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POSITIVE GROUND STATE SOLUTIONS FOR SOME NON-AUTONOMOUS KIRCHHOFF TYPE PROBLEMS

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ABSTRACT. In this paper, we study the existence of positive ground state solutions for non-autonomous Kirchhoff type problems:

$$-\Big(1+b\int_{\mathbb{R}^3}|\nabla u|^2\Big)\Delta u+u=a(x)|u|^{p-1}u \quad \text{in } \mathbb{R}^3,$$

where $b>0$, $3<p<5$ and $a:\mathbb{R}^3\to\mathbb{R}$ is such that

$$\lim_{|x|\to\infty}a(x)=a_\infty>0,$$

but no symmetry property on $a(x)$ is required.

1. Introduction and main results. In the present paper, we consider the existence of positive ground state solutions for non-autonomous Kirchhoff type problems:

$$(SK) \qquad -\Big(1+b\int_{\mathbb{R}^3}|\nabla u|^2\Big)\Delta u+u=a(x)|u|^{p-1}u \quad \text{in } \mathbb{R}^3,$$

where $b>0$ is a real parameter, $3<p<5$ and $a(x)$ satisfies conditions which will be stated later, but with no symmetry property on $a(x)$.

Kirchhoff type problems in the bounded domain $\Omega\subset\mathbb{R}^3$ are often referred to as being nonlocal because of the presence of the integral over the entire domain Ω. This is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, i.e.,

$$u_{tt}-\Big(a+b\int_\Omega|\nabla u|^2\Big)\Delta u=f(x,u),$$

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where Ω is a bounded domain in $\mathbb{R}^N$, u denotes displacement, $f(x,u)$ is external force and a is initial tension, while b is related to intrinsic properties of the string. This type of equation was first proposed by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string by taking into account the subsequent change in string length caused by oscillations.

Kirchhoff type problems in the bounded domain have been studied by many authors. Alves et al. [**1**], Ma-Rivera [**18**] and Yang-Zhang [**25**] obtained the existence of positive solutions via variational methods, while Mao-Zhang [**19**] and Zhang-Perera [**26**] obtained sign changing solutions via invariant sets of descent flow. Recently, many authors, such as Chen [**7**], Jin-Wu [**12**], Li-Wu [**15**], Li et al. [**16**], Nie-Wu [**20**], and Wu [**24**], have been more interested in Kirchhoff type problems in the unbounded domain or in $\mathbb{R}^N$. The concentration behavior of positive solutions has been studied by He-Zou [**9**], He et al. [**11**] and Wang et al. [**21**]. A result with Hartree-type nonlinearities may be found in Lü [**17**]. Ground state solutions for Kirchhoff type problems with critical nonlinearities is considered in He-Zou [**10**].

We now consider non-autonomous Kirchhoff type problems (SK) and obtain a representation of Palais-Smale sequences for this problem. Similar results may be found in [**3, 23**]. In this paper, we make the following assumption on $a(x)$:

$$(\mathrm{a}_1) \qquad \lim_{|x|\to+\infty} a(x) = a_\infty > 0, \qquad \alpha(x) := a(x) - a_\infty \in L^{6/(5-p)}(\mathbb{R}^3),$$

and consider the functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1} \tag{1.1}$$

on $H^1(\mathbb{R}^3)$. This functional is well defined under assumption (a_1). Since a symmetry assumption is not available, we carefully investigate the behavior of Palais-Smale sequences and obtain a representation of Palais-Smale sequences for the functional I. However, this is quite different from the work of Benci and Cerami [**3**]. Direct calculation for the nonlocal term

$$\int_{\mathbb{R}^3}|\nabla u|^2$$

does not show that I' is weakly, sequentially continuous in $H^1(\mathbb{R}^3)$. For any Palais-Smale sequences $\{u_n\}$ with $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, we do not know whether

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 \longrightarrow \int_{\mathbb{R}^3} |\nabla u|^2$$

holds. In fact, this is the case if $u \neq 0$ and $3 < p < 5$ in equation (SK), see Lemma 3.1 herein and [**14,** Lemma 3.2], where this idea was introduced by Li and Ye. In this case, the representation of Palais-Smale sequences is quite normal and is similar to that of the Schrödinger-Poisson systems obtained by Cerami and Vaira [**6**]. However, when $u = 0$, the situation is quite different and

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 \longrightarrow 0$$

may be invalid. A new representation for the Palais-Smale sequences must be described in another form, which is presented in Lemma 3.2.

Palais-Smale sequences for the functional I are closely related to the solution of the problem at infinity:

$$(NS_\infty) \qquad -\Delta u + u = a_\infty |u|^{p-1} u \quad \text{in } \mathbb{R}^3.$$

Thus, we denote the unique radial solution of equation (NS_∞) by w and set

$$m_\infty := \left(\frac{1}{2} - \frac{1}{p+1}\right)\|w\|^2.$$

Additionally, if we make the assumption:

$$(\mathrm{a}_2) \qquad a(x) \geq a_\infty \quad \text{for any } x \in \mathbb{R}^3,$$

and $a(x) > a_\infty$ holds on a positive measure set, then the next problem:

$$(\mathrm{NS}) \qquad -\Delta u + u = a(x)|u|^{p-1} u \quad \text{in } \mathbb{R}^3,$$

admits a ground state solution. The ground state solution of (NS) with assumption (a_2) is denoted by w_a and

$$m_a := \left(\frac{1}{2} - \frac{1}{p+1}\right)\|w_a\|^2 < m_\infty.$$

Now, we state the result on the existence of ground state solutions for problem (SK).

