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IDEALS IN CROSS SECTIONAL C^*-ALGEBRAS OF FELL BUNDLES

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ABSTRACT. With each Fell bundle over a discrete group G we associate a partial action of G on the spectrum of the unit fiber. We discuss the ideal structure of the corresponding full and reduced cross-sectional C^*-algebras in terms of the dynamics of this partial action.

Introduction. The discussion of the ideal structure of crossed products by a discrete group by means of the dynamical properties of the action goes far back (see, for instance, [**9, 18, 22**]).

Archbold and Spielberg discussed [**8**] the relation between the ideal structure of the full crossed product and that of the base algebra, under the assumption of topological freeness. More recently, the definitions of topological freeness and several related results were extended to different settings: by Exel, Laca and Quigg for partial actions on commutative C^*-algebras [**12**], by Lebedev [**17**] and later by Giordano and Sierakowski [**14**], for partial actions on arbitrary C^*-algebras, and by Kwaśniewski [**16**] for crossed products by Hilbert C^*-bimodules.

We show in this article that a Fell bundle $\mathcal{B}$ over a discrete group G gives rise to a partial action of G on the spectrum of the unit fiber. This partial action agrees with those discussed in the above-mentioned work, and we generalize some of those results to this context.

This work is organized as follows. After establishing some background and notation in Section 1, in Section 2 we introduce a partial action $\widehat{\alpha}$ on the spectrum of the unit fiber of a Fell bundle $\mathcal{B}$ over a discrete group. When $\mathcal{B}$ is the Fell bundle corresponding to a partial action γ, then $\widehat{\alpha}$ agrees with $\widehat{\gamma}$, as defined in [**5,** Section 7] or [**17**], and when $\mathcal{B}$ is the Fell bundle associated [**2**] with the crossed-product

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by a Hilbert C^*-bimodule, then $\widehat{\alpha}$ is the homeomorphism $\widehat{h}$ discussed in **[16]**.

Following familiar lines, we establish in Section 3 a bijective correspondence between the family of $\widehat{\alpha}$-invariant open sets in the spectrum of the unit fiber and the set of ideals in $\mathcal{B}$ (Proposition 3.8 and Proposition 3.10) This enables us to show that, when $\widehat{\alpha}$ is topologically free, its minimality is equivalent to the simplicity of $C^*_r(\mathcal{B})$ (Corollary 3.12). We then proceed to generalize some of the results of Giordano and Sierakowski **[14]** to our setting (Theorem 3.19) which concern the connection among the exactness property, the residual intersection property, the structure ideal of $\mathcal{B}$, and that of $C^*_r(\mathcal{B})$.

Finally, Section 4 contains some applications to the theory of Fell bundles with commutative unit fiber.

1. Preliminaries. In this section, we establish some notation and recall some basic definitions and facts regarding the spectrum of a C^*-algebra and the Rieffel correspondence. We refer the reader to **[19]** for further details.

If A is a C^*-algebra, we denote by $\mathcal{I}(A)$ the lattice of ideals in A and by $\operatorname{Prim} A$ the primitive space of A, that is, $\operatorname{Prim} A$ is the set of primitive ideals with the hull-kernel topology. The spectrum of A, which we denote by $\widehat{A}$, consists of the unitary equivalence classes of irreducible representations of A with the initial topology for the map

$$k : \widehat{A} \longrightarrow \operatorname{Prim}(A), \text{ given by } k([\pi]) = \ker \pi \text{ for all } [\pi] \in \widehat{A}, \tag{1.1}$$

that is, a subset S of $\widehat{A}$ is open if and only if $S = k^{-1}(O)$, where O is open in $\operatorname{Prim} A$. We will usually drop the brackets and denote $[\pi] \in \widehat{A}$ by π.

Suppose now that A and B are C^* -algebras and that X is an $A-B$ imprimitivity bimodule. We denote by $\langle\ ,\ \rangle_L$ and $\langle\ ,\ \rangle_R$ the left and right inner products on X, respectively.

An irreducible representation $\pi : B \to B(\mathcal{H}_\pi)$ induces an irreducible representation $\operatorname{Ind}_X \pi$ of A as follows. Let $X \otimes_B \mathcal{H}_\pi$ be the Hilbert space obtained as the completion of the algebraic tensor product $X \odot_B \mathcal{H}_\pi$ with respect to the norm induced by the inner product determined by

$$\langle x \otimes h, y \otimes k \rangle := \langle \pi(\langle y, x \rangle_R) h, k \rangle, \tag{1.2}$$

for $x, y \in X$ and $h, k \in \mathcal{H}_\pi$.

Then $\operatorname{Ind}_X \pi : A \to B(X \otimes_B \mathcal{H}_\pi)$ is defined by

$$\operatorname{Ind}_X \pi(a)(x \otimes h) = ax \otimes h, \tag{1.3}$$

for $a \in A$, $x \in X$, and $h \in \mathcal{H}_\pi$.

Since $\operatorname{Ind}_X \pi$ is irreducible as well, the imprimitivity bimodule X yields a map

$$\operatorname{Ind}_X : \widehat{B} \longrightarrow \widehat{A} \tag{1.4}$$

that turns out to be a homeomorphism.

The imprimitivity bimodule X also yields the Rieffel correspondence

$$h_X : \mathcal{I}(B) \longrightarrow \mathcal{I}(A),$$

which is a lattice isomorphism determined by the equation

$$h_X(I)X = XI, \quad \text{for all } I \in \mathcal{I}(B), \tag{1.5}$$

where

$$XI = \overline{\operatorname{span}}\{xi : x \in X, i \in I\}$$

and

$$h_X(I)X = \overline{\operatorname{span}}\{jx : x \in X, j \in h_X(I)\}.$$

These constructions are connected by the relation ([**19,** subsection 3.24])

$$\ker \operatorname{Ind}_X \pi = h_X(\ker \pi). \tag{1.6}$$

If J is an ideal in A, we denote the canonical projection on A/J by P_J. Let X_J be the set

$$X_J = \{\pi \in \widehat{A} : \pi|_J \neq 0\}. \tag{1.7}$$

Then the map $J \mapsto X_J$ is a bijection from $\mathcal{I}(A)$ onto the topology on $\widehat{A}$.

In addition, the maps

$$r_J : X_J \longrightarrow \widehat{J}$$

and

$$q_J : \widehat{A} \setminus X_J \longrightarrow \widehat{A/J},$$

determined, respectively, by

$$r_J(\pi) = \pi|_J \quad \text{and} \quad q_J(\pi) \circ P_J = \pi, \tag{1.8}$$

are homeomorphisms.

If X is an $A - B$ imprimitivity bimodule and J is an ideal in B, then $XJ = h_X(J)X$, and X/XJ is an $A/h_X(J) - B/J$ imprimitivity bimodule. Furthermore, the diagram

$$\begin{tikzcd} \widehat{B/J} \arrow[r, "\mathrm{Ind}_{X/XJ}"] & \widehat{A/h_X(J)} \\ \widehat{B} \setminus X_J \arrow[u, "q_J"] \arrow[r, "\mathrm{Ind}_X"'] & \widehat{A} \setminus X_{h_X(J)} \arrow[u, "q_{h_X(J)}"'] \end{tikzcd} \tag{1.9}$$

commutes.

2. The partial action associated with a Fell bundle.

Notation 2.1. Throughout this work, $\mathcal{B} = (B_t)_{t\in G}$ will denote a Fell bundle over a discrete group G. We will make use of the usual notation:

$$X^* = \{x^* : x \in X\} \subseteq B_{t^{-1}},$$

$$X_1X_2 \cdots X_n = \overline{\text{span}}\{x_1x_2 \cdots x_n : x_i \in X_i\} \subseteq B_{t_1t_2\cdots t_n},$$

for $X \subseteq B_t$ and $X_i \subseteq B_{t_i}$, where $t, t_i \in G$ and $i = 1, \cdots, n$.

In this setting, B_t is a Hilbert C^* -bimodule over B_e, for left and right multiplication and inner products given by

$$\langle b_1, b_2\rangle_L = b_1b_2^*, \ \langle b_1, b_2\rangle_R = b_1^*b_2. \tag{2.1}$$

We denote by $C^*(\mathcal{B})$ the cross-sectional C^*-algebra of $\mathcal{B}$, and by $C_c(\mathcal{B})$ the dense *-subalgebra of compactly supported cross sections.

The map $E : C_c(\mathcal{B}) \to B_e$ consisting of evaluation at e extends to a conditional expectation $E : C^*(\mathcal{B}) \to B_e$.

We next recall some definitions and results related to the reduced cross-sectional C^*-algebra of a Fell bundle. Further details and proofs can be found in [**10**].

Let $\ell^2(\mathcal{B})$ denote the right Hilbert C^*-module over B_e consisting of those sections ξ such that

$$\sum_{t\in G} \xi^*(t)\xi(t)$$

converges in B_e.

Thus, $\ell^2(\mathcal{B})$ is the direct sum of the right B_e-Hilbert C^*-modules $\{B_t : t \in G\}$. Let

$$j_t : B_t \longrightarrow \ell^2(\mathcal{B})$$

be the inclusion map, that is,

$$j_t(b) = b\delta_t, \quad \text{for } t \in G \text{ and } b \in B_t, \tag{2.2}$$

where $b\delta_t(s) = \delta_{s,t}b$, $\delta_{s,t}$ is the Kronecker delta. Then j_t is adjointable, and its adjoint is evaluation at t.

Each $b_t \in B_t$ defines an adjointable operator $\Lambda_{b_t} \in \mathcal{L}(\ell^2(\mathcal{B}))$, given by

$$\Lambda_{b_t}(\xi)(s) = b_t\xi(t^{-1}s), \quad \text{for all } \xi \in \ell^2(\mathcal{B}),\ s \in G.$$

The reduced C^*-algebra $C_r^*(\mathcal{B})$ of the Fell bundle $\mathcal{B}$ is the C^*-subalgebra of $\mathcal{L}(\ell^2(\mathcal{B}))$ generated by $\{\Lambda_b : b \in \mathcal{B}\}$. The correspondence

$$b_t \longmapsto \Lambda_{b_t}$$

extends to a *-homomorphism

$$\Lambda : C^*(\mathcal{B}) \longrightarrow C_r^*(\mathcal{B})$$

verifying ([**10,** subsection 3.6])

$$\ker \Lambda = \{c \in C^*(\mathcal{B}) : E(c^*c) = 0\}. \tag{2.3}$$

We will often view B_e as a C^*-subalgebra of $C_r^*(\mathcal{B})$ by identifying $a \in B_e$ with $\Lambda_a \in C_r^*(\mathcal{B})$.

