |f(v)|^2$ almost everywhere in $\mathbb{R}^N$ and then $|f(v_n)|^2 \rightharpoonup |f(v)|^2$ in $D^{1,p}(\mathbb{R}^N)$. Thus, there exist measures $\mu$ and $\nu$ such that $|\nabla f^2(v_n)|^p \rightharpoonup \mu$, $f^{2(p^*)}(v_n) \rightharpoonup \nu$. Let $x_j$ be a singular point of the measures $\mu$ and $\nu$. We define a function $\phi(x) \in C_0^\infty(\mathbb{R}^N)$ such that $\phi(x) = 1$ in $B(x_j, \epsilon)$, $\phi(x) = 0$ in $\mathbb{R}^N \setminus B(x_j, 2\epsilon)$ and $|\nabla\phi| \leq 2/\epsilon$ in $\mathbb{R}^N$. Let

$$\widetilde{w}_n = \sqrt[p]{1 + 2^{p-1}|f(v_n)|^p}\, f(v_n)\phi.$$

Then $\{\widetilde{w}_n\}$ is bounded in $D^{1,p}(\mathbb{R}^N)$. Obviously, $\langle J'(v_n), \widetilde{w}_n\phi\rangle \to 0$, i.e.,

$$\begin{aligned}
&- \lim_{n\to\infty} \Bigg[a \int_{\mathbb{R}^N} \sqrt[p]{1 + 2^{p-1}|f(v_n)|^p}\, f(v_n)|\nabla v_n|^{p-2}\nabla v_n \nabla\phi\, dx \\
&\qquad + \frac{b}{p}\bigg(\int_{\mathbb{R}^N} \frac{|\nabla v_n|^p}{1+2^{p-1}|f(v_n)|^p}\, dx\bigg)\bigg(\int_{\mathbb{R}^N} \frac{f(v_n)|\nabla v_n|^{p-2}\nabla v_n \nabla\phi}{(1+2^{p-1}|f(v_n)|^p)^{(p-1)/p}}\, dx\bigg)\Bigg] \\
&= \lim_{n\to\infty} \bigg\{a \int_{\mathbb{R}^N} \bigg(1 + \frac{2^{p-1}|f(v_n)|^p}{1 + 2^{p-1}|f(v_n)|^p}\bigg)|\nabla v_n|^p \phi\, dx
\end{aligned} \tag{2.7}$$

$$+\frac{b}{p}\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p}{1+2^{p-1}|f(v_n)|^p}\,dx\right)\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p\phi}{1+2^{p-1}|f(v_n)|^p}\,dx\right)$$
$$-\alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^q\phi\,dx-\beta\int_{\mathbb{R}^N}f^{2(p^*)}(v_n)\phi\,dx\bigg\}.$$

On the other hand, by the Hölder inequality and $(f_4)$, we have that

$$\begin{aligned}
0&\le\lim_{\epsilon\to 0}\lim_{n\to\infty}\left|a\int_{\mathbb{R}^N}\sqrt[p]{1+2^{p-1}|f(v_n)|^p}f(v_n)|\nabla v_n|^{p-2}\nabla v_n\nabla\phi\,dx\right|\\
&\le C\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n||\nabla v_n|^{p-1}|\nabla\phi|\,dx\\
&\le C\lim_{\epsilon\to 0}\lim_{n\to\infty}\left[\left(\int_{\mathbb{R}^N}|\nabla v_n|^p dx\right)^{(p-1)/p}\left(\int_{\mathbb{R}^N}|v_n\nabla\phi|^p dx\right)^{1/p}\right]\\
&\le C\lim_{\epsilon\to 0}\left(\int_{B(x_j,2\epsilon)}|v|^{p^*}dx\right)^{1/p^*}=0.
\end{aligned}\tag{2.8}$$

Similarly, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\lim_{n\to\infty}\left[\frac{b}{p}\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p}{1+2^{p-1}|f(v_n)|^p}\,dx\right)\right.\\
&\qquad\left.\cdot\left(\int_{\mathbb{R}^N}\frac{f(v_n)|\nabla v_n|^{p-2}\nabla v_n\nabla\phi}{(1+2^{p-1}|f(v_n)|^p)^{(p-1)/p}}\,dx\right)\right]=0.
\end{aligned}\tag{2.9}$$

From equations (2.7)–(2.9), by the weak continuity of $\mathcal{F}(v)$, we obtain

$$\begin{aligned}
0&=\lim_{n\to\infty}\left\{a\int_{\mathbb{R}^N}\left(1+\frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p}\right)|\nabla v_n|^p\phi\,dx\right.\\
&\quad+\frac{b}{p}\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p}{1+2^{p-1}|f(v_n)|^p}\,dx\right)\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p\phi}{1+2^{p-1}|f(v_n)|^p}\,dx\right)\\
&\quad\left.-\alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^q\phi\,dx-\beta\int_{\mathbb{R}^N}f^{2(p^*)}(v_n)\phi\,dx\right\}\\
&\ge\lim_{\varepsilon\to 0}\left[\frac{a}{2}\int_{\mathbb{R}^N}\phi|\nabla f^2(v_n)|^p dx-\alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^q\phi\,dx\right.\\
&\quad\left.-\beta\int_{\mathbb{R}^N}f^{2(p^*)}(v_n)\phi\,dx\right]=\frac{a}{2}\mu_j-\beta\nu_j.
\end{aligned}\tag{2.10}$$

Hence, $2\beta\nu_j \geq a\mu_j$. Combining this with Lemma 2.3, we obtain $\nu_j \geq 2^{-1}\beta^{-1}aS\nu_j^{p/p^*}$. This result implies that

(I) $\nu_j = 0$, or
(II) $\nu_j \geq (2^{-1}\beta^{-1}aS)^{N/p}$.