Theorem 1.1. *Let* (a_1) *and* (a_2) *hold, and assume that*

$$0 < b \leq \frac{{m_\infty}^{\mu} - m_a^{\mu}}{\nu\, m_a^{1+\mu}}, \tag{1.2}$$

where $\mu = (p-3)/(p+1)$ *and* $\nu = 2(p+1)/(p-1)$. *Then, problem* (SK) *has a positive ground state solution.*

We remark that a similar result has been obtained by Cerami and Vaira [**6**] for some non-automous Schrödinger-Poisson systems.

This paper is organized as follows. In Section 2, we will introduce the variational setting and some basic lemmas. In Section 3, we obtain a representation of Palais-Smale sequences for the functional I and establish the compactness conditions. Finally, the proof of Theorem 1.1 is presented in Section 4.

2. Notation and variational setting. In this section, we give the variational setting for problem (SK) and use the following notation: $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard scalar product and norm

$$(u,v) = \int_{\mathbb{R}^3} \big(\nabla u \nabla v + uv\big), \quad \|u\| = (u,u)^{1/2}.$$

The dual space of $H^1(\mathbb{R}^3)$ is denoted by H^{-1}. A Lebesgue space is denoted by $L^q(\mathbb{R}^3)$, $1 \leq q \leq +\infty$, and the norm in $L^q(\mathbb{R}^3)$ is denoted by $|u|_q$. C and C_i are various positive constants. Moreover, in what follows, we always assume that $a_\infty = 1$, without any loss of generality.

To prove the existence of a ground state solution, we set

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : G(u) = 0\},$$

where

$$G(u) = \langle I'(u), u\rangle = \|u\|^2 + b\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\bigg)^2 - \int_{\mathbb{R}^3} a(x)|u|^{p+1}.$$

We remark that

$$I|_{\mathcal{N}}(u) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \tag{2.1}$$
$$= \frac{1}{4}\|u\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} a(x)|u|^{p+1}.$$

The next lemma contains some properties of $\mathcal{N}$.

Lemma 2.1. *The following statements hold*:

(i) $\mathcal{N}$ *is a* C^1 *regular manifold diffeomorphic to the sphere of* $H^1(\mathbb{R}^3)$;
(ii) I *is bounded from below on* $\mathcal{N}$ *by a positive constant*;
(iii) u *is a critical point of* I *if and only if* u *is a critical point of* I *constrained on* $\mathcal{N}$.

Proof.

(i) Let $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ be such that $\|u\| = 1$. Then there exists a unique $t \in \mathbb{R}^+\backslash\{0\}$ for which $tu \in \mathcal{N}$. Indeed, considering that t must satisfy $t > 0$ such that

$$0 = \langle I'(tu), tu\rangle = t^2\|u\|^2 + bt^4\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - t^{\,p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1},$$

set

$$m := b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2, \qquad n := \int_{\mathbb{R}^3} a(x)|u|^{p+1}.$$

We must find a positive solution of $t^2(1+mt^2 - nt^{p-1}) = 0$ with $m > 0$ and $n > 0$. In fact, since $p > 3$, the equation

$$1 + mt^2 - nt^{p-1} = 0$$

has a unique solution $t = t(u) > 0$, and the corresponding point $t(u)u \in \mathcal{N}$, which is called the *projection* of u on $\mathcal{N}$, is such that

$$I(t(u)u) = \max_{t>0} I(tu).$$

Let $u \in \mathcal{N}$. Then,

$$0 = \|u\|^2 + b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \int_{\mathbb{R}^3} a(x)|u|^{p+1} \geq \|u\|^2 - C_0\|u\|^{p+1},$$

which implies that

$$\|u\| \geq C_1 > 0. \tag{2.2}$$

As is well-known, $I \in C^2\big(H^1(\mathbb{R}^3), \mathbb{R}\big)$ and

$$\begin{aligned}\langle I''(u)v, w\rangle = &\int_{\mathbb{R}^3} (\nabla v \nabla w + vw) + 2b \int_{\mathbb{R}^3} \nabla u \nabla w \int_{\mathbb{R}^3} \nabla u \nabla v \\ &+ b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla v \nabla w - p \int_{\mathbb{R}^3} a(x)|u|^{p-1} vw.\end{aligned}$$

Therefore, G becomes a C^1 functional, and we have

$$\begin{aligned}\langle G'(u), u\rangle &= \langle I''(u)u, u\rangle \\ &= 2\|u\|^2 + 4b\Big(\int_{\mathbb{R}^3} |\nabla u|^2\Big)^2 - (p+1)\int_{\mathbb{R}^3} a(x)|u|^{p+1} \\ &= (1-p)\|u\|^2 + (3-p)b\Big(\int_{\mathbb{R}^3} |\nabla u|^2\Big)^2 \\ &\leq (1-p)\|u\|^2 \leq (1-p)C_1^2 < 0.\end{aligned} \tag{2.3}$$