We denote by D_t the ideal in B_e defined by $D_t = B_t B_t^*$. Since the structure described above makes B_t into a $D_t - D_{t^{-1}}$ imprimitivity bimodule, B_t yields, as in equation (1.4), a homeomorphism

$$\operatorname{Ind}_{B_t} : \widehat{D}_{t^{-1}} \longrightarrow \widehat{D_t}.$$

We will denote by X_t, r_t and q_t, respectively, the set X_{D_t} and the maps r_{D_t} and q_{D_t} defined in equations (1.7) and (1.8). Notice that $X_e = \widehat{B_e}$. Finally, we denote by $\widehat{\alpha}_t$ the homeomorphism that causes the diagram

$$\begin{tikzcd} X_{t^{-1}} \arrow[r, "\hat{\alpha}_t"] \arrow[d, "r_{t^{-1}}"'] & X_t \arrow[d, "r_t"] \\ \widehat{D}_{t^{-1}} \arrow[r, "\operatorname{Ind}_{B_t}"'] & \widehat{D}_t \end{tikzcd}$$

commute, that is,

$$\widehat{\alpha}_t : X_{t^{-1}} \longrightarrow X_t \text{ is given by } \widehat{\alpha}_t = r_t^{-1} \circ \operatorname{Ind}_{B_t} \circ r_{t^{-1}}, \tag{2.4}$$

for all $t \in G$.

Remark 2.2. If $\pi \in X_{t^{-1}}$ is a representation of D_e on $\mathcal{H}_\pi$, then $\widehat{\alpha}_t(\pi)$ is the representation of D_e on $B_t \otimes_{D_{t^{-1}}} \mathcal{H}_\pi$, given by

$$\big(\widehat{\alpha}_t(\pi)a\big)(b \otimes h) = ab \otimes h, \tag{2.5}$$

for all $a \in D_e$, $b \in B_t$ and $h \in \mathcal{H}_\pi$.

Proof. When $a \in D_t$, the result follows straightforwardly from the definition, and equation (2.5) clearly defines an extension of $\operatorname{Ind}_{B_t}(\pi|_{D_{t^{-1}}})$ to a representation of D_e. $\square$

Proposition 2.3. *Given a Fell bundle $\mathcal{B} = (B_t)_{t\in G}$ over a discrete group G, let $\widehat{\alpha}_t$ be the homeomorphism defined in equation* (2.4)*, for $t \in G$. Then*

$$\widehat{\alpha} := \big(\{X_t\}_{t\in G}, \{\widehat{\alpha}_t\}_{t\in G}\big)$$

is a partial action of G on $\widehat{B_e}$.

Proof. Clearly, $\widehat{\alpha}_t$ is a homeomorphism between open subsets of X, so it remains to show that $\widehat{\alpha}_{st}$ extends $\widehat{\alpha}_s\widehat{\alpha}_t$, for all $s,t \in G$.

We first show that $\operatorname{dom}\widehat{\alpha}_s\widehat{\alpha}_t \subseteq \operatorname{dom}\widehat{\alpha}_{st}$. Let $\pi \in \operatorname{dom}\widehat{\alpha}_s\widehat{\alpha}_t$, and assume that $\pi \notin \operatorname{dom}\widehat{\alpha}_{st}$, that is, $\pi|_{D_{(st)^{-1}}} = 0$. We will show that this implies that $\widehat{\alpha}_t(\pi)|_{D_{s^{-1}}} = 0$, which contradicts the fact that $\pi \in \operatorname{dom}\widehat{\alpha}_s\widehat{\alpha}_t$.

In fact, let $d \in D_{s^{-1}}$. Then, for $b \in B_t$ and $h \in \mathcal{H}_\pi$, we have

$$\|\widehat{\alpha}_t(\pi)(d)(b \otimes h)\|^2 = \langle db \otimes h, db \otimes h\rangle = \langle \pi(b^*d^*db)h, h\rangle = 0,$$

because $b^*d^*db \in B_t^*D_{s^{-1}}B_t = B_t^*B_s^*B_sB_t \subseteq B_{st}^*B_{st} = D_{(st)^{-1}}$.

We now show that $\widehat{\alpha}_{st} = \widehat{\alpha}_s\widehat{\alpha}_t$ on $\operatorname{dom}\widehat{\alpha}_s\widehat{\alpha}_t$, namely, we will show that, if $\pi \in \operatorname{dom}\widehat{\alpha}_s\widehat{\alpha}_t$ is a representation on $\mathcal{H}_\pi$, then the map

$$U : B_s \otimes_{D_{s^{-1}}} B_t \otimes_{D_{t^{-1}}} \mathcal{H}_\pi \longrightarrow B_{st} \otimes_{D_{(st)^{-1}}} \mathcal{H}_\pi,$$

defined by

$$U(b_s \otimes b_t \otimes h) = b_sb_t \otimes h,$$

for $b_s \in B_s$, $b_t \in B_t$ and $h \in \mathcal{H}_\pi$, is a unitary operator intertwining $\widehat{\alpha}_s\widehat{\alpha}_t(\pi)$ and $\widehat{\alpha}_{st}(\pi)$.

In order to check that the definition of U makes sense, first notice that

$$\begin{aligned} B_s \otimes_{D_{s^{-1}}} B_t \otimes_{D_{t^{-1}}} \mathcal{H}_\pi &= B_s \otimes_{D_{s^{-1}}} D_{s^{-1}}B_t \otimes_{D_{t^{-1}}} \mathcal{H}_\pi \\ &= B_s \otimes_{D_{s^{-1}}} D_{s^{-1}}(B_tB_t^*B_t) \otimes_{D_{t^{-1}}} \mathcal{H}_\pi \\ &= B_s \otimes_{D_{s^{-1}}} B_t(B_t^*D_{s^{-1}}B_t) \otimes_{D_{t^{-1}}} \mathcal{H}_\pi. \end{aligned}$$

This implies that the map

$$\widetilde{U} : B_s \times B_t \times \mathcal{H}_\pi \longrightarrow B_{st} \otimes_{D_{(st)^{-1}}} \mathcal{H}_\pi$$

defined by $\widetilde{U}(b_s, b_t, h) = b_sb_t \otimes b_{st}$ is balanced: given $b_s \in B_s$, $b_t \in B_t$, $e \in B_t^*D_{s^{-1}}B_t$, $c \in D_{t^{-1}}$ and $h \in \mathcal{H}_\pi$, we have that $ec \in B_t^*D_{s^{-1}}B_tD_{t^{-1}} = B_t^*D_{s^{-1}}B_t \subseteq D_{(st)^{-1}}$. Therefore,

$$\begin{aligned} \widetilde{U}(b_s, b_tec, h) &= b_sb_tec \otimes h = b_sb_t \otimes \pi(ec)h \\ &= b_sb_t \otimes \pi(e)\pi(c)h \\ &= b_sb_te \otimes \pi(c)h \\ &= \widetilde{U}(b_s, b_t, \pi(c)h). \end{aligned}$$

Additionally, U is an isometry because, if $b_s, c_s \in B_s$, $b_t, c_t \in B_t$, and $h, h' \in \mathcal{H}_\pi$, then

$$\begin{aligned}\langle b_s \otimes b_t \otimes h, c_s \otimes c_t \otimes h' \rangle &= \langle \big(\widehat{\alpha}_t(\pi)(c_s^* b_s)\big)(b_t \otimes h), c_t \otimes h' \rangle \\ &= \langle c_s^* b_s b_t \otimes h, c_t \otimes h' \rangle \\ &= \langle \pi(c_t^* c_s^* b_s b_t) h, h' \rangle \\ &= \langle b_s b_t \otimes h, c_s c_t \otimes h' \rangle \\ &= \langle U(b_s \otimes b_t \otimes h), U(c_s \otimes c_t \otimes h') \rangle.\end{aligned}$$

Furthermore, U is onto because its image is a non-zero $\widehat{\alpha}_{st}(\pi)$-invariant subspace of $B_{st} \otimes \mathcal{H}$. Finally, it is apparent that U intertwines $\widehat{\alpha}_s \widehat{\alpha}_t(\pi)$ and $\widehat{\alpha}_{st}(\pi)$. □

Definition 2.4. Let $\mathcal{B}$ be a Fell bundle over a discrete group G. The partial action $\widehat{\alpha}$ in Proposition 2.3 will be called the *partial action associated with* $\mathcal{B}$.

Example 2.5 (Crossed products by Hilbert C^*-bimodules)**.** When $\mathcal{B}$ is the Fell bundle associated to a Hilbert C^*-bimodule X over a C^*-algebra A as in [**2,** subsection 2.6], the associated partial action $\widehat{\alpha}$ is the partial homeomorphism $\widehat{h}$ discussed in [**16**]. When the C^*-algebra A is commutative, it also agrees with the partial homeomorphism induced by the partial action θ in [**3,** subsection 1.9].

Example 2.6 (Partial crossed products)**.** If $\gamma = (\{\gamma_t\}_{t\in G}, \{D_t\}_{t\in G})$ is a partial action of a discrete group G on a C^*-algebra A, then the Fell bundle $\mathcal{B}_\gamma$ associated with γ has fibers $B_t = \{t\} \times D_t$ with the obvious structure of Banach space, and product and involution given by:

$$\begin{aligned}(r, d_r)(s, d_s) &= (rs, \gamma_r(\gamma_{r^{-1}}(d_r) d_s)), \\ (r, d_r)^* &= (r^{-1}, \gamma_{r^{-1}}(d_r^*)).\end{aligned}$$

The unit fiber of $\mathcal{B}_\gamma$ is identified with A in the obvious way.