In order to obtain the possible concentration of mass at infinity, we similarly define a cut-off function $\phi_R \in C_0^\infty(\mathbb{R}^N)$ such that $\phi_R(x) = 0$ on $|x| < R$ and $\phi_R(x) = 1$ on $|x| > R+1$. Since $\{\phi_R w_n\}$ is bounded in $D^{1,p}(\mathbb{R}^N)$, we have

(2.11)
$$\begin{aligned} &-\lim_{n\to\infty}\Bigg[a\int_{\mathbb{R}^N}\sqrt[p]{1+2^{p-1}|f(v_n)|^p}f(v_n)|\nabla v_n|^{p-2}\nabla v_n\nabla\phi_R\,dx \\ &\qquad+\frac{b}{p}\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p}{1+2^{p-1}|f(v_n)|^p}\,dx\right)\left(\int_{\mathbb{R}^N}\frac{f(v_n)|\nabla v_n|^{p-2}\nabla v_n\nabla\phi_R}{(1+2^{p-1}|f(v_n)|^p)^{(p-1)/p}}\,dx\right)\Bigg] \\ &\quad=\lim_{n\to\infty}\Bigg\{a\int_{\mathbb{R}^N}\left(1+\frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p}\right)|\nabla v_n|^p\phi_R\,dx \\ &\qquad+\frac{b}{p}\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p}{1+2^{p-1}|f(v_n)|^p}\,dx\right)\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p\phi_R}{1+2^{p-1}|f(v_n)|^p}\,dx\right) \\ &\qquad\qquad-\alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^q\phi_R\,dx-\beta\int_{\mathbb{R}^N}f^{2(p^*)}(v_n)\phi_R\,dx\Bigg\}. \end{aligned}$$

It is easy to prove that

$$\lim_{R\to\infty}\lim_{n\to\infty}a\int_{\mathbb{R}^N}\sqrt[p]{1+2^{p-1}|f(v_n)|^p}f(v_n)|\nabla v_n|^{p-2}\nabla v_n\nabla\phi_R\,dx=0$$

and

$$\begin{aligned} \lim_{R\to\infty}\lim_{n\to\infty}\Bigg[&\frac{b}{p}\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p}{1+2^{p-1}|f(v_n)|^p}dx\right) \\ &\cdot\left(\int_{\mathbb{R}^N}\frac{f(v_n)|\nabla v_n|^{p-2}\nabla v_n\nabla\phi_R}{(1+2^{p-1}|f(v_n)|^p)^{(p-1)/p}}\,dx\right)\Bigg]=0. \end{aligned}$$

Using the weak continuity of $\mathcal{F}$, we have

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{|x|>R}k(x)|f(v_n)|^q\phi_R\,dx=0.$$

Therefore, by

$$
\begin{aligned}
0 = \lim_{n\to\infty} \Bigg\{ a \int_{\mathbb{R}^N} \left(1 + \frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p}\right) |\nabla v_n|^p \phi_R \, dx \\
+ \frac{b}{p} \left( \int_{\mathbb{R}^N} \frac{|\nabla v_n|^p}{1+2^{p-1}|f(v_n)|^p} \, dx \right) \\
\left( \int_{\mathbb{R}^N} \frac{|\nabla v_n|^p \phi_R}{1+2^{p-1}|f(v_n)|^p} \, dx - \beta \int_{\mathbb{R}^N} f^{2(p^*)}(v_n) \phi_R \, dx \right) \Bigg\}.
\end{aligned}
$$

By Lemma 2.3, we have that either

(III) $\nu_\infty = 0$, or
(IV) $\nu_\infty \geq (2^{-1}\beta^{-1} a S)^{N/p}$.

Next, we claim that (II) and (IV) cannot occur if $\alpha, \beta$ are properly chosen. In fact, from the weak lower semicontinuity of the norm and the weak continuity of $\mathcal{F}$, we have