(ii) Let $u \in \mathcal{N}$. Using equations (2.1) and (2.2), we obtain

$$\begin{aligned}I(u) &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u\|^2 + \Big(\frac{1}{4} - \frac{1}{p+1}\Big) b\Big(\int_{\mathbb{R}^3} |\nabla u|^2\Big)^2 \\ &\geq \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u\|^2 > C_2 > 0.\end{aligned}$$

(iii) On one hand, if $u \not\equiv 0$ is a critical point of I, $I'(u) = 0$, then $u \in \mathcal{N}$. On the other hand, let u be a critical point of I constrained on $\mathcal{N}$. Then there exists $\lambda \in \mathbb{R}$ such that $I'(u) = \lambda G'(u)$. Hence, from

$$0 = G(u) = \langle I'(u), u\rangle = \lambda G'(u)$$

and equation (2.3), $\lambda = 0$ is implied and then $I'(u) = 0$ follows. $\square$

Setting

$$m := \inf\{I(u) : u \in \mathcal{N}\},$$

as a consequence of Lemma 2.1 (ii), m turns out to be a positive number.

We consider problem (NS), i.e.,

$$-\Delta u + u = a(x)|u|^{p-1}u, \tag{NS}$$

setting

$$\begin{aligned} I_a(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1} \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} (1+\alpha(x))|u|^{p+1}, \end{aligned}$$

where $\alpha(x) := a(x) - a_\infty = a(x) - 1$, and

$$\begin{aligned} \mathcal{N}_a &:= \Big\{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \|u\|^2 = \int_{\mathbb{R}^3} a(x)|u|^{p+1}\Big\} \\ &= \Big\{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \|u\|^2 = \int_{\mathbb{R}^3} \big(1+\alpha(x)\big)|u|^{p+1}\Big\}. \end{aligned}$$

When $a(x) = a_\infty = 1$, equation (NS) becomes

$$-\Delta u + u = |u|^{p-1}u. \tag{NS_∞}$$

In this case, we use the notation I_∞ and $\mathcal{N}_\infty$, respectively, for the functional and natural constraints, namely,

$$I_\infty(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}$$

and

$$\mathcal{N}_\infty := \Big\{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \|u\|^2 = \int_{\mathbb{R}^3} |u|^{p+1}\Big\}.$$

The following well-known propositions, which provide results regarding the existence of positive solutions of (NS_∞) and (NS), are useful in obtaining our theorem.

Proposition 2.2. *Problem* (NS_∞) *has a positive, ground state solution* $w \in H^1(\mathbb{R}^3)$*, radially symmetric about the origin, unique up to translations, decaying exponentially, together with its derivatives, as* $|x| \to +\infty$.

Proposition 2.3. *Let* (a_2) *hold. Then, equation* (NS) *has a positive, ground state solution* $w_a \in H^1(\mathbb{R}^3)$.

Proposition 2.2 can be found in [**4, 8, 13**]. Proposition 2.3 can be proved using a minimization argument and the concentration-compactness principle.

Since w and w_a are ground state solutions, setting

$$m_\infty =: \inf\{I_\infty(u), u \in \mathcal{N}_\infty\},$$
$$m_a =: \inf\{I_a(u), u \in \mathcal{N}_a\},$$

we obtain $I_\infty(u) \geq I_\infty(w) = m_\infty$ for all u solutions of equation (NS_∞) and also $I_a(u_a) \geq I_a(w_a) = m_a$ for all u_a solutions of equation (NS). Under condition (a_2), we also have that $m_\infty > m_a$, where

$$m_\infty = I_\infty(w) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|w\|^2$$

and

$$m_a = I_a(w_a) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|w_a\|^2.$$

Similar to Lemma 2.1 (i), it is possible to show that, for any function $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, there exists a unique function $\tau u \in \mathcal{N}_a$ such that

$$I_a(\tau u) = \max_{t>0} I_a(tu).$$

Lemma 2.4. *Let $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ and tu, τu, $(t, \tau > 0)$ are its projections on $\mathcal{N}$, $\mathcal{N}_a$, respectively. Then,*

$$\tau \leq t. \tag{2.4}$$

Proof. Let $u \in H^1(\mathbb{R}^3)\backslash\{0\}$. Since $\tau u \in \mathcal{N}_a$ and $tu \in \mathcal{N}$, we have

$$\tau^2\|u\|^2 = \tau^{p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1},$$
$$t^2\|u\|^2 = -t^4 b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + t^{p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1}.$$

It follows from (a_1) that

$$\tau^{p-1} = \frac{\|u\|^2}{\int\limits_{\mathbb{R}^3} a(x)|u|^{p+1}}$$
$$= \frac{-t^2 b\left(\int\limits_{\mathbb{R}^3} |\nabla u|^2\right)^2 + t^{p-1}\int\limits_{\mathbb{R}^3} a(x)|u|^{p+1}}{\int\limits_{\mathbb{R}^3} a(x)|u|^{p+1}} \leq t^{p-1},$$

which implies that $\tau \leq t$. □

3. Representation of Palais-Smale sequences. In this section, we investigate the behavior of Palais-Smale sequences of I.