The partial action γ induces a partial action $\widehat{\gamma}$ on $\widehat{A}$ that was defined in [**5,** Section 7] and [**6**] and further discussed in [**17**]. The partial action $\widehat{\gamma}$ is given by

$$\widehat{\gamma}_t(\pi) = \pi \circ \gamma_{t^{-1}} \quad \text{for } \pi \in \widehat{A},$$

and it agrees with the partial action associated with the Fell bundle $\mathcal{B}_\gamma$. In fact, it is easily checked that, if $\pi \in \widehat{D}_{t^{-1}}$ is a representation on a Hilbert space $\mathcal{H}_\pi$, then the map

$$U : B_t \otimes_{D_{t^{-1}}} \mathcal{H}_\pi \longrightarrow \mathcal{H}_\pi, \text{ given by } U((t, d_t) \otimes h) = \pi(\gamma_{t^{-1}}(d_t))(h),$$

for $d_t \in D_t$, $t \in G$ and $h \in \mathcal{H}_\pi$, is a unitary operator intertwining $\operatorname{Ind}_{B_t} \pi$ and $\pi \circ \gamma_{t^{-1}}$.

Example 2.7 (Fell bundles with commutative unit fiber)**.** We now assume that the Fell bundle $\mathcal{B}$ has commutative unit fiber, that is, $B_e = C_0(X)$ for a locally compact Hausdorff space X. We identify X with $\widehat{B_e}$ in the usual way: $x \in X$ is viewed as $[\pi_x] \in \widehat{B_e}$, where π_x is evaluation at x.

If $I_x = \ker \pi_x$, then $x \in X_{t^{-1}}$ if and only if $B_t^* B_t \not\subseteq I_x$, that is, (**[19,** subsection 3.3]), $x \in X_{t^{-1}}$ if and only if $B_t I_x \neq B_t$. Therefore, if $b_t(x)$ denotes the image of an element b_t of B_t under the quotient map on $B_t / B_t I_x$, then

$$\widehat{B_e} \setminus X_{t^{-1}} = \{x \in X : b_t(x) = 0 \text{ for all } b_t \in B_t\}.$$

Additionally, if $x \in X_{t^{-1}}$, we have, by equation (1.6),

$$I_{\hat{\alpha}_t(x)} B_t = I_{\hat{\alpha}_t(x)} D_t B_t = \ker(\operatorname{Ind}_{B_t} \pi_x) B_t = B_t \ker \pi_x = B_t I_x.$$

Therefore,

$$(ab_t)(x) = \begin{cases} a(\widehat{\alpha}_t(x)) b_t(x) & \text{if } x \in X_{t^{-1}} \\ 0 & \text{otherwise,} \end{cases} \tag{2.6}$$

for $a \in B_e$ and $b_t \in B_t$.

3. Topological freeness and ideals in the cross-sectional C^*-algebras. In this section, we show that some well-known results relating topological freeness and the ideal structure of crossed products carry over to our setting.

Proposition 3.1. *Let $\mathcal{B} = (B_t)_{t \in G}$ be a Fell bundle over a discrete group G, and let ρ be a representation of $C^*(\mathcal{B})$ on a Hilbert space $\mathcal{K}$. Suppose that $\sigma : B_e \to B(\mathcal{H})$ is an irreducible subrepresentation of $\rho|_{B_e}$, and let $\mathcal{H}_t = \overline{\operatorname{span}}\, \rho(B_t)\mathcal{H}$, for each $t \in G$. Then:*

(i) *$\mathcal{H}_t$ is $\rho(B_e)$-invariant for all $t \in G$.*
(ii) *$\mathcal{H}_t = \{0\}$ if $\sigma \notin X_{t^{-1}}$, and $\mathcal{H}_t \perp \mathcal{H}$ if $\sigma \notin X_t$.*
(iii) *If $\sigma \in X_t \cap X_{t^{-1}}$ and $\widehat{\alpha}_t(\sigma) \neq \sigma$, then $\mathcal{H}_t \perp \mathcal{H}$.*

Proof. Statement (i) is apparent. As for (ii), consider the orthogonal decompositions

$$\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}, \qquad \rho|_{B_e} = \sigma \oplus \sigma^{\perp}.$$

Notice that any element in B_t can be written as xb_ty, where $x \in D_t$, $b_t \in B_t$, and $y \in D_{t^{-1}}$. Additionally, if $\sigma \notin X_{t^{-1}}$, then $\sigma|_{D_{t^{-1}}} = 0$, and, for any $h \in \mathcal{H}$,

$$\rho(xb_ty)(h) = \rho(xb_t)(\sigma(y)(h) + \sigma^{\perp}(y)h) = 0,$$

which shows that $\mathcal{H}_t = \{0\}$.

If $\sigma \notin X_t$, then, for x, b_t, y as above, and $h, h' \in \mathcal{H}$,

$$\begin{aligned} \langle \rho(xb_ty)h, h'\rangle &= \langle \rho(b_ty)h, \rho(x^*)h'\rangle \\ &= \langle \rho(b_ty)h, \sigma(x^*)h' + \sigma^{\perp}(x^*)h'\rangle \\ &= \langle \rho(b_ty)h, \sigma(x^*)h'\rangle \\ &= 0, \end{aligned}$$

which completes the proof of (ii). In order to prove (iii), we now assume that $\sigma \in X_t \cap X_{t^{-1}}$. Let σ_t denote the subrepresentation of $\rho|_{B_e}$ on $\mathcal{H}_t$, that is,

$$\sigma_t(c)h_t = \rho(c)h_t,$$

for all $c \in B_e$ and $h_t \in \mathcal{H}_t$. Then the map

$$U : B_t \otimes_{D_{t^{-1}}} \mathcal{H} \longrightarrow \mathcal{H}_t \text{ given by } U(b_t \otimes h) = \rho(b_t)h$$

is a unitary operator intertwining σ_t and $\widehat{\alpha}_t(\sigma)$. In fact, if $b_t, c_t \in B_t$, and $h, k \in \mathcal{H}$, then

$$\langle b_t \otimes h, c_t \otimes k\rangle = \langle \sigma(c_t^*b_t)h, k\rangle = \langle \rho(c_t^*b_t)h, k\rangle = \langle \rho(b_t)h, \rho(c_t)k\rangle.$$

Therefore, if $\sigma \neq \widehat{\alpha}_t(\sigma)$, then σ and σ_t are irreducible non-equivalent subrepresentations of $\rho|_{B_e}$. It now follows from [**7,** subsection 12.15] that $\mathcal{H}$ and $\mathcal{H}_t$ are orthogonal. ☐

Definition 3.2. Recall from [**12,** subsection 2.2] that a partial action θ of a discrete group G on a locally compact topological space X is *topologically free* if, for any finite subset S of $G \setminus \{e\}$, the set

$$\bigcup_{t\in S}\{x \in \operatorname{dom}\theta_t : \theta_t(x) = x\}$$

has empty interior. Equivalently, θ is topologically free if the set

$$F_t = \{x \in \operatorname{dom}\theta_t : \theta_t(x) = x\}$$

has empty interior for any t in $G \setminus \{e\}$.

Theorem 3.3. *Suppose that $\mathcal{B} = (B_t)_{t\in G}$ is a Fell bundle over a discrete group G, A is a C^*-algebra and*

$$\phi : C^*(\mathcal{B}) \longrightarrow A$$

is a *-homomorphism, and let $J := \ker\phi \cap B_e$.

If the partial action $\widehat{\alpha}$ associated with $\mathcal{B}$ is topologically free on $\widehat{B_e} \setminus X_J$, then

$$\|\phi(c)\| \geq \|\phi(E(c))\|, \quad \text{for all } c \in C^*(\mathcal{B}). \tag{3.1}$$

Proof. Since it suffices to show that equation (3.1) holds when c belongs to the dense $\star$-subalgebra $C_c(\mathcal{B})$ of compactly supported cross sections, we assume that

$$c = \sum_{t\,\in\,\operatorname{supp}(c)} c(t)\delta_t,$$

where $\operatorname{supp}(c)$ is a finite subset of G. In order to show the statement, we will prove that

$$\|\phi(c)\| \geq \|\phi(E(c))\| - \epsilon, \tag{3.2}$$

for all $\epsilon > 0$.

Fix $\epsilon > 0$. Note that

$$\begin{aligned}\|\phi(E(c))\| = \|E(c) + J\|_{B_e/J} &= \max\{\|\tau(E(c) + J)\| : \tau \in \widehat{B_e/J}\}\\ &= \max\{\|\sigma(E(c))\| : \sigma \in \widehat{B_e} \setminus X_J\}.\end{aligned} \tag{3.3}$$

In addition, since the map $\sigma \mapsto \|\sigma(E(c))\|$ is lower semicontinuous on $\widehat{B_e} \setminus X_J$ ([**19,** A30]), we can choose a set V that is open in $\widehat{B_e} \setminus X_J$

such that

$$\|\sigma(E(c)\| \geq \|\phi(E(c))\| - \epsilon, \tag{3.4}$$

for all $\sigma \in V$.

Now, since $\widehat{\alpha}$ is topologically free on $\widehat{B_e} \setminus X_J$, the set

$$F = \bigcup_{\substack{t \in \operatorname{supp}(c) \\ t \neq e}} \{\sigma \in X_{t^{-1}} : \widehat{\alpha}_t(\sigma) = \sigma\} \tag{3.5}$$

does not contain V. Thus, we can choose a representation $\sigma \in V$ on a Hilbert space $\mathcal{H}$ such that $\sigma \notin F$.

Let $\widetilde{\phi} : B_e/J \to \phi(B_e)$ be the canonical isomorphism induced by $\phi|_{B_e}$, and let ψ_0 be a state of $\phi(B_e)$ associated with the irreducible representation $q_J(\sigma) \circ (\widetilde{\phi})^{-1}$, where q_J is as in equation (1.8). Extend ψ_0 to a pure state ψ on $\phi(C^*(\mathcal{B}))$. The GNS construction for ψ yields a representation π of $\phi(C^*(\mathcal{B}))$ on a Hilbert space $\mathcal{K}$ containing a closed subspace $\mathcal{H}$ such that $q_J(\sigma) \circ (\widetilde{\phi})^{-1}$ is the subrepresentation of $\pi|_{\phi(B_e)}$ on $\mathcal{H}$.