$$
\begin{aligned}
0 > c &= \lim_{n\to+\infty} \left( J(v_n) - \frac{1}{2(p^*)} \left\langle J'(v_n), \sqrt[p]{1+2^{p-1}|f(v_n)|^p} f(v_n) \right\rangle \right) \\
&= \lim_{n\to+\infty} \Bigg\{ a \int_{\mathbb{R}^N} \left[ \frac{1}{p} - \frac{1}{2(p^*)} \left(1 + \frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p}\right) \right] |\nabla v_n|^p dx \\
&\qquad + \left(\frac{1}{p} - \frac{1}{p^*}\right) \frac{b}{2p} \left( \int_{\mathbb{R}^N} |f'(v_n)|^p |\nabla v_n|^p dx \right)^2 \\
&\qquad\qquad + \left( \frac{1}{2(p^*)} - \frac{1}{q} \right) \alpha \int_{\mathbb{R}^N} k(x) |f(v_n)|^p dx \Bigg\} \\
&\geq \lim_{n\to+\infty} \left\{ \frac{a}{N} \int_{\mathbb{R}^N} |\nabla v_n|^p dx - \frac{\alpha}{qr} \int_{\mathbb{R}^N} k(x) |f(v_n)|^q dx \right\} \\
&\geq \frac{a}{N} \int_{\mathbb{R}^N} |\nabla v|^p dx - \frac{\alpha}{qr} \left( \int_{\mathbb{R}^N} |k(x)|^r dx \right)^{1/r} \left( \int_{\mathbb{R}^N} |f(v)|^{2(p^*)} dx \right)^{q/2(p^*)} \\
&\geq \frac{a}{2N} \int_{\mathbb{R}^N} |\nabla f^2(v)|^p dx - \frac{\alpha}{pr} \|k(x)\|_r \|f^2(v)\|_{p^*}^{q/2} \\
&\geq \frac{aS}{2N} \|f^2(v)\|_{p^*}^p - \frac{\alpha}{qr} \|k(x)\|_r \|f^2(v)\|_{p^*}^{q/2}.
\end{aligned}
$$

This inequality implies that

$$
\|f^2(v_n)\|_{p^*} \leq c \alpha^{2/(2p-q)}.
$$

Therefore,

(2.12)

$$\begin{aligned}
0 > c &= \lim_{n\to+\infty}\bigg(J(v_n) - \frac{1}{2(p^*)}\Big\langle J'(v_n), \sqrt[p]{1+2^{p-1}|f(v_n)|^p}f(v_n)\Big\rangle\bigg)\\
&= \lim_{n\to+\infty}\bigg\{a\int_{\mathbb{R}^N}\bigg[\frac{1}{p} - \frac{1}{2(p^*)}\bigg(1 + \frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p}\bigg)\bigg]|\nabla v_n|^p dx\\
&\qquad + \bigg(\frac{1}{p} - \frac{1}{p^*}\bigg)\frac{b}{2p}\bigg(\int_{\mathbb{R}^N}|f'(v_n)|^p|\nabla v_n|^p dx\bigg)^2\\
&\qquad\qquad + \bigg(\frac{1}{2(p^*)} - \frac{1}{q}\bigg)\alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^p dx\bigg\}\\
&\geq \lim_{n\to+\infty}\bigg\{\frac{a}{N}\int_{\mathbb{R}^N}|\nabla v_n|^p dx - \frac{\alpha}{qr}\int_{\mathbb{R}^N}k(x)|f(v_n)|^q dx\bigg\}\\
&\geq \lim_{R\to+\infty}\lim_{n\to+\infty}\bigg\{\frac{a}{N}\int_{\mathbb{R}^N}|\nabla v_n|^p\phi_R\,dx - \frac{\alpha}{qr}\|k(x)\|_r\|f^2(v)\|_{p^*}^{q/2}\bigg\}\\
&\geq \frac{a}{2N}\mu_\infty - c\alpha^{q/(2p-q)}\\
&\geq \frac{a}{2N}(2\beta)^{(p-N)/p}S^{N/p} - c\alpha^{q/(2p-q)}.
\end{aligned}$$

However, if $\alpha > 0$ is given, we can choose $\beta_0$ small enough such that, for every $0 < \beta < \beta_0$, the last term on the right-hand side of equation (2.12) is greater than zero, which is a contradiction. Similarly, if $\beta > 0$ is given, we can take $\alpha_0$ small enough such that, for every $0 < \alpha < \alpha_0$, the last term on the right-hand side of equation (2.12) is greater than zero. Similarly, we are able to prove that (II) cannot occur for each $j$. Hence,

$$\int_{\mathbb{R}^N} f^{2(p^*)}(v_n)\,dx \longrightarrow \int_{\mathbb{R}^N} f^{2(p^*)}(v)\,dx \quad \text{as} \quad n\to+\infty$$

and

$$\int_{\mathbb{R}^N} k(x)(|f(v_n)|^q - |f(v)|^q)\,dx \leq \|k(x)\|_r\||f(v_n)|^q - |f(v)|^q\|_{2(p^*)/q} \longrightarrow 0$$

as $n\to+\infty$. Thus, from the weak lower semicontinuity of the norm and $f\in C^\infty$, we have