Lemma 3.1. *Let $\{u_n\}$ be a $(PS)_c$ sequence of I. Then there exists $u \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$. Moreover, if $u \neq 0$ where $c \leq m$, then*

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 \longrightarrow \int_{\mathbb{R}^3} |\nabla u|^2.$$

Proof. Let $u_n \in H^1(\mathbb{R}^3)$ be a $(PS)_c$ sequence of I, that is,

$$I(u_n) \longrightarrow c \quad \text{and} \quad I'(u_n) \longrightarrow 0 \quad \text{in } H^{-1}.$$

Thus, for n large enough,

$$\begin{aligned} c + o(1)\|u_n\| &\geq I(u_n) - \frac{1}{p+1}\langle I'(u_n), u_n\rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_n\|^2 \\ &\quad + \left(\frac{1}{4} - \frac{1}{p+1}\right) b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 \\ &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_n\|^2. \end{aligned}$$

Then, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$; hence, passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$.

Without loss of generality, we assume that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 \longrightarrow A^2,$$

for some $A \in \mathbb{R}$. If $u \neq 0$, we see that

$$\int_{\mathbb{R}^3} |\nabla u|^2 \leq A^2.$$

Suppose that

$$\int_{\mathbb{R}^3} |\nabla u|^2 < A^2.$$

By $I'(u_n) \to 0$, we have

$$\int_{\mathbb{R}^3} \nabla u \nabla \varphi + u\varphi + bA^2 \int_{\mathbb{R}^3} \nabla u \nabla \varphi - \int_{\mathbb{R}^3} a(x)|u|^{p-1} u\varphi = 0,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^3)$. Take $\varphi = u$. Then $\langle I'(u), u\rangle < 0$. The term $p > 3$ implies that $\langle I'(tu), tu\rangle > 0$ for small $t > 0$. Hence, there exists a $t_0 \in (0,1)$ satisfying $\langle I'(t_0 u), t_0 u\rangle = 0$. We can see that $I(t_0 u) = \max_{t\in[0,1]} I(tu)$. Therefore,

$$\begin{aligned}
c &\leq I(t_0 u) - \frac{1}{4}\langle I'(t_0 u), t_0 u\rangle \\
&= \frac{t_0^2}{4} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \left(\frac{1}{4} - \frac{1}{p+1}\right) t_0^{p+1} \int_{\mathbb{R}^3} a(x)|u|^{p+1} \\
&< \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \left(\frac{1}{4} - \frac{1}{p+1}\right) \int_{\mathbb{R}^3} a(x)|u|^{p+1} \\
&\leq \liminf_{n\to\infty} \left[\frac{1}{4} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2) + \left(\frac{1}{4} - \frac{1}{p+1}\right) \int_{\mathbb{R}^3} a(x)|u_n|^{p+1}\right] \\
&= \liminf_{n\to\infty} \left(I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle\right) = c,
\end{aligned}$$

which is impossible. Therefore,

$$\int_{\mathbb{R}^3} |\nabla u|^2 = A^2,$$

and the proof is complete. □

Lemma 3.2. *Let $\{u_n\}$ be a $(PS)_c$ sequence of I constrained on $\mathcal{N}$, i.e., $u_n \in \mathcal{N}$, and*

$$I(u_n) \longrightarrow c, \qquad I'|_{\mathcal{N}}(u_n) \longrightarrow 0 \quad \text{strongly in } H^{-1}.$$

Then, passing to a subsequence, one of the following two cases hold.

Case 1. *There exists a solution $\overline{u} \neq 0$ of problem (SK), a number $k \in \mathbb{N} \cup \{0\}$, k function u^1, u^2, …, u^k of $H^1(\mathbb{R}^3)$ and k sequences of points $\{y_n^j\} \subset \mathbb{R}^3$, $1 \leq j \leq k$, such that*

(a) $|y_n^j| \to \infty$, $|y_n^j - y_n^i| \to \infty$ *if* $i \neq j$, $n \to +\infty$;

(b) $u_n - \displaystyle\sum_{j=1}^{k} u^j(\cdot - y_n^j) \to \overline{u}$ *in* $H^1(\mathbb{R}^3)$;

(c) $I(u_n) \to I(\overline{u}) + \displaystyle\sum_{j=1}^{k} I_\infty(u^j)$;

(d) u^j *are nontrivial weak solutions of equation* (NS_∞).