We now define $\rho : C^*(\mathcal{B}) \longrightarrow B(K)$ by $\rho = \pi \circ \phi$. If $Q \in B(\mathcal{K}, \mathcal{H})$ is the orthogonal projection on $\mathcal{H}$, then

$$Q\rho(b)Q^* = Q\pi(\phi(b))Q^* = Q(\pi(\widetilde{\phi}(b + J))Q^* = q_J(\sigma)(b + J) = \sigma(b), \tag{3.6}$$

for all $b \in B_e$, which shows that σ is an irreducible subrepresentation of $\rho|_{B_e}$.

We now set $\mathcal{H}_t = \overline{\operatorname{span}}\rho(B_t)(\mathcal{H})$. By Proposition 3.1, we have, since $\sigma \notin F$, that $\mathcal{H}_t \perp \mathcal{H}$ for all $t \in \operatorname{supp}(c)$ such that $t \neq e$. Therefore,

$$\begin{aligned}
\|\phi(c)\| &\geq \|\pi \circ \phi(c)\| = \|\rho(c)\| \\
&\geq \|Q\rho(c)Q^*\| = \|Q\rho(E(c))Q^*\| \\
&= \|\sigma(E(c))\| \geq \|\phi(E(c))\| \\
&\quad - \epsilon.
\end{aligned}$$

□

Corollary 3.4. *Suppose that $\mathcal{B} = (B_t)_{t \in G}$ is a Fell bundle over a discrete group G such that the partial action associated with $\mathcal{B}$ is topologically free. Then*:

(i) *if I is an ideal in $C^*(\mathcal{B})$ such that $I \cap B_e = \{0\}$, then $I \subset \ker \Lambda$, where*

$$\Lambda : C^*(\mathcal{B}) \longrightarrow C^*_r(\mathcal{B})$$

is the canonical surjective map.

(ii) *If I is an ideal in $C^*_r(\mathcal{B})$ such that $I \cap B_e = \{0\}$, then $I = \{0\}$. Consequently, a representation of $C^*_r(\mathcal{B})$ is faithful if and only if its restriction to B_e is faithful.*

Proof.

(i) Since the restriction of the quotient map

$$P_I : C^*(\mathcal{B}) \longrightarrow C^*(\mathcal{B})/I$$

to B_e is injective, we have by Theorem 3.3 that

$$\|P_I(E(c))\| \leq \|P_I(c)\| \quad \text{for all } c \in C^*(\mathcal{B}).$$

Consequently, $E(I) \subseteq I \cap B_e = \{0\}$ and $I \subset \ker \Lambda$, see equation (2.3).

(ii) Let $J = \Lambda^{-1}(I)$. Then $J \lhd C^*(B)$ and $\Lambda(J \cap B_e) \subseteq I \cap B_e = \{0\}$.

Therefore, $J \cap B_e \subseteq \ker \Lambda \cap B_e = \{0\}$. It now follows from (i) that $J \subseteq \ker \Lambda$. Hence, $I = \Lambda(J) = \{0\}$. $\square$

Definition 3.5 (cf., [**5**])**.** Let $\mathcal{B}$ be a Fell bundle over a discrete group G. A subset $\mathcal{J} \subseteq \mathcal{B}$ is an *ideal* of $\mathcal{B}$ if it is a Fell bundle over G with the inherited structure, and if $\mathcal{J}\mathcal{B} = \mathcal{J} = \mathcal{B}\mathcal{J}$. An ideal I in B_e is said to be $\mathcal{B}$-*invariant* if $B_t I B_t^* \subseteq I$, for all $t \in G$.

Proposition 3.6. *Let $\mathcal{B}$ be a Fell bundle over a discrete group G, and let I be an ideal in B_e. Then the following statements are equivalent*:

(i) *I is a $\mathcal{B}$-invariant ideal.*
(ii) *$B_t I B_t^* = I \cap B_t B_t^*$ for all $t \in G$.*
(iii) *$B_t I = I B_t$ for all $t \in G$.*
(iv) *$\mathcal{I} = (I B_t)_{t \in G}$ is an ideal of $\mathcal{B}$.*

Proof. Suppose that I is $\mathcal{B}$-invariant. Then $B_t I B_t^* \subseteq I$, and, since $B_t I B_t^* \subseteq B_t B_e B_t^* = B_t B_t^*$, we have that $B_t I B_t^* \subseteq I \cap B_t B_t^*$.

On the other hand, since $B_t^*IB_t \subseteq I$, we have that

$$I \cap B_tB_t^* = IB_tB_t^* = B_tB_t^*IB_tB_t^* \subseteq B_tIB_t^*.$$

Thus, (i) implies (ii).

Now, if (ii) holds, then

$$B_tI = B_tB_t^*B_tI = B_tIB_t^*B_t = (I \cap B_tB_t^*)B_t = (IB_tB_t^*)B_t = IB_t,$$

which implies (iii). Clearly $\mathcal{I}$ is a right ideal, and it is apparent that it is also a left ideal if (iii) holds. Finally, suppose that $\mathcal{I}$ is an ideal in $\mathcal{B}$. Then

$$B_tIB_t^* \subseteq \mathcal{I} \cap B_e = I. \qquad \square$$

Remark 3.7. If $J \lhd C^*(\mathcal{B})$ or $J \lhd C_r^*(\mathcal{B})$, then $J \cap B_e$ is a $\mathcal{B}$-invariant ideal.

Proof. In both cases, $J_eB_t = J_t = B_tJ_e$, where $J_t = J \cap B_t$ for all $t \in G$. It is clear that $J_t \supseteq J_eB_t$ and $J_t \supseteq B_tJ_e$. On the other hand, since J_t is a Hilbert C^* sub-bimodule of B_t, we have that $J_t = J_tJ_t^*J_t \subseteq J_eB_t \cap B_tJ_e$. $\square$

Proposition 3.8. *Let $\mathcal{B}$ be a Fell bundle over a discrete group G. The map $I \mapsto \mathcal{I} = (I_t)_{t\in G}$, where $I_t = IB_t$ is an isomorphism from the lattice of $\mathcal{B}$-invariant ideals of B_e onto that of the ideals of $\mathcal{B}$. Its inverse is given by $\mathcal{I} \mapsto \mathcal{I} \cap B_e$.*

Proof. Assume that I is $\mathcal{B}$-invariant. Then, by Proposition 3.6, $\mathcal{I} = (I_t)$ is an ideal in $\mathcal{B}$, and the correspondence $I \mapsto \mathcal{I}$ is injective because $I_e = I$. Conversely, if $\mathcal{I}$ is an ideal of $\mathcal{B}$, let $I_t := \mathcal{I} \cap B_t$, for all $t \in G$. Since $\mathcal{I}$ is a Fell bundle and a right ideal of $\mathcal{B}$, we have:

$$I_t = I_eI_t \subseteq I_eB_t \subseteq \mathcal{I} \cap B_t = I_t.$$

Then $I_t = I_eB_t$, and, analogously, $I_t = B_tI_e$. Thus, I_e is a $\mathcal{B}$-invariant ideal of B_e, and $\mathcal{I} = (I_eB_t)$.

Finally, it is clear that both maps preserve inclusion, which implies they are lattice isomorphisms. $\square$

Definition 3.9. Recall that, if α is a partial action of G on a set X, then a set $S \subset X$ is said to be *α-invariant* if

$$\alpha_t(S \cap \operatorname{dom} \alpha_t) = S \cap \operatorname{dom} \alpha_{t^{-1}}, \text{ for all } t \in G.$$

Proposition 3.10. *Let $\mathcal{B}$ be a Fell bundle over a discrete group G, and let $\widehat{\alpha}$ be the partial action on $\widehat{B_e}$ associated with $\mathcal{B}$. Then the map $J \mapsto X_J$ is an isomorphism from the lattice of $\mathcal{B}$-invariant ideals in B_e to that of open $\widehat{\alpha}$-invariant sets in $\widehat{B_e}$.*

Proof. Since it is well known that the correspondence $J \mapsto X_J$ is a lattice isomorphism from $\mathcal{I}(B_e)$ to the topology of $\widehat{B_e}$, the proof amounts to showing that an ideal J in B_e is $\mathcal{B}$-invariant if and only if the open set X_J is $\widehat{\alpha}$-invariant.

First, assume that J is $\mathcal{B}$-invariant. If $\sigma \in X_J \cap X_{t^{-1}}$, then $\sigma|_{JD_{t^{-1}}} \neq 0$. In addition, $B_tJ = JB_t$ is a $D_tJ - JD_{t^{-1}}$ imprimitivity bimodule, and it follows that $\operatorname{Ind}_{B_tJ}(\sigma|_{JD_{t^{-1}}}) \neq 0$.

On the other hand, if σ is a representation on a Hilbert space $\mathcal{H}_\sigma$, then the map

$$b_tj \otimes_{D_{t^{-1}}J} h \longmapsto b_tj \otimes_{D_{t^{-1}}} h$$

extends to a unitary operator from $B_tJ \otimes_{D_{t^{-1}}J} \mathcal{H}_\sigma$ onto $B_t \otimes_{D_{t^{-1}}} \mathcal{H}_\sigma$ that intertwines $\operatorname{Ind}_{B_t}(\sigma|_{D_{t^{-1}}})|_{D_tJ}$ and $\operatorname{Ind}_{B_tJ}(\sigma|_{D_{t^{-1}}J})$. This shows that $\widehat{\alpha}_t(\sigma)|_J \neq 0$, that is, that $\widehat{\alpha}_t(\sigma) \in X_J$.

Assume now that X_J is $\widehat{\alpha}$-invariant. Then

$$B_tJ = B_tD_{t^{-1}}J = h_{B_t}(D_{t^{-1}}J)B_t,$$

for all $t \in G$.