$$
\begin{aligned}
o(1)\|v_n\| &= \langle J'(v_n), w_n\rangle \\
&= a\int_{\mathbb{R}^N}\left(1+\frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p}\right)|\nabla v_n|^p dx \\
&\quad + \frac{b}{p}\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p}{1+2^{p-1}|f(v_n)|^p}dx\right)^2 \\
&\quad - \alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^q dx - \beta\int_{\mathbb{R}^N}f^{2(p^*)}(v_n)\,dx \\
&= a\|v_n\|^p + a\int_{\mathbb{R}^N}\frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p}|\nabla v_n|^p dx \\
&\quad + \frac{b}{p}\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p}{1+2^{p-1}|f(v_n)|^p}dx\right)^2 \\
&\quad - \alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^q dx - \beta\int_{\mathbb{R}^N}f^{2(p^*)}(v_n)\,dx \\
&\geq a\|v_n - v\|^p + a\|v\|^p + a\int_{\mathbb{R}^N}\frac{2^{p-1}|f(v)|^p}{1+2^{p-1}|f(v)|^p}|\nabla v|^p dx \\
&\quad + \frac{b}{p}\left(\int_{\mathbb{R}^N}\frac{|\nabla v|^p}{1+2^{p-1}|f(v)|^p}dx\right)^2 - \alpha\int_{\mathbb{R}^N}k(x)|f(v)|^p dx \\
&\quad - \beta\int_{\mathbb{R}^N}f^{2(p^*)}(v)\,dx \\
&= a\|v_n - v\|^p + o(1)\|v\|,
\end{aligned}
$$

since $J'(v) = 0$. Thus, we have proved that $\{v_n\}$ strongly converges to $v$ in $D^{1,p}(\mathbb{R}^N)$. □

**3. Proof of Theorem 1.1.** In this section, we prove the existence of infinitely many solutions of equation (1.1) which tend to 0. Let $X$ be a Banach space, and denote

$$
\begin{aligned}
&\Sigma := \{A \subset X \setminus \{0\} : \\
&\qquad A \text{ is closed in } X \text{and symmetric with respect to the orgin}\}.
\end{aligned}
$$

For $A \in \Sigma$, we define genus $\gamma(A)$ as

$$
\gamma(A) := \inf\{m \in N : \text{there exists } \varphi \in C(A, R^m \backslash \{0\}, -\varphi(x) = \varphi(-x))\}.
$$

If there is no mapping $\varphi$ as above for any $m \in N$, then $\gamma(A) = +\infty$. Let $\Sigma_k$ denote the family of closed, symmetric subsets $A$ of $X$ such that $0 \notin A$ and $\gamma(A) \geq k$. We list some properties of the genus, see **[25, 45]**.

**Proposition 3.1.** *Let $A$ and $B$ be closed, symmetric subsets of $X$ which do not contain the origin. Then, the following conditions hold.*

(i) *If there exists an odd continuous mapping from $A$ to $B$, then $\gamma(A) \leq \gamma(B)$;*
(ii) *if there is an odd homeomorphism from $A$ to $B$, then $\gamma(A) = \gamma(B)$;*
(iii) *if $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$;*
(iv) *then, $n$-dimensional sphere $S^n$ has a genus of $n+1$ by the Borsuk-Ulam theorem;*
(v) *if $A$ is compact, then $\gamma(A) < +\infty$, and there exists a $\delta > 0$ such that $U_\delta(A) \in \Sigma$ and $\gamma(U_\delta(A)) = \gamma(A)$, where $U_\delta(A) = \{x \in X : \|x - A\| \leq \delta\}$.*

The next version of the symmetric mountain pass lemma is due to Kajikiya **[25]**.

**Lemma 3.2.** *Let $E$ be an infinite-dimensional space and $J \in C^1(E, R)$, and suppose that the following conditions hold.*

($C_1$) *$J(u)$ is even, bounded from below, $J(0) = 0$ and $J(u)$ satisfies the local Palais-Smale condition, i.e., for some $\overline{c} > 0$, in the case when every sequence $\{u_k\}$ in $E$ satisfying $\lim_{k\to\infty} J(u_k) = c < \overline{c}$ and $\lim_{k\to\infty} \|J'(u_k)\|_{E^*} = 0$ has a convergent subsequence;*
($C_2$) *for each $k \in N$, there exists an $A_k \in \Sigma_k$ such that $\sup_{u\in A_k} J(u) < 0$.*

*Then, either* ($R_1$) *or* ($R_2$) *holds.*

($R_1$) *There exists a sequence $\{u_k\}$ such that $J'(u_k) = 0$, $J'(u_k) < 0$ and $\{u_k\}$ converges to $0$;*
($R_2$) *there exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $J'(u_k) = 0$, $J(u_k) < 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $J'(v_k) = 0$, $J(v_k) < 0$, $\lim_{k\to\infty} J(v_k) = 0$ and $\{v_k\}$ converges to a non-zero limit.*

**Remark 3.3.** Lemma 3.2 provides a sequence $\{u_k\}$ of critical points such that $J(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