Case 2. *There exist $\overline{u}^0 \in H^1(\mathbb{R}^3)$, $\{\overline{y}_n^0\} \subset \mathbb{R}^3$, a number $k \in \mathbb{N} \cup \{0\}$, k function u^1, u^2, …, u^k of $H^1(\mathbb{R}^3)$ and k sequences of points $\{y_n^j\} \subset \mathbb{R}^3$, $1 \leq j \leq k$, such that*

(a) $|\overline{y}_n^0| \to \infty$, $|y_n^j| \to \infty$, $|\overline{y}_n^0 - y_n^j| \longrightarrow \infty$ *and* $|y_n^j - y_n^i| \to \infty$ *if* $i \neq j$, $n \to +\infty$;

(b) $u_n - \displaystyle\sum_{j=1}^{k} u^j(\cdot - y_n^j) - \overline{u}^0(\cdot - \overline{y}_n^0) \to 0$ *in* $H^1(\mathbb{R}^3)$;

(c) $I(u_n) \to J(\overline{u}^0) + \displaystyle\sum_{j=1}^{k} I_\infty(u^j)$, *where*

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\bigg)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1};$$

(d) u^j *are nontrivial weak solutions of equation* (NS_∞), $\overline{u}^0$ *is a nontrivial weak solution of the next problem*:

$$(SK_\infty) \qquad -\Big(1 + b\int_{\mathbb{R}^3} |\nabla u|^2\Big)\Delta u + u = |u|^{p-1}u \quad \text{in } \mathbb{R}^3.$$

Moreover, in both cases we admit $k = 0$, and the cases hold without u^j.

Proof. By equation (2.1), we obtain

$$I(u_n) \geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_n\|^2;$$

hence, $\{u_n\}$ is bounded, and we can prove that

$$I'(u_n) \longrightarrow 0 \quad \text{in } H^1(\mathbb{R}^3). \tag{3.1}$$

In fact, we have

$$o(1) = I'|_{\mathcal{N}}(u_n) = I'(u_n) - \lambda_n G'(u_n), \tag{3.2}$$

for some $\lambda_n \in \mathbb{R}$. Thus, taking the scalar product with u_n, we obtain

$$o(1) = \langle I'|_{\mathcal{N}}(u_n), u_n\rangle = \langle I'(u_n), u_n\rangle - \lambda_n \langle G'(u_n), u_n\rangle. \tag{3.3}$$

Since $u_n \in \mathcal{N}$, i.e., $\langle I'(u_n), u_n\rangle = 0$, and, by equation (2.3),

$$\langle G'(u_n), u_n\rangle \leq -\alpha < 0,$$

for some constant $\alpha > 0$. Thus, it follows from equation (3.3) that $\lambda_n \to 0$ for $n \to +\infty$. Moreover, by the boundedness of $\{u_n\}$, $G'(u_n)$ is bounded, and this implies that $\lambda_n G'(u_n) \to 0$, so equation (3.1) follows from equation (3.2).

Since u_n is bounded in $H^1(\mathbb{R}^3)$, there exists $\overline{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence $u_n \rightharpoonup \overline{u}$ in $H^1(\mathbb{R}^3)$ and in $L^{p+1}(\mathbb{R}^3)$, $u_n(x) \to \overline{u}(x)$ almost everywhere on $\mathbb{R}^3$. It follows from Lemma 3.1 that $I'(\overline{u}) = 0$, and hence, $\overline{u}$ is a weak solution of equation (SK).

Case 1. $\overline{u} \neq 0$. If $u_n \to \overline{u}$ in $H^1(\mathbb{R}^3)$, then the proof is complete. Thus, we can assume that u_n does not converge strongly to $\overline{u}$ in $H^1(\mathbb{R}^3)$. We set

$$z_n^1(x) = u_n(x) - \overline{u}(x).$$

Clearly, $z_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, but not strongly. We claim that

$$(*) \qquad I(u_n) = I(\overline{u}) + I_\infty(z_n^1) + o(1),$$
$$\langle I'_\infty(z_n^1), z_n^1\rangle = o(1) \quad \text{and} \quad I'_\infty(z_n^1) = o(1).$$

In fact, we have

$$\|u_n\|^2 = \|z_n^1 + \overline{u}\,\|^2 = \|z_n^1\|^2 + \|\,\overline{u}\,\|^2 + o(1). \tag{3.4}$$

It follows from [**5**] that

$$|u_n|_{p+1}^{p+1} = |\,\overline{u}\,|_{p+1}^{p+1} + |z_n^1|_{p+1}^{p+1} + o\,(1), \tag{3.5}$$

and, using [**2,** Lemma A.2] and [**22,** Lemma 8.1], respectively, we obtain

$$\alpha(x)|z_n^1|^{p-1}z_n^1 \longrightarrow 0 \quad \text{in } H^{-1}, \tag{3.6}$$

$$|u_n|^{p-1}u_n = |\overline{u}|^{p-1}\overline{u} + |z_n^1|^{p-1}z_n^1 + o\,(1) \quad \text{in } H^{-1}. \tag{3.7}$$

It follows from equations (3.4), (3.5), (3.6), (3.7) and Lemma 3.1 that
(3.8)

$$\begin{aligned} I(u_n) &= \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla u_n|^2\bigg)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|u_n|^{p+1} \\ &= \frac{1}{2}\|z_n^1\|^2 + \frac{1}{2}\|\,\overline{u}\,\|^2 + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla\overline{u}\,|^2\bigg)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|\,\overline{u}\,|^{p+1} \\ &\quad - \frac{1}{p+1}\int_{\mathbb{R}^3} |z_n^1|^{p+1} + o\,(1) \\ &= I(\overline{u}) + I_\infty(z_n^1) + o\,(1). \end{aligned}$$