Now, since the Rieffel correspondence is a lattice isomorphism,

$$\begin{aligned} h_{B_t}(D_{t^{-1}}J) &= h_{B_t}\left(\bigcap\{\ker \pi|_{D_{t^{-1}}} : \pi \in X_J^c \cap X_{t^{-1}}\}\right) \\ &= \bigcap\{h_{B_t}(\ker \pi|_{D_{t^{-1}}}) : \pi \in X_J^c \cap X_{t^{-1}}\} \\ &= \bigcap\{\ker \operatorname{Ind}_{B_t}(\pi|_{D_{t^{-1}}}) : \pi \in X_J^c \cap X_{t^{-1}}\} \end{aligned}$$

$$= D_t \cap \bigcap\{\ker \widehat{\alpha}_t(\pi) : \pi \in X_J^c \cap X_{t^{-1}}\}$$
$$= D_t \cap \bigcap\{\ker \pi : \pi \in X_J^c \cap X_t\} = D_t J.$$

Thus, $B_t J = J D_t B_t = J B_t$. □

Definition 3.11. Recall that a partial action α on a topological space X is said to be *minimal* if X does not have α-invariant open proper subsets.

Corollary 3.12. *Let $\mathcal{B} = (B_t)_{t\in G}$ be a Fell bundle with associated partial action $\widehat{\alpha}$. Consider the following statements*:

(i) *$C_r^*(\mathcal{B})$ is simple.*
(ii) *The Fell bundle $\mathcal{B}$ has no non-trivial ideals.*
(iii) *B_e has no non-trivial $\mathcal{B}$-invariant ideals.*
(iv) *The partial action $\widehat{\alpha}$ is minimal.*

Then we have (i) $\Rightarrow$ (ii) *if and only if* (iii) *if and only if* (iv) *and, if $\widehat{\alpha}$ is topologically free, then we also have* (iv) $\Rightarrow$ (i), *so in this case all the statements are equivalent.*

Proof. Since all open proper subsets of $\widehat{B_e}$ can be written as X_J for some non-trivial ideal J in B_e, Proposition 3.8 and Proposition 3.10 show that (ii), (iii) and (iv) are equivalent.

Assume now that $C_r^*(\mathcal{B})$ is simple, and let $\mathcal{J} \ntrianglelefteq \mathcal{B}$. Then $C_r^*(\mathcal{J}) \lhd C_r^*(\mathcal{B})$ by [**5,** subsection 3.2]. In addition, since $\mathcal{J} \neq \mathcal{B}$, we have that

$$E(C_r^*(\mathcal{J})) = \mathcal{J} \cap B_e \neq B_e,$$

by Proposition 3.8. This implies that $C_r^*(\mathcal{J}) \neq C_r^*(\mathcal{B})$. Therefore, $C_r^*(\mathcal{J}) = \{0\}$. We now have that

$$0 \subseteq \mathcal{J} \subseteq C_r^*(\mathcal{J}) = \{0\};$$

hence, $\mathcal{J} = \{0\}$, and therefore, (i) implies (ii).

Suppose now that (iv) holds and that $\widehat{\alpha}$ is topologically free. Let $J \ntrianglelefteq C_r^*(\mathcal{B})$, and set $J_e = J \cap B_e$.

By Remark 3.7, J_e is $\mathcal{B}$-invariant. Now, by Proposition 3.10, $X_{J_e} = \emptyset$, which implies that $J_e = \{0\}$. It now follows from Corollary 3.4 that $J = \{0\}$, which implies that $C_r^*(\mathcal{B})$ is simple. □

Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_t)_{t\in G}$ be Fell bundles over a discrete group G. A map

$$\phi : \mathcal{A} \longrightarrow \mathcal{B}$$

is said to be a morphism if

$$\phi|_{A_t} : A_t \longrightarrow B_t$$

is linear for all $t \in G$, and $\phi(aa') = \phi(a)\phi(a')$, $\phi(a^*) = \phi(a)^*$ for all $a, a' \in \mathcal{A}$, which implies that ϕ is norm decreasing. A morphism ϕ induces a homomorphism

$$\phi_c : C_c(\mathcal{A}) \longrightarrow C_c(\mathcal{B}),$$

given by $\phi_c(f)(t) := \phi(f(t))$. The map ϕ_c is a $\| \|_1$-continuous $*$-homomorphism, so it extends to a homomorphism of Banach $*$-algebras

$$\phi_1 : L^1(\mathcal{A}) \longrightarrow L^1(\mathcal{B}),$$

and hence, to a C^*-algebra homomorphism

$$\phi_* : C^*(\mathcal{A}) \longrightarrow C^*(\mathcal{B}).$$

Thus, we have a functor

$$(\mathcal{A} \xrightarrow{\phi} \mathcal{B}) \longmapsto (C^*(\mathcal{A}) \xrightarrow{\phi_*} C^*(\mathcal{B})),$$

that turns out to be exact ([**4,** 3.1]).

If we now consider reduced C^*-algebras instead of full C^*-algebras, we obtain another functor. In fact, suppose that

$$E_{\mathcal{A}} : C^*(\mathcal{A}) \longrightarrow A_e$$

is the canonical conditional expectation and that

$$\Lambda_{\mathcal{A}} : C^*(\mathcal{A}) \longrightarrow C^*_r(\mathcal{A})$$

is the canonical homomorphism. Since $\ker \Lambda_A = \{x \in C^*(\mathcal{A}) : E_{\mathcal{A}}(x^*x) = 0\}$, and the diagram

$$\begin{tikzcd} C^*(\mathcal{A}) \arrow[r, "\phi_*"] \arrow[d, "E_{\mathcal{A}}"'] & C^*(\mathcal{B}) \arrow[d, "E_{\mathcal{B}}"] \\ A_e \arrow[r, "\phi|_{B_e}"'] & B_e \end{tikzcd}$$

is commutative, we have that $\phi_*(\ker\Lambda_A) \subseteq \ker\Lambda_{\mathcal{B}}$. It follows that there exists a unique homomorphism $\phi_r : C_r^*(\mathcal{A}) \to C_r^*(\mathcal{B})$ such that

$$\begin{tikzcd} C^*(\mathcal{A}) \arrow[r, "\phi_*"] \arrow[d, "\Lambda_{\mathcal{A}}"'] & C^*(\mathcal{B}) \arrow[d, "\Lambda_{\mathcal{B}}"] \\ C_r^*(\mathcal{A}) \arrow[r, "\phi_r"'] & C_r^*(\mathcal{B}) \end{tikzcd}$$

commutes. Thus, we have another functor

$$(\mathcal{A} \xrightarrow{\phi} \mathcal{B}) \mapsto (C_r^*(\mathcal{A}) \xrightarrow{\phi_r} C_r^*(\mathcal{B})).$$

If ϕ is injective or surjective, then so is ϕ_r (**[5,** subsection 3.2]). However, if we consider the exact sequence of Fell bundles

$$0 \longrightarrow \mathcal{I} \xrightarrow{i} \mathcal{B} \xrightarrow{p} \mathcal{B}/\mathcal{I} \longrightarrow 0 \ ,$$

where $\mathcal{I}$ is an ideal in $\mathcal{B}$, then the induced sequence

$$0 \longrightarrow C_r^*(\mathcal{I}) \xrightarrow{i_r} C_r^*(\mathcal{B}) \xrightarrow{p_r} C_r^*(\mathcal{B}/\mathcal{I}) \longrightarrow 0$$

is not exact in general, because $C_r^*(\mathcal{I})$ does not necessarily agree with $\ker p_r$.

We remark that, since $\ker\Lambda_{\mathcal{A}} = \{x \in C^*(\mathcal{A}) : E_{\mathcal{A}}(x^*x) = 0\}$, we can define a map

$$C_r^*(\mathcal{A}) \longrightarrow A_e$$

such that

$$\Lambda_A(x) \longmapsto E_A(x),$$

for all $\Lambda_{\mathcal{A}}(x) \in C_r^*(\mathcal{A})$. This map is itself a faithful conditional expectation (**[10,** subsection 2.12]) with range A_e, which we will also denote by $E_{\mathcal{A}}$.

Let $\mathcal{I}(\mathcal{B})$ and $\mathcal{I}(C_r^*(\mathcal{B}))$ denote the lattice of ideals of the Fell bundle $\mathcal{B}$ and in $C_r^*(\mathcal{B})$, respectively. Since, for every $\mathcal{I} \in \mathcal{I}(\mathcal{B})$, we may identify $C_r^*(\mathcal{I})$ with the closure of $C_c(\mathcal{I})$ in $C_r^*(\mathcal{B})$, there is an order-preserving map

$$\mu : \mathcal{I}(\mathcal{B}) \longrightarrow \mathcal{I}(C_r^*(\mathcal{B}))$$

given by $\mu(\mathcal{I}) := C_r^*(\mathcal{I})$.

We now consider the maps

$$\nu_1, \nu_2 : \mathcal{I}(C_r^*(\mathcal{B})) \longrightarrow \mathcal{I}(\mathcal{B}),$$

given as follows. $\nu_1(J)$ is the ideal of $\mathcal{B}$ corresponding to $J \cap B_e$ by Proposition 3.6 (and Remark 3.7), that is, $\nu_1(J) = (J_t)_{t\in G}$, where $J_t = J \cap B_t$. Also, define $\nu_2(J)$ to be the ideal of $\mathcal{B}$ generated by $E_{\mathcal{B}}(J)$. Then, both ν_1 and ν_2 are left inverses for μ which implies that μ is injective. However, μ is not surjective in general. Clearly, a necessary condition for μ to be onto is that $\nu_1 = \nu_2$, that is, that $J \cap B_e = E_{\mathcal{B}}(J)$ for all $J \in \mathcal{I}(C_r^*(\mathcal{B}))$.

Definition 3.13 (cf., [**21**])**.** Let $\mathcal{B} = (B_t)_{t\in G}$ be a Fell bundle over a discrete group G. An ideal J of $C_r^*(\mathcal{B})$ is said to be *diagonal invariant* if $E_{\mathcal{B}}(J) \subseteq J$, that is, $E_{\mathcal{B}}(J) = J \cap B_e$.