In order to obtain infinitely many solutions we need some lemmas. Let $J(v)$ be the previously defined functional. Then

$$\begin{aligned}
J(v) :=& \frac{a}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{b}{2}\bigg(\int_{\mathbb{R}^N} \frac{1}{p}|f'(v)|^p |\nabla v|^p dx\bigg)^2 \\
&- \frac{\alpha}{q}\int_{\mathbb{R}^N} k(x)|f(v)|^q dx - \frac{\beta}{2(p^*)}\int_{\mathbb{R}^N} |f(v)|^{2(p^*)} dx \\
\geq& \frac{a}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx \\
&- \frac{\alpha}{p}\bigg(\int_{\mathbb{R}^N} |k(x)|^r dx\bigg)^{1/r}\bigg(\int_{\mathbb{R}^N} f^{2(p^*)}(v)\, dx\bigg)^{q/(2(p^*))} \\
&- \frac{\beta}{2(p^*)}\int_{\mathbb{R}^N} |f(v)|^{2(p^*)} dx \\
\geq& \frac{a}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx - \alpha c_1 \|f(v)\|^{q/2} - \beta c_2 \|f(v)\|^{2(p^*)} \\
\geq& \frac{a}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx - \alpha c_1 \|v\|^{q/2} - \beta c_2 \|v\|^{p^*}.
\end{aligned}$$

Let

$$Q(t) := \frac{a}{p}t^p - \alpha c_1 t^{q/2} - \beta c_2 t^{p^*}.$$

Then, it is easy to see that, given $\beta > 0$, there exists an $\alpha_1 > 0$ small enough such that, for every $0 < \alpha < \alpha_1$, there exist $t_0$ and $t_1$ such that $0 < t_0 < t_1$, $Q(t) > 0$ for $t_0 < t < t_1$, $Q(t) < 0$ for $t > t_1$ and $0 < t < t_0$. Similarly, given $\alpha > 0$, we can choose $\beta_1 > 0$ with the property that $t_0, t_1$ as above exist for each $0 < \beta < \beta_1$. Clearly, $Q(t_0) = 0 = Q(t_1)$. Following the same idea as in **[19]**, we consider the truncated functional

$$\begin{aligned}
\widetilde{J}(v) =& \frac{a}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{b}{2}\bigg(\int_{\mathbb{R}^N} \frac{1}{p}|f'(v)|^p |\nabla v|^p dx\bigg)^2 \\
&- \frac{\alpha}{q}\int_{\mathbb{R}^N} k(x)|f(v)|^q dx - \frac{\beta}{2(p^*)}\varphi(v)\int_{\mathbb{R}^N} |f(v)|^{2(p^*)} dx,
\end{aligned} \tag{3.1}$$

where $\varphi(v) = \chi(\|v\|)$ and $\chi : \mathbb{R}^+ \to [0,1]$ is a non-increasing $C^\infty$

function such that $\chi(t) = 1$ if $t \le t_0$ and $\chi(t) = 0$ if $t \ge t_1$. Thus,

$$\widetilde{J}(v) \ge \overline{Q}(\|v\|),$$

where

$$\overline{Q}(t) = \frac{a}{p}t^p - \alpha c_1 t^{q/2} - \beta c_2 t^{p^*}\varphi(t).$$

It is clear that $\widetilde{J}(v) \in C^1$ and is bounded from below.

Using the above arguments, we obtain the following.

**Lemma 3.4.** *Let $\widetilde{J}(v)$ be as defined in equation* (3.1). *Then,*

(i) *if $\widetilde{J}(v) < 0$, then $\|v\| \le T_0$ and $\widetilde{J}(v) = J(v)$;*
(ii) *for any $\beta > 0$, there exists an $\alpha^*$ such that, if $0 < \alpha < \alpha^*$ and $c < 0$, then $\widetilde{J}(v)$ satisfies $(PS)_c$;*
(iii) *for any $\alpha > 0$, there exists a $\beta^*$ such that, if $0 < \beta < \beta^*$ and $c < 0$, then $\widetilde{J}(v)$ satisfies $(PS)_c$.*

**Remark 3.5.** Denote

$$K_c = \{v \in D^{1,p}(\mathbb{R}^N) : \widetilde{J}'(v) = 0,\ \widetilde{J}(v) = c\}.$$

If $\alpha, \beta$ are as in Lemma 3.4 (ii) or (iii), then it follows from $(PS)_c$ that $K_c$ $(c < 0)$ is compact.

**Lemma 3.6.** *Let*

$$K_c = \{v \in D^{1,p}(\mathbb{R}^N) : \widetilde{J}'(v) = 0,\ \widetilde{J}(v) = c\}.$$

*Then, for any $m \in \mathbb{R}$, there is an $\varepsilon_m < 0$ such that $\gamma(\widetilde{J}^{\varepsilon_m}) \ge m$.*

*Proof.* Denote by $D_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = (\int_\Omega |\nabla u|^p)^{1/p}$. Extending functions in $D_0^{1,p}(\Omega)$ by 0 outside $\Omega$, we may assume that $D_0^{1,p}(\Omega) \subset D^{1,p}(\mathbb{R}^N)$. Let $X_m$ be an $m$-dimensional subspace of $D_0^{1,p}(\Omega)$. For any $v \in X_m$, $v \neq 0$, use $v = r_m w$ with $w \in X_m$ and $\|w\| = 1$. From the assumptions of $k(x)$, it is easy to see that, for every $w \in X_m$ with $\|w\| = 1$, there exist $d_m > 0$ such that

$$\int_\Omega k(x)|w|^{q/2}dx \ge d_m.$$

Thus, $0 < r_m < T_0$. Since all the norms are equivalent and $0 < |\Omega| < +\infty$, using $(f_1)$ and $(f_4)$, we obtain