For all $h \in H^1(\mathbb{R}^3)$, we have

$$\begin{aligned} o\,(1)\|h\| = \langle I'(u_n), h\rangle &= \langle u_n, h\rangle + b\int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} \nabla u_n \nabla h \\ &\quad - \int_{\mathbb{R}^3} a(x)|u_n|^{p-1}u_n h \\ &= \langle\,\overline{u}, h\rangle + b\int_{\mathbb{R}^3} |\nabla\overline{u}\,|^2 \int_{\mathbb{R}^3} \nabla\overline{u}\nabla h \\ &\quad - \int_{\mathbb{R}^3} a(x)|\overline{u}|^{p-1}uh \\ &\quad + \langle z_n^1, h\rangle - \int_{\mathbb{R}^3} |z_n^1|^{p-1}z_n^1 h + o\,(1)\|h\| \\ &= \langle I'(\overline{u}), h\rangle + \langle I'_\infty(z_n^1), h\rangle + o\,(1)\|h\|, \end{aligned}$$

so

$$I'_\infty(z_n^1) = o\,(1) \quad \text{in } H^{-1}. \tag{3.9}$$

Moreover, by $I'(\overline{u})=0$, we obtain

$$0=\langle I'(u_n),u_n\rangle = \langle I'(\overline{u}),\overline{u}\,\rangle+\langle I'_\infty(z_n^1),z_n^1\rangle+o\,(1) = \langle I'_\infty(z_n^1),z_n^1\rangle+o\,(1). \tag{3.10}$$

Now, claim $(*)$ holds.

Letting

$$\delta:=\limsup_{n\to+\infty}\left(\sup\int_{y\in\mathbb{R}^3}|z_n^1|^{p+1}\right),$$

we have $\delta>0$. In fact, if $\delta=0$ holds, then by **[22,** Lemma 1.21], $z_n^1\to 0$ in $L^{p+1}(\mathbb{R}^3)$ would hold, contradicting the fact that u_n does not converge strongly to $\overline{u}$ in $L^{p+1}(\mathbb{R}^3)$. Then, we may assume the existence of $\{y_n^1\}\subset\mathbb{R}^3$, such that

$$\int_{B_1(y_n^1)}|z_n^1|^{p+1}>\frac{\delta}{2}.$$

Then, we consider $z_n^1(\cdot+y_n^1)$, and, letting $z_n^1(\cdot+y_n^1)\rightharpoonup u^1$ in $H^1(\mathbb{R}^3)$, $z_n^1(x+y_n^1)\to u^1(x)$ almost everywhere on $\mathbb{R}^3$. Since

$$\int_{B_1(0)}|z_n^1(x+y_n^1)|^{p+1}>\frac{\delta}{2},$$

the Rellich theorem implies

$$\int_{B_1(0)}|u^1(x)|^{p+1}>\frac{\delta}{2}.$$

Thus, $u^1\neq 0$. It follows from $z_n^1\rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ that $\{y_n^1\}$ must be unbounded and, up to a subsequence, we can assume that $|y_n^1|\to+\infty$. Moreover, by equation (3.9), we have $I'_\infty(u^1)=0$. Finally, let

$$z_n^2(x)=z_n^1(x)-u^1(x-y_n^1).$$

Then, by equations (3.4), (3.5) and the Brezis-Lieb lemma, we obtain

$$\|z_n^2\|^2=\|u_n\|^2-\|\overline{u}\|^2-\|u^1\|^2+o\,(1),$$
$$|z_n^2|_{p+1}^{p+1}=|u_n|_{p+1}^{p+1}-|\overline{u}|_{p+1}^{p+1}-|u^1|_{p+1}^{p+1}+o\,(1),$$

which implies

$$I_\infty(z_n^2)=I_\infty(z_n^1)-I_\infty(u^1)+o\,(1).$$

Hence, by claim $(*)$, we obtain

$$I(u_n) = I(\overline{u}) + I_\infty(z_n^1) + o\,(1) = I(\overline{u}) + I_\infty(u^1) + I_\infty(z_n^2) + o\,(1).$$

It is easy to prove that

$$I'_\infty(z_n^2) = o\,(1) \quad \text{in } H^{-1}.$$

Now, if $z_n^2 \to 0$ in $H^1(\mathbb{R}^3)$, we are done. Otherwise, $z_n^2 \rightharpoonup 0$ not strongly, so we repeat the argument. By iterating this procedure, we obtain sequences of points $y_n^j \in \mathbb{R}^3$ such that

$$|y_n^j| \longrightarrow +\infty, \qquad |y_n^j - y_n^i| \longrightarrow +\infty,$$

if $i \neq j$ and a sequence of

$$z_n^j(x) = z_n^{j-1}(x) - u^{j-1}(x - y_n^{j-1})$$

with $j \geq 2$ such that

$$z_n^j(x + y_n^j) \rightharpoonup u^j(x) \quad \text{in } H^1(\mathbb{R}^3), \qquad I'(u^j) = 0.$$

Thus,

$$I(u_n) = I(\overline{u}) + \sum_{j=1}^{k} I_\infty(u^j) + o\,(1).$$

Since $I_\infty(u^j) \geq m_\infty$ for all j and $I(u_n)$ is bounded, the iteration must stop at some finite index k.