In [**14**], Giordano and Sierakowski thoroughly discussed the above correspondence μ. In what follows, we generalize their methods and results to the context of Fell bundles.

Given an ideal J of $C_r^*(\mathcal{B})$, let $\mathcal{J}^{(1)} := \nu_1(J)$ and $J^{(1)} := \mu\nu_1(J)$, for μ and ν_1 as above. Then $J^{(1)} \subseteq J$, as it is the closure of the subset $C_c(\mathcal{J}^{(1)})$ of J.

Similarly, we define

$$\mathcal{J}^{(2)} := \nu_2(J)$$

and

$$J^{(2)} := \mu\nu_2(J).$$

Then $\mathcal{J}^{(2)}$ is the ideal of $\mathcal{B}$ generated by $E_{\mathcal{B}}(J)$ and $J^{(2)} = C_r^*(\mathcal{J}^{(2)})$. Note that the unit fiber of $\mathcal{J}^{(2)}$ is the invariant ideal of B_e generated by the ideal $E_{\mathcal{B}}(J)$ of B_e. Since $E_{\mathcal{B}}$ is the identity on $J \cap B_e$, it follows that $\mathcal{J}^{(1)} \subseteq \mathcal{J}^{(2)}$. Therefore, $J^{(1)} \subseteq J \cap J^{(2)}$.

Definition 3.14 (cf., [**14,** Definition 3.1])**.** Let $\mathcal{B} = (B_t)_{t\in G}$ be a Fell bundle over the discrete group G, and let $\mathcal{I} = (I_t)_{t\in G}$ be an ideal of $\mathcal{B}$. Then:

(i) $\mathcal{B}$ is said to have the exactness property at $\mathcal{I} \lhd \mathcal{B}$ if the sequence

$$0 \longrightarrow C_r^*(\mathcal{I}) \xrightarrow{i_r} C_r^*(\mathcal{B}) \xrightarrow{p_r} C_r^*(\mathcal{B}/\mathcal{I}) \longrightarrow 0$$

is exact.

(ii) $\mathcal{B}$ is said to have the intersection property at $\mathcal{I}$ if the intersection of B_e/I_e with any nonzero ideal in $C_r^*(\mathcal{B}/\mathcal{I})$ is also nonzero.

If $\mathcal{B}$ has the exactness property at every ideal $\mathcal{I} \in \mathcal{I}(\mathcal{B})$, we say that $\mathcal{B}$ has the exactness property, and if it has the intersection property at every ideal $\mathcal{I} \in \mathcal{I}(\mathcal{B})$, we say that $\mathcal{B}$ has the *residual intersection property.*

In view of Definition 3.14, the second statement of Corollary 3.4 could be restated in the following way: $\mathcal{B}$ has the intersection property whenever its associated partial action is topologically free. More generally, we have:

Proposition 3.15. *Let $\mathcal{B} = (B_t)_{t\in G}$ be a Fell bundle over a discrete group G. Suppose that $\mathcal{J} = (J_t)_{t\in G}$ is an ideal of $\mathcal{B}$, and let $X := \widehat{B}_e \setminus X_{J_e}$. If the partial action of $\mathcal{B}$ is topologically free on X, then $\mathcal{B}$ has the intersection property at the ideal $\mathcal{J}$.*

Proof. The unit fiber of the quotient bundle $\mathcal{B}/\mathcal{J}$ is B_e/J_e, whose spectrum is homeomorphic to $\widehat{B_e} \setminus X_{J_e} = X$. On the other hand, it is easily checked that the partial action associated to the Fell bundle $\mathcal{B}/\mathcal{J}$ agrees with that induced by the partial action of $\mathcal{B}$. Now, by the commutativity of diagram (1.9) and the fact that the partial action associated with $\mathcal{B}$ is topologically free on X, we conclude that the partial action associated to $\mathcal{B}/\mathcal{J}$ is topologically free. Finally, we apply part (ii) in Corollary 3.4. $\square$

Corollary 3.16. *If the partial action of the Fell bundle $\mathcal{B}$ is topologically free on every invariant closed subset of $\widehat{B}_e$, then $\mathcal{B}$ has the residual intersection property.*

Proposition 3.17. *Let $\mathcal{B} = (B_t)_{t\in G}$ be a Fell bundle over a discrete group G, and let $J \in \mathcal{I}(C_r^*(\mathcal{B}))$.*

(i) *If $\mathcal{B}$ has the exactness property at $\mathcal{J}^{(2)}$, then $J \subseteq J^{(2)}$. If, in addition, J is diagonal invariant, then $J^{(1)} = J = J^{(2)}$.*

(ii) *If $\mathcal{B}$ has the exactness property and the intersection property at $\mathcal{J}^{(1)}$, then $J^{(1)} = J = J^{(2)}$.*

Proof. Let

$$0 \longrightarrow \mathcal{J}^{(2)} \xrightarrow{\ i\ } \mathcal{B} \xrightarrow{\ p\ } \mathcal{B}/\mathcal{J}^{(2)} \longrightarrow 0$$

be the exact sequence associated with the ideal $\mathcal{J}^{(2)}$ of $\mathcal{B}$, and suppose that $\mathcal{B}$ has the exactness property at $\mathcal{J}^{(2)}$. Then the diagram

$$\begin{tikzcd}
0 \arrow[r] & C_r^*(\mathcal{J}^{(2)}) \arrow[r, "i_r"] \arrow[d, "E_{\mathcal{J}^{(2)}}"] & C_r^*(\mathcal{B}) \arrow[r, "p_r"] \arrow[d, "E_{\mathcal{B}}"] & C_r^*(\mathcal{B}/\mathcal{J}^{(2)}) \arrow[r] \arrow[d, "E_{\mathcal{B}/\mathcal{J}^{(2)}}"] & 0 \\
0 \arrow[r] & \mathcal{J}^{(2)} \cap B_e \arrow[r, "i"'] & B_e \arrow[r, "p"'] & B_e/(\mathcal{J}^{(2)} \cap B_e) \arrow[r] & 0
\end{tikzcd}$$

is commutative and has exact rows. If $x \in J^+$, then $E_{\mathcal{B}}(x) \in \mathcal{J}^{(2)} \cap B_e^+$, which implies that $E_{\mathcal{B}/\mathcal{J}^{(2)}} p_r(x) = 0$. Since $p_r(x) \in C_r^*(\mathcal{B}/\mathcal{J}^{(2)})^+$ and $E_{\mathcal{B}/\mathcal{J}^{(2)}}$ is faithful, then $p_r(x) = 0$. Then $x \in C_r^*(\mathcal{J}^{(2)})$ because of the exactness of the first row at $C_r^*(\mathcal{B})$. This shows that $J \subseteq J^{(2)}$. Since the inclusion $J^{(1)} \subseteq J$ always holds, and the definition of diagonal invariance requires precisely that $\mathcal{J}^{(1)} = \mathcal{J}^{(2)}$, which implies that $J^{(1)} = J^{(2)}$, we conclude that $J^{(1)} = J = J^{(2)}$.

Suppose now that $\mathcal{B}$ has both the exactness and the residual intersection properties at $\mathcal{J}^{(1)}$. Let

$$q : \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{J}^{(1)}$$

be the quotient map. In order to prove that $J^{(1)} = J = J^{(2)}$, it suffices to show that $J^{(1)} = J$, for, in this case, we have that $E(J) \subseteq J^{(1)}$, and, consequently, that $J^{(2)} = J^{(1)}$. In other words, we must show that $q_r(J) = \{0\}$. Since $\mathcal{B}$ is exact at $\mathcal{J}^{(1)}$, we have $\ker q_r = J^{(1)}$. Let

$$\overline{q}_r : C_r^*(\mathcal{B})/J^{(1)} \longrightarrow C_r^*(\mathcal{B}/\mathcal{J}^{(1)})$$

be the isomorphism induced by q_r. Since $\mathcal{B}$ has the intersection property at $\mathcal{J}^{(1)}$, in order to prove that $q_r(J) = \{0\}$, it suffices to show that $q_r(J) \cap B_e/(J \cap B_e) = \{0\}$, or, equivalently, that

$$J/J^{(1)} \cap (B_e + J^{(1)})/J^{(1)} = \{0\}, \tag{3.7}$$

since

$$J/J^{(1)} \cap (B_e + J^{(1)})/J^{(1)} = \bar{q}_r^{-1}(q_r(J) \cap B_e/(J \cap B_e)).$$

Let $x \in J$ and $b \in B_e$ be such that

$$x + J^{(1)} = b + J^{(1)} \in J/J^{(1)} \cap (B_e + J^{(1)})/J^{(1)}.$$

Then $x - b \in J^{(1)} \subseteq J$, which implies that $b \in J \cap B_e \subseteq J^{(1)}$ and $x \in J^{(1)}$, so equation (3.7) holds, and (ii) follows. □

Lemma 3.18. *If the map*

$$\mu : \mathcal{I}(\mathcal{B}) \longrightarrow \mathcal{I}(C_r^*(\mathcal{B}))$$

given by

$$\mathcal{I} \longmapsto C_r^*(\mathcal{I})$$

is a lattice isomorphism and $\mathcal{B}$ has the exactness property at $\mathcal{J} \in \mathcal{I}(\mathcal{B})$, then

$$\mu_{\mathcal{J}} : \mathcal{I}(\mathcal{B}/\mathcal{J}) \longrightarrow \mathcal{I}(C_r^*(\mathcal{B}/\mathcal{J}))$$

given by

$$\mathcal{I} \longmapsto C_r^*(\mathcal{I}/\mathcal{J})$$

is also a lattice isomorphism.