$$\begin{aligned}
\widetilde{J}(v) &= J(v) \\
&= \frac{a}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{b}{2}\bigg(\int_{\mathbb{R}^N} \frac{1}{p}|f'(v)|^p|\nabla v|^p dx\bigg)^2 \\
&\quad - \frac{\alpha}{q}\int_{\mathbb{R}^N} k(x)|f(v)|^q dx - \frac{\beta}{2(p^*)}\int_{\mathbb{R}^N} |f(v)|^{2(p^*)} dx \\
&\leq \frac{a}{p}\int_{\Omega} |\nabla v|^p dx + \frac{b}{2p^2}\bigg(\int_{\Omega} |f'(v)|^p|\nabla v|^p dx\bigg)^2 \\
&\quad - \frac{\alpha}{q}\int_{\Omega} k(x)(c|v|^{q/2} + c)\, dx - \frac{\beta}{2(p^*)}\int_{\Omega} (v^{2(p^*)} + c)\, dx \\
&\leq \frac{a}{p}\int_{\Omega} |\nabla v|^p dx + \frac{b}{2p^2}\bigg(\int_{\Omega} |\nabla v|^p dx\bigg)^2 \\
&\quad - \frac{c\alpha}{q}\int_{\Omega} k(x)|v|^{q/2} dx - \frac{c\beta}{2(p^*)}\int_{\Omega} v^{2(p^*)} dx - c \\
&\leq \frac{a}{p}\, r_m^p + \frac{b}{2p^2}\, r_m^{2p} - \alpha d_m r_m^{q/2} - \beta c r_m^{p^*} - c \\
&:= \varepsilon_m.
\end{aligned}$$

Hence, we may choose $r_m \in (0, T_0)$ small enough such that $\widetilde{J}(v) < \varepsilon_m < 0$. Let

$$S_{r_m} = \{v \in D^{1,p}(\mathbb{R}^N);\ \|v\| = r_m\}.$$

Then, $S_{r_m} \cap X_m \subset \widetilde{J}^{\varepsilon_m}$. Using Proposition 3.1 (ii), we obtain $\gamma(\widetilde{J}^{\varepsilon_m}) \geq \gamma(S_{r_m} \cap X_m) \geq m$. Therefore, we can denote $\Gamma_m = \{A \in \Sigma; \gamma(A) \geq m\}$, and let

$$c_m := \inf_{A \in \Gamma_m} \sup_{v \in A} \widetilde{J}(v). \tag{3.2}$$

Then

$$-\infty < c_m \leq \varepsilon_m < 0, \quad m \in \mathbb{N}, \tag{3.3}$$

because $\widetilde{J}^{\varepsilon_m} \in \Gamma_m$ and $\widetilde{J}$ is bounded from below. □

**Lemma 3.7.** *Let $\alpha, \beta$ be as in Lemma* 3.4 (ii) *or* (iii)*. Then, all $c_m$ given by equation* (3.2) *are critical values of $\widetilde{J}$ and $c_m \to 0$ as $m \to \infty$.*

*Proof.* It is clear that $c_m \leq c_{m+1}$. By equation (3.3), we have $c_m < 0$. Hence, $c_m \to \overline{c} \leq 0$. Moreover, as $(PS)_c$ is satisfied, it follows from a standard argument [**45**] that all $c_m$ are critical values of $\widetilde{J}$. We claim that $\overline{c} = 0$. If $\overline{c} < 0$, then, by Remark 3.5, $K_{\overline{c}}$ is compact and $K_{\overline{c}} \in \Sigma$. It follows that $\gamma(K_{\overline{c}}) = m_0 < +\infty$, and there exists a $\delta > 0$ such that $\gamma(K_{\overline{c}}) = \gamma(N_\delta(K_{\overline{c}})) = m_0$. By the deformation lemma, there exist $\epsilon > 0$ ($\overline{c} + \epsilon < 0$) and an odd homeomorphism $\eta$ such that

$$\eta(\widetilde{J}^{\overline{c}+\varepsilon_m} \setminus N_\delta(K_{\overline{c}})) \subset \widetilde{J}^{\overline{c}-\epsilon}. \tag{3.4}$$

Since $c_m$ is increasing and converges to $\overline{c}$, there exists an $m \in \mathbb{N}$ such that $c_m > \overline{c} - \epsilon$ and $c_{m+m_0} \leq \overline{c}$. There exists an $A \in \Gamma_{m+m_0}$ such that $\sup_{u\in A} \widetilde{J}(u) < \overline{c} + \epsilon$. By Proposition 3.1, we have

$$\gamma(\overline{A \setminus N_\delta(K_{\overline{c}})}) \geq \gamma(A) - \gamma(N_\delta(K_{\overline{c}})), \qquad \gamma(\eta(\overline{A \setminus N_\delta(K_{\overline{c}})})) \geq m. \tag{3.5}$$

Therefore, we have

$$\eta(\overline{A \setminus N_\delta(K_{\overline{c}})}) \in \Gamma_m.$$

Consequently,

$$\sup_{u\in\eta(\overline{A\setminus N_\delta(K_{\overline{c}})})} \widetilde{J}(u) \geq c_m > \overline{c} - \epsilon. \tag{3.6}$$

On the other hand, by equations (3.4) and (3.5), we have

$$\eta(\overline{A \setminus N_\delta(K_{\overline{c}})}) \subset \eta(\widetilde{J}^{\overline{c}+\epsilon} \setminus N_\delta(K_{\overline{c}})) \subset \widetilde{J}^{\overline{c}}, \tag{3.7}$$

which contradicts equation (3.6). Hence, $c_n \to 0$ as $n \to \infty$. $\square$

Now, we provide the proof of Theorem 2.2.