Case 2. $\overline{u} = 0$, i.e., $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ not strongly ($\{u_n\} \subset \mathcal{N}$ and $I(u_n) \to c$, so $c > 0$). We claim that

$$(**) \qquad \begin{aligned} I(u_n) &= J(u_n) + o\,(1), \\ \langle J'(u_n), u_n \rangle &= o\,(1) \end{aligned}$$

and

$$J'(u_n) = o\,(1) \quad \text{in } H^{-1}.$$

In fact, it follows from equation (3.6) that

(3.11)
$$\begin{aligned} I(u_n) &= \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|u_n|^{p+1} \\ &= \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1} + o\,(1) \\ &= J(u_n) + o\,(1). \end{aligned}$$

For all $h \in H^1(\mathbb{R}^3)$, we have

$$\begin{aligned} o\,(1)\|h\| = \langle I'(u_n), h\rangle &= \langle u_n, h\rangle + b\int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} \nabla u_n \nabla h \\ &\quad - \int_{\mathbb{R}^3} a(x)|u_n|^{p-1}u_n h \\ &= \langle u_n, h\rangle + b\int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} \nabla u_n \nabla h \\ &\quad - \int_{\mathbb{R}^3} |u_n|^{p-1}u_n h + o\,(1)\|h\| \\ &= \langle J'(u_n), h\rangle + o\,(1)\|h\|, \end{aligned}$$

which implies

$$J'(u_n) = o\,(1) \quad \text{in } H^{-1}. \tag{3.12}$$

Moreover,

$$0 = \langle I'(u_n), u_n\rangle = \langle J'(u_n), u_n\rangle + o\,(1). \tag{3.13}$$

Then, claim $(**)$ holds.

Letting

$$\delta := \liminf_{n\to+\infty}\left(\sup_{y\in\mathbb{R}^3}\int_{B_1(y)} |u_n|^{p+1}\right),$$

we have $\delta > 0$. In fact, if $\delta = 0$ holds, then, by **[22,** Lemma 1.21], $u_n \to 0$ in $L^{p+1}(\mathbb{R}^3)$ would hold, contradicting the fact that u_n does not converge strongly to 0 in $L^{p+1}(\mathbb{R}^3)$. We may assume the existence of $\{\overline{y}_n^0\} \subset \mathbb{R}^3$, such that

$$\int_{B_1(\overline{y}_n^{\,0})} |u_n|^{p+1} > \frac{\delta}{2}.$$

Next, we consider $u_n(\cdot+\overline{y}_n^0)$, and let $u_n(\cdot+\overline{y}_n^0) \rightharpoonup \overline{u}^0$ in $H^1(\mathbb{R}^3)$. Then

$$u_n(x+\overline{y}\,_n^0) \longrightarrow \overline{u}\,^0(x),$$

almost everywhere on $\mathbb{R}^3$. Since

$$\int_{B_1(0)} |u_n(x+\overline{y}\,_n^0)|^{p+1} > \frac{\delta}{2},$$

the Rellich theorem implies

$$\int_{B_1(0)} |\,\overline{u}\,^0(x)|^{p+1} > \frac{\delta}{2}.$$

Thus, $\overline{u}\,^0 \neq 0$. It follows from $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ that $\{\overline{y}\,_n^0\}$ must be unbounded and, up to a subsequence, we can assume that $|\overline{y}\,_n^0| \to +\infty$.

Moreover, similar to Lemma 3.1, by $u_n(\cdot+\overline{y}\,_n^0) \rightharpoonup \overline{u}\,^0 \neq 0$ in $H^1(\mathbb{R}^3)$ and equation (3.12), we can obtain that $J'(\overline{u}\,^0)=0$ and

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 = \int_{\mathbb{R}^3} |\nabla u_n(\cdot+\overline{y}\,_n^0)|^2 \longrightarrow \int_{\mathbb{R}^3} |\nabla\,\overline{u}\,^0|^2.$$

Finally, let

$$z_n^1(x) = u_n(x) - \overline{u}\,^0(x-\overline{y}\,_n^0).$$

Then, by the Brezis-Lieb lemma, we obtain

$$\begin{aligned}\|z_n^1\| &= \|u_n\|^2 - \|\,\overline{u}\,^0\|^2 + o\,(1),\\ |z_n^1|_{p+1}^{p+1} &= |u_n|_{p+1}^{p+1} - |\,\overline{u}\,^0|_{p+1}^{p+1} + o\,(1),\end{aligned}$$

which implies that

$$\begin{aligned}J(u_n) &= \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla u_n|^2\bigg)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1}\\ &= \frac{1}{2}\|z_n^1\|^2 + \frac{1}{2}\|\,\overline{u}\,^0\|^2 + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla\overline{u}\,^0|^2\bigg)^2\\ &\quad - \frac{1}{p+1}\int_{\mathbb{R}^3} |z_n^1|^{p+1} - \frac{1}{p+1}\int_{\mathbb{R}^3} |\,\overline{u}\,^0|^{p+1}\\ &= J(\overline{u}\,^0) + I_\infty(z_n^1) + o\,(1).\end{aligned}$$