Proof. Let

$$\mathcal{I}_{\mathcal{J}} := \{\mathcal{I} \in \mathcal{I}(\mathcal{B}) : \mathcal{J} \subseteq \mathcal{I}\}$$

and

$$\mathcal{I}_{\mu(\mathcal{J})} := \{I \in \mathcal{I}(C_r^*(\mathcal{B})) : \mu(\mathcal{J}) \subseteq I\}.$$

Then the restriction of μ to $\mathcal{I}_{\mathcal{J}}$ gives rise to an isomorphism between $\mathcal{I}_{\mathcal{J}}$ and $\mathcal{I}_{\mu(\mathcal{J})}$. On the other hand, the map

$$\eta_1 : \mathcal{I} \longmapsto \mathcal{I}/\mathcal{J}$$

is an isomorphism from $\mathcal{I}_{\mathcal{J}}$ onto $\mathcal{I}(\mathcal{B}/\mathcal{J})$, as is the map

$$\eta_2 : I \longmapsto I/C_r^*(\mathcal{J})$$

from $\mathcal{I}_{\mu(\mathcal{J})}$ onto $\mathcal{I}(C_r^*(\mathcal{B})/C_r^*(\mathcal{J}))$. Moreover, since $\mathcal{B}$ is exact at $\mathcal{J}$, the quotient map

$$p : \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{J}$$

induces an isomorphism

$$\overline{p}_r : C_r^*(\mathcal{B})/C_r^*(\mathcal{J}) \longrightarrow C_r^*(\mathcal{B}/\mathcal{J})$$

which, in turn, induces an obvious lattice isomorphism

$$\eta_3 : \mathcal{I}(C_r^*(\mathcal{B})/C_r^*(\mathcal{J})) \longrightarrow \mathcal{I}(C_r^*(\mathcal{B}/\mathcal{I})).$$

Then $\mu_{\mathcal{J}}$ is an isomorphism because $\mu_{\mathcal{J}} = \eta_3\eta_2\mu|_{\mathcal{I}_{\mathcal{J}}}\eta_1^{-1}$. □

Theorem 3.19. *Let $\mathcal{B} = (B_t)_{t\in G}$ be a Fell bundle over a discrete group G. Let $\mu : \mathcal{I}(\mathcal{B}) \to \mathcal{I}(C_r^*(\mathcal{B}))$ be the lattice homomorphism given by $\mu(\mathcal{I}) = C_r^*(\mathcal{I})$. Then the following statements are equivalent*:

(i) *the map μ is an isomorphism of lattices.*
(ii) *$\mathcal{B}$ has the exactness property and every $J \in \mathcal{I}(C_r^*(\mathcal{B}))$ is diagonal invariant.*
(iii) *$\mathcal{B}$ has the exactness and residual intersection properties.*

Proof. It follows from Proposition 3.17 that either statement (ii) or (iii) implies (i). Suppose that μ is a lattice isomorphism. Then any ideal of $C_r^*(\mathcal{B})$ is of the form $C_r^*(\mathcal{I})$, and therefore, is diagonal invariant. Recall from the comments preceding Definition 3.13 that the inverse of μ is given by $J \mapsto J \cap B_e$. To show that (i) implies (ii), we must prove that $\mathcal{B}$ has the exactness property at any ideal $\mathcal{I} = (I_t)_{t\in G}$ of $\mathcal{B}$. The quotient map

$$p : \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{I}$$

induces a surjective homomorphism

$$p_r : C_r^*(\mathcal{B}) \longrightarrow C_r^*(\mathcal{B}/\mathcal{I}),$$

whose kernel contains $C_r^*(\mathcal{I})$. Then

$$I_e = E_{\mathcal{B}}(C_r^*(\mathcal{I})) \subseteq E_{\mathcal{B}}(\ker(p_r)) = \ker(p_r) \cap B_e,$$

the last equation following from the diagonal invariance of $\ker(p_r)$. But,

$$\ker(p_r) \cap B_e = \ker(p|_{B_e}) = I_e = C_r^*(\mathcal{I}) \cap B_e.$$

Then $\ker(p_r) = C_r^*(\mathcal{I})$.

To conclude that (i) also implies (iii) we must show that $\mathcal{B}$ has the residual intersection property. So, pick an element $\mathcal{J} = (J_t)_{t\in G} \in \mathcal{I}(\mathcal{B})$, and suppose that $I \lhd C_r^*(\mathcal{B}/\mathcal{I})$ is such that

$$I \cap \frac{B_e}{J_e} = \{0\}.$$

By Lemma 3.18, there is a unique $\mathcal{I} = (I_t)_{t\in G} \lhd \mathcal{B}$ such that $\mathcal{J} \subseteq \mathcal{I}$ and $I = C_r^*(\mathcal{I}/\mathcal{J})$. Then

$$\{0\} = I \cap \frac{B_e}{J_e} = \frac{I_e \cap B_e}{J_e},$$

that is, $J_e = I_e$. Since, by Proposition 3.8, this implies that $\mathcal{I} = \mathcal{J}$, it follows that $I = \{0\}$. $\square$

Corollary 3.20. *Let $\mathcal{B} = (B_t)_{t\in G}$ be a Fell bundle over a discrete group G. Then the correspondences*

$$\mathcal{J} \longmapsto C^*(\mathcal{J}) \quad \textit{and} \quad \mathcal{J} \longmapsto C_r^*(\mathcal{J})$$

are injective lattice homomorphisms from the lattice of ideals in $\mathcal{B}$ to the lattices $\mathcal{I}(C^*(\mathcal{B}))$ and $\mathcal{I}(C_r^*(\mathcal{B}))$ of ideals in $C^*(\mathcal{B})$ and $C_r^*(\mathcal{B})$, respectively. If $\mathcal{B}$ has the exactness property and its associated partial action is topologically free on every $\widehat{\alpha}$-invariant closed subset of $\widehat{B_e}$, then

$$\mathcal{I}(\mathcal{B}) \longrightarrow \mathcal{I}(C_r^*(\mathcal{B}))$$

is a lattice isomorphism.

Proof. Let $\mathcal{J} = (J_t)_{t\in J}$ be an ideal in $\mathcal{B}$. By [**4,** subsection 3.1],

$$\overline{C_c(\mathcal{J})} = C^*(\mathcal{J}) \lhd C^*(\mathcal{B}),$$

where $\overline{C_c(\mathcal{J})}$ is the closure of $C_c(\mathcal{J})$ in $C^*(\mathcal{B})$. It follows that $B_e \cap C^*(\mathcal{J}) = \mathcal{J} \cap B_e$, which takes care of the injectivity, in view of Proposition 3.8. The rest of the proof follows immediately from Theorem 3.19 and Corollary 3.16. $\square$

Example 3.21 (Ideal structure of quantum Heisenberg manifolds)**.** The family

$$\{D^c_{\mu,\nu} : c \in \mathbb{Z}, \quad c > 0, \quad \mu,\ \nu \in \mathbb{T}\}$$

of quantum Heisenberg manifolds was constructed [**20**] as a deformation of the Heisenberg manifold M_c for a positive integer c. The C^*-algebra $D^c_{\mu\nu}$ was shown [**2**] to be the crossed product of $C(\mathbb{T}^2)$ by a Hilbert C^*-bimodule $X^c_{\mu\nu}$, where $\mathbb{T}$ denotes the unit circle. Since $X^c_{\mu\nu}$ is full in both the left and the right, $\widehat{\alpha}$ turns out to be a homeomorphism, which was shown in [**3**] (see also Example 2.5) to be given by

$$\widehat{\alpha}(x,y) = (x+2\mu, y+2\nu) \quad \text{for all } (x,y) \in \mathbb{T}^2.$$

Let $G_{\mu\nu}$ denote the abelian free group

$$G_{\mu\nu} = \mathbb{Z} + 2\mu\mathbb{Z} + 2\nu\mathbb{Z}.$$

Rieffel showed [**20,** subsection 6.2] that $D^c_{\mu\nu}$ is simple if and only if $\operatorname{rank} G_{\mu\nu} = 3$. On the other hand, when $\operatorname{rank} G_{\mu\nu} = 1$, the C^*-algebra $D^c_{\mu\nu}$ is Morita equivalent to the commutative C^*-algebra $C(M_c)$ ([**1,** subsection 2.8]), and, consequently, has the same ideal structure. We now discuss the case in which $\operatorname{rank} G_{\mu\nu} = 2$. First note that the action $\widehat{\alpha}$ is free in that case. In fact, $\widehat{\alpha}_n(x,y) = (x,y)$ if and only if $2n\mu$ and $2n\nu$ are integers, which implies that $n = 0$ or $\operatorname{rank} G_{\mu\nu} = 1$.

Additionally, $C(\mathbb{T}^2) \rtimes X^c_{\mu\nu}$ has the exactness property by [**4,** subsection 3.1] because it is the cross-sectional C^*-algebra of a Fell bundle $\mathcal{B}$ over the amenable group $\mathbb{Z}$. Thus, we are under the assumptions of Lemma 3.20, and there is a lattice isomorphism between $\mathcal{I}(D^c_{\mu\nu})$ and the lattice of $\widehat{\alpha}$-invariant open sets of the two-torus.

4. Fell bundles with commutative unit fiber. Throughout this section, we will assume that the unit fiber of the Fell bundle $\mathcal{B}$ is commutative, that is, $B_e = C_0(X)$, for some locally compact Hausdorff space X. We will use the identifications and facts established in Example 2.7. Let $j_t : B_t \to \ell^2(\mathcal{B})$ be the inclusion map described in equation (2.2). Exel proved [**10**] that, for any $c \in C^*_r(\mathcal{B})$ and $t \in B_t$, there is a unique element $\widehat{c}(t) \in B_t$, called the Fourier coefficient of c corresponding to t, such that

$$j_t^* c j_e(a) = \widehat{c}(t)a \quad \text{for all } a \in B_e.$$

He also showed that $c = 0$ if and only if $\widehat{c} = 0$ ([**10,** subsections 2.6, 2.7, 2.12]).

Lemma 4.1. *Let $a \in B_e$ and $c \in C^*_r(\mathcal{B})$. Then $\widehat{ac} = a\widehat{c}$ and $\widehat{ca} = \widehat{c}a$.*

Consequently, c commutes with a if and only if $a\widehat{c}(t) = \widehat{c}(t)a$ for all $t \in G$.

Proof. Note that $\Lambda_a j_e(a') = j_e(aa')$ for all $a' \in B_e$. Then

$$\widehat{ca}(t)a' = j_t^* caj_e(a') = j_t^* cj_e(aa') = j_t^* cj_e(a)a' = \widehat{c}(t)aa',$$

and it follows that $\widehat{ca} = \widehat{c}a$.