*Proof of Theorem* 2.2. By Lemma 3.4 (i), $\widetilde{J}(v) = J(v)$ if $\widetilde{J} < 0$. This and Lemma 3.7 give the result. $\square$

*Proof of Theorem* 1.1. The proof follows from Theorem 2.2, since $u_m = f(v_m) \neq u_n = f(v_n)$ if $v_m \neq v_n$ and $f \in C^\infty$. $\square$

## REFERENCES

**1**. G. Autuori, F. Colasuonno and P. Pucci, *On the existence of stationary solutions for higher order p-Kirchhoff problems*, Comm. Cont. Math. **16** (2014), 1450002.

**2**. G. Autuori, A. Fiscella and P. Pucci, *Stationary Kirchhoff problems involving a fractional operator and a critical nonlinearity*, submitted.

**3**. H. Brezis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.

**4**. H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical exponents*, Comm. Pure Appl. Math. **34** (1983), 437–477.

**5**. J. Chabrowski, *On multiple solutions for the nonhomogeneous p-Laplacian with a critical Sobolev exponent*, Diff. Int. Eq. **8** (1995), 705–716.

**6**. ———, *Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents*, Calc. Var. **3** (1995), 493–512.

**7**. J. Chen and S. Li, *On multiple solutions of a singular quasi-linear equation on unbounded domain*, J. Math. Anal. Appl. **275** (2002), 733–746.

**8**. M. Chipot and B. Lovat, *Some remarks on nonlocal elliptic and parabolic problems*, Nonlin. Anal. **30** (1997), 4619–4627.

**9**. F. Colasuonno and P. Pucci, *Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations*, Nonlin. Anal. **74** (2011), 5962–5974.

**10**. M. Colin and L. Jeanjean, *Solutions for a quasilinear Schrödinger equations: A dual approach*, Nonlin. Anal. **56** (2004), 213–226.

**11**. F.J.S.A. Corrêa, *On positive solutions of nonlocal and nonvariational elliptic problems*, Nonlin. Anal. **59** (2004), 1147–1155.

**12**. F.J.S.A. Corrêa and G.M. Figueiredo, *On a elliptic equation of p-Kirchhoff-type via variational methods*, Bull. Austral. Math. Soc. **74** (2006), 263–277.

**13**. F.J.S.A. Corrêa and R.G. Nascimento, *On a nonlocal elliptic system of p-Kirchhoff-type under Neumann boundary condition*, Math. Comp. Model. **49** (2009), 598–604.

**14**. G.W. Dai and R.F. Hao, *Existence of solutions for a $p(x)$-Kirchhoff-type equation*, J. Math. Anal. Appl. **359** (2009), 275–284.

**15**. P. D'Ancona and Y. Shibata, *On global solvability of non-linear viscoelastic equations in the analytic category*, Math. Meth. Appl. Sci. **17** (1994), 477–489.

**16**. P. D'Ancona and S. Spagnolo, *Global solvability for the degenerate Kirchhoff equation with real analytic data*, Invent. Math. **108** (1992), 247–262.

**17**. A. Ferrero and F. Gazzola, *Existence of solutions for singular critical growth semilinear elliptic equations*, J. Diff. Eq. **177** (2001), 494–522.

**18**. J. Garcia Azorero and I. Peral, *Hardy inequalities and some critical elliptic and parabolic problems*, J. Diff. Eq. **144** (1998), 441–476.

**19**. ———, *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term*, Trans. Amer. Math. Soc. **323** (1991), 877–895.

**20**. N. Ghoussoub and C. Yuan, *Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents*, Trans. Amer. Math. Soc. **352** (2000), 5703–5743.

**21**. X. He and W. Zou, *Infinitely many positive solutions for Kirchhoff-type problems*, Nonlin. Anal. **70** (2009), 1407–1414.

**22**. X. He and W. Zou, *Multiplicity of solutions for a class of Kirchhoff-type problems*, Acta Math. Appl. Sinica **26** (2010), 387–394.

**23**. X.M. He and W.M. Zou, *Infinitely many arbitrarily small solutions for sigular elliptic problems with critical Sobolev-Hardy exponents*, Proc. Edinburgh Math. Soc. **52** (2009), 97–108.

**24**. J. Jin and X. Wu, *Infinitely many radial solutions for Kirchhoff-type problems in* $\mathbb{R}^N$, J. Math. Anal. Appl. **369** (2010), 564–574.

**25**. R. Kajikiya, *A critical-point theorem related to the symmetric mountain-pass lemma and its applications to elliptic equations*, J. Funct. Anal. **225** (2005), 352–370.

**26**. G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

**27**. M.A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, Pergamon, Elmsford, NY, 1964.

**28**. S. Li and W. Zou, *Remarks on a class of elliptic problems with critical exponents*, Nonlin. Anal. **32** (1998), 769–774.

**29**. Y. Li, F. Li and J. Shi, *Existence of a positive solution to Kirchhoffty peproblems without compactness conditions*, J. Diff. Eq. **253** (2012), 2285–2294.

**30**. S. Liang and J. Zhang, *Existence of solutions for Kirchhoff-type problems with critical nonlinearity in* $\mathbb{R}^3$, Nonlin. Anal. **17** (2014), 126–136.

**31**. J.L. Lions, *On some equations in boundary value problems of mathematical physics*, North-Holland Math. Stud. **30**, North Holland, Amsterdam, 1978.

**32**. P.L. Lions, *The concentration compactness principle in the calculus of variations, The locally compact case*, I, II, Ann. Inst. Poincare Anal. Nonlin. **1** (1984), 109–145, 223–283.

**33**. D.C. Liu, *On a p-Kirchhoff equation via fountain theorem and dual fountain theorem*, Nonlin. Anal. **72** (2010), 302–308.

**34**. J.Q. Liu, Y.Q. Wang and Z.Q. Wang, *Soliton solutions to quasilinear Schrödinger equations* II, J. Diff. Eq. **187** (2003), 473–493.

**35**. ______, *Solutions for quasilinear Schrödinger equations via the Nehari method*, Comm. Partial Diff. Eq. **29** (2004), 879–901.

**36**. J.Q. Liu and Z.Q. Wang, *Soliton solutions for quasilinear Schrödinger equations*, Proc. Amer. Math. Soc. **131** (2003), 441–448.

**37**. T.F. Ma and J.E. Munoz Rivera, *Positive solutions for a nonlinear nonlocal elliptic transmission problem*, Appl. Math. Lett. **16** (2003), 243–248.

**38**. A. Moameni, *Existence of soliton solutions for a quasilinear Schrödinger quation involving critical exponent in* $\mathbb{R}^N$, J. Diff. Eq. **229** (2006), 570–587.

**39**. K. Nishihara, *On a global solution of some quasilinear hyperbolic equation*, Tokyo J. Math. **7** (1984), 437–459.

**40**. I. Peral, *Multiplicity of solutions for the p-Laplacian*, in *Lecture notes at the second school on nonlinear functional analysis and applications to differential equations at* ICTP, A. Ambrosetti, K.C. Chang and I. Eklend, eds., Trieste, 1997.

**41**. K. Perera and Z. Zhang, *Nontrivial solutions of Kirchhoff-type problems via the Yang index*, J. Diff. Eq. **221** (2006), 246–255.

**42**. M. Poppenberg, K. Schmitt and Z.Q. Wang, *On the existence of solition solutions to quasilinear Schrödinger equations*, Calc. Var. Partial Diff. Eq. **14** (2002), 329–344.

**43**. P. Pucci and S. Saldi, *Critical stationary Kirchhoff equations in* $\mathbb{R}^N$ *involving nonlocal operators*, Rev. Mat. Iber. **32** (2016), 1–22.

**44**. P. Pucci, M. Xiang and B. Zhang, *Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff-type equations involving the fractional p-Laplacian in* $\mathbb{R}^N$, Calc. Var. **54** (2015), 2785–2806.

**45**. P.H. Rabinowitz, *Minimax methods in critical-point theory with applications to differential equations*, CBME Regional Conference Series in Mathematics **65**, American Mathematical Society, Providence, RI, 1986.

**46**. E.A. Silva and M.S. Xavier, *Multiplicity of solutions for quasilinear elliptic problems involving critical Sobolev exponents*, Ann. Inst. Poincare Anal. Nonlin. **20** (2003), 341–358.

**47**. J. Wang, L. Tian, J. Xu and F. Zhang, *Multiplicity and concentration of positive solutions for a Kirchhoff-type problem with critical growth*, J. Diff. Eq. **253** (2012), 2314–2351.

**48**. Y.J. Wang, Y.M. Zhang and Y.T. Shen, *Multiple solutions for quasilinear Schrödinger equations involving critical exponent*, Appl. Math. Comp. **216** (2010), 849–856.

**49**. M. Willem, *Minimax theorems*, Birkhäuser, Boston, 1996.

**50**. M.Z. Wu and Z.D. Yang, *Existence and concentration of solutions for a p-Laplace equation with potentials in* $\mathbb{R}^N$, Elect. J. Diff. Eq. **2010** (2010), 1–11.

**51**. X. Wu, *Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in* $\mathbb{R}^N$, Nonlin. Anal. **12** (2011), 1278–1287.

Changchun Normal University, College of Mathematics, Changchun, Jilin, 130032 P.R. China and Jilin University, Key Laboratory of Symbolic Computation and Knowledge Engineering of Ministry of Education, Changchun 130012 P.R. China

**Email address**: **liangsihua@126.com**

Nanjing Normal University, Institute of Mathematics, School of Mathematical Science, Nanjing, Jiangsu, 210046 P.R. China

**Email address**: **jihuiz@jlonline.com**