On the other hand, as is easily checked, $j_t^* \Lambda_a(\xi) = a\xi(t)$ for all $\xi \in \ell^2(\mathcal{B})$. Therefore, if $a' \in B_e$:

$$\widehat{ac}(t)a' = j_t^* acj_e(a') = aj_t^* cj_e(a') = a\widehat{c}(t)a',$$

which shows that $\widehat{ac}(t) = a\widehat{c}(t)$. The last statement follows from the first one and from the fact that $ac = ca$ if and only if $\widehat{ac - ca} = 0$. $\square$

Lemma 4.2. *Let $b_t \in B_t$, and*

$$F_t = \{x \in X_{t^{-1}} : \widehat{\alpha}_t(x) = x\}.$$

Then $b_t \in B_e'$ if and only if $b_t(x) = 0$ for all $x \notin F_t$.

Proof. Since $ab_t = b_t a$ if and only if $(ab_t - b_t a)(x) = 0$ for all $x \in X$, we have that $b_t \in B_e'$ if and only if $b_t(x)a(\widehat{\alpha}_t(x)) = b_t(x)a(x)$ for all $x \in X_{t^{-1}}$ and $a \in B_e$. Thus, $b_t \in B_e'$ if $b_t(x) = 0$ for all $x \notin F_t$.

Conversely, if $b_t \in B_e'$ and $x \in X_{t^{-1}} \setminus F_t$, we can pick an element $a \in B_e$ such that $a(x) \neq 0 = a(\widehat{\alpha}_t(x))$. Then $b_t(x)a(x) = 0$, which shows that $b_t(x) = 0$. $\square$

Zeller-Meier showed that, if α is an action of a discrete group G on a commutative C^*-algebra A, then A is a maximal commutative C^*-subalgebra of the reduced crossed product $A \rtimes_{\alpha,r} G$ if and only if α is topologically free on $\widehat{A}$ (**[22,** Proposition 4.14]). The previous results allow us to generalize that result in the following way.

Proposition 4.3. *Let B_e' be the commutant of B_e in $C_r^*(\mathcal{B})$. Then $B_e' = B_e$ if and only if $\widehat{\alpha}$ is topologically free.*

Proof. Let $c \in C_r^*(\mathcal{B})$. By Lemmas 4.1 and 4.2, we have $c \in B_e'$ if and only if $\widehat{c}(t) = 0$ outside F_t for all $t \in G$. Then if, for all $t \neq e$, the interior of F_t is empty, we have $\widehat{c}(t) = 0$, so $c \in B_e$, and therefore,

$B_e' = B_e$. On the other hand, if there exists $t \neq e$ such that F_t has a non empty interior, then there exists $a \in D_{t^{-1}}$, $a \neq 0$, such that $a(x) = 0$ for all $x \notin F_t$. Since $B_t a \neq 0$, there exists $b_t' \in B_t$ such that $0 \neq b_t' a =: b_t \in B_t$. Now $b_t(x) = 0$ for all $x \notin F_t$, and therefore, $b_t \in B_e' \setminus B_e$. □

Corollary 4.4. *The partial action $\widehat{\alpha}$ is topologically free if and only if B_e is a maximal commutative C^*-subalgebra of $C_r^*(\mathcal{B})$, and consequently, it is a Cartan subalgebra of $C_r^*(\mathcal{B})$.*

4.1. The case of partial crossed products. We consider next a partial action on a commutative C^*-algebra $A = C_0(X)$, where X a locally compact Hausdorff space. It is clear from Example 2.6 that, in this case, the partial action $\widehat{\alpha}$ associated to the Fell bundle agrees with α when X is identified in the usual way with $\widehat{A}$. In what follows, we will write α to denote either one.

Theorem 4.5. *Suppose that α is a partial action of a discrete group G on a commutative C^*-algebra. Consider the following statements*:

(i) *A is a maximal commutative C^*-subalgebra of $A \rtimes_{\alpha,r} G$.*
(ii) *α is a topologically free.*
(iii) *If I is an ideal in $A \rtimes_\alpha G$ with $A \cap I = \{0\}$, then $I \subseteq \ker \Lambda$, where*
$$\Lambda : A \rtimes_\alpha G \longrightarrow A \rtimes_{\alpha,r} G$$
is the canonical map.
(iv) *If I is a non-zero ideal of $A \rtimes_{\alpha,r} G$, then $A \cap I \neq \{0\}$.*
(v) *If a representation $\phi : A \rtimes_{\alpha,r} G \to B(H)$ is faithful when restricted to A, then ϕ is faithful.*

Then we have that (i) *if and only if* (ii) *if and only if* (iii) $\Rightarrow$ (iv) *if and only if* (v).

Proof. Corollary 4.4 shows that (i) and (ii) are equivalent. In addition, Corollary 3.4 proves that (ii) implies (iii), and its proof shows that (iii) implies (iv). Since (iv) and (v) are obviously equivalent, we are left with the proof of the fact that (iii) implies (ii). We will adapt the proof for global actions [**8,** Theorem 2], which in turn essentially follows **[15]**, to our setting. Suppose (iii) holds. Let X be a locally compact Hausdorff topological space such that $A = C_0(X)$.

Given $x \in X$, let $o(x)$ denote the α-orbit of x:

$$o(x) := \{\alpha_t(x) : t \text{ such that } x \in X_{t^{-1}}\}.$$

Let $H^x := \ell^2(o(x))$ with its canonical orthonormal basis $\{e_y : y \in o(x)\}$. Consider

$$v^x : G \longrightarrow B(H^x)$$

defined by

$$v_t^x(e_y) = \begin{cases} e_{\alpha_t(y)} & \text{if } y \in X_{t^{-1}} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, v_t^x is a partial isometry with initial space $\ell^2(o(x) \cap X_{t^{-1}})$ and final space $\ell^2(o(x) \cap X_t)$.

We claim that v^x is a partial representation of G. Let us first note that $(v_t^x)^* = v_{t^{-1}}^x$, since

$$\begin{aligned} \langle v_t^x(e_y), e_z \rangle &= \begin{cases} 1 & \text{if } y \in X_{t^{-1}} \text{ and } z = \alpha_t(y) \\ 0 & \text{otherwise} \end{cases} \\ &= \langle e_y, v_{t^{-1}}^x(e_z) \rangle. \end{aligned}$$

We next show that

$$v_r^x v_s^x v_{s^{-1}}^x(e_y) = v_{rs}^x v_{s^{-1}}^x(e_y), \quad \text{for all } r, s \in G,\ y \in o(x).$$

In fact, we have, on one hand, that

$$v_r^x v_s^x v_{s^{-1}}^x(e_y) = \begin{cases} e_{\alpha_r(y)} & \text{if } y \in X_s \cap X_{r^{-1}} \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$v_{rs}^x v_{s^{-1}}^x(e_y) = \begin{cases} 0 & \text{if } y \notin X_s \cap \alpha_s(X_{s^{-1}r^{-1}} \cap X_{s^{-1}}) = X_{r^{-1}} \cap X_s \\ e_{\alpha_r(y)} & \text{otherwise.} \end{cases}$$

We now define the representation $\pi^x : A \to B(H^x)$ by $\pi^x(a)(e_y) = a(y)e_y$ for all $a \in A$ and $y \in o(x)$.

We claim that the pair (π^x, v^x) is a covariant representation of the system (A, α). In fact, if $a \in C_0(X_{t^{-1}})$, $y \in o(x)$:

$$\begin{aligned}\pi^x(\alpha_t(a))(e_y) &= \alpha_t(a)(y)e_y \\ &= \begin{cases} a(\alpha_{t^{-1}}(y))e_y & \text{if } y \in X_t \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

On the other hand, $v_t^x \pi^x(a) v_{t^{-1}}^x(e_y) = 0$ if $y \notin X_t$, and if $y \in X_t$:

$$\begin{aligned}v_t^x \pi^x(a) v_{t^{-1}}^x(e_y) &= v_t^x(a(\alpha_{t^{-1}}(y))e_{\alpha_{t^{-1}}(y)}) \\ &= a(\alpha_{t^{-1}}(y))e_y.\end{aligned}$$

Let

$$\rho^x : A \rtimes_\alpha G \longrightarrow B(H^x)$$

be the integrated form $\rho^x = \pi^x \rtimes v^x$ of the covariant representation (π^x, v^x). If

$$I = \bigcap_{x \in X} \ker \rho^x,$$

then $I \cap A = 0$ since if $a \in A$ and $\rho^x(a) = 0$ for all $x \in X$, then

$$0 = \rho^x(a)(e_y) = a(y)e_y, \quad \text{for all } x \in X, y \in o(x),$$

which shows that $a = 0$. Since we are assuming that (iii) holds, $I \subseteq \ker \Lambda$.

Let $t \neq e$ and $a \in A$ be such that

$$\operatorname{supp}(a) \subseteq \{x \in X \cap X_{t^{-1}} : \alpha_t(x) = x\}.$$

Then we have, for $x \in X$, $y \in o(x)$:

- if $y \in \operatorname{supp}(a)$ then $\alpha_t(y) = y$, and

$$\rho^x(a\delta_e - a\delta_t)(e_y) = a(y)e_y - a(\alpha_t(y))e_{\alpha_t(y)} = 0.$$

- if $y \notin \operatorname{supp}(a)$ then $\alpha_t(y) \notin \operatorname{supp}(a)$, and therefore, we have

$$\rho^x(a\delta_e - a\delta_t)(e_y) = a(y)e_y - a(\alpha_t(y))e_{\alpha_t(y)} = 0.$$

From the computations above, we conclude that $a\delta_e - a\delta_t \in I$. Therefore, $a\delta_e - a\delta_t \in \ker \Lambda$. Then

$$a = E(a\delta_e - a\delta_t) = 0,$$

from which it follows that the set

$$\{x \in X \cap X_{t^{-1}} : \alpha_t(x) = x\}$$

has empty interior.

$\square$

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