For all $h \in H^1(\mathbb{R}^3)$, we have

$$o\,(1)\|h\| = \langle J'(u_n(\cdot)), h(\cdot)\rangle = \langle u_n(\cdot), h(\cdot)\rangle$$

$$
\begin{aligned}
&\quad + b\int_{\mathbb{R}^3}|\nabla u_n(\cdot)|^2\int_{\mathbb{R}^3}\nabla u_n(\cdot)\nabla h(\cdot) \\
&\quad - \int_{\mathbb{R}^3}|u_n(\cdot)|^{p-1}u_n(\cdot)h(\cdot) \\
&= \langle \overline{u}^{\,0}(\cdot-\overline{y}^{\,0}_{n}), h(\cdot)\rangle \\
&\quad + b\int_{\mathbb{R}^3}|\nabla\overline{u}^{\,0}(\cdot-\overline{y}^{\,0}_{n})|^2\int_{\mathbb{R}^3}\nabla\overline{u}^{\,0}(\cdot-\overline{y}^{\,0}_{n})\nabla h(\cdot) \\
&\quad - \int_{\mathbb{R}^3}|\,\overline{u}^{\,0}(\cdot-\overline{y}^{\,0}_{n})|^{p-1}u(\cdot-\overline{y}^{\,0}_{n})h(\cdot) \\
&\quad + \langle z_n^1(\cdot), h\rangle - \int_{\mathbb{R}^3}|z_n^1(\cdot)|^{p-1}z_n^1(\cdot)h(\cdot) + o\,(1)\|h\| \\
&= \langle J'(\overline{u}^{\,0}(\cdot-\overline{y}^{\,0}_{n})), h(\cdot)\rangle + \langle I'_\infty(z_n^1), h\rangle + o\,(1)\|h\|,
\end{aligned}
$$

from which it follows that

$$I'_\infty(z_n^1) = o\,(1) \quad \text{in } H^{-1}.$$

Moreover, by $J'(\overline{u}^{\,0}) = 0$, we obtain

$$0 = \langle J'(u_n), u_n\rangle = \langle I'_\infty(z_n^1), z_n^1\rangle + o\,(1),$$

which yields

$$\langle I'_\infty(z_n^1), z_n^1\rangle = o\,(1).$$

Now, if $z_n^1 \to 0$ in $H^1(\mathbb{R}^3)$, then we are done. Otherwise, $z_n^1 \rightharpoonup 0$, not strongly, and we repeat the argument. By iterating this procedure in a manner similar to that of Case 1, we obtain sequences of points $y_n^j \in \mathbb{R}^3$ such that

$$|y^{\,j}_{n}| \longrightarrow +\infty, \qquad |y^{\,j}_{n} - y_n^i| \longrightarrow +\infty$$

if $i \neq j$ and a sequence of

$$z_n^j(x) = z_n^{j-1}(x) - u^{j-1}(x - y^{\,j-1}_{n})$$

with $j \geq 2$ such that

$$z_n^j(x + y^{\,j}_{n}) \rightharpoonup u^j(x) \quad \text{in } H^1(\mathbb{R}^3),$$

$I'_\infty(u^j) = 0$. Thus,

$$I(u_n) = J(\overline{u}^0) + \sum_{j=1}^{k} I_\infty(u^j) + o\,(1).$$

Since $I_\infty(u^j) \geq m_\infty$ for all j and $I(u_n)$ is bounded, the iteration must stop at some finite index k. $\square$

Remark 3.3. Problem (SK_∞) has nontrivial solutions for $3 < p < 5$, see **[9]**. To our knowledge, if $\overline{u}^0$ is a nontrivial solution of this problem, $J(\overline{u}^0) \geq m_{SK_\infty}$, where m_{SK_∞} is the ground state solution energy. We obtain

$$\begin{aligned} J(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2 \\ - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1} > \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1} \\ = I_\infty(u), \end{aligned}$$

which implies that $m_{SK_\infty} > m_\infty$.

Remark 3.4. We have given a nontrivial solution $\overline{u}$ of problem (SK) in Case 1. If Case 2 occurs, the smallest level c must be

$$c = m_{SK_\infty};$$

Case 2 holds for $k = 0$. Alternatively, if u $(u = u^+ - u^-)$ is a solution of (SK_∞), then any of u^+ or u^- are not. Because of this, and the fact that we know little about the bounded state solution of this problem, the next level c which satisfies Case 2 is unknown.

Corollary 3.5. *Assume that $\{u_n\}$ is a $(PS)_c$ sequence. Then $\{u_n\}$ is relatively compact for all $c \in (0, m_\infty]$.*

Proof. Let us consider a $(PS)_c$ sequence $\{u_n\}$ and apply Lemma 3.2 to it, assuming that $I_\infty(u^j) \geq m_\infty$ for all j and $I(\overline{u}) > 0$. Thus,

$$\lim_{n\to+\infty} I(u_n) = c \leq m_\infty.$$

Case 1 (c) gives $k = 0$, and Case 2 cannot hold for $m_\infty < m_{SK_\infty}$. Therefore, $u_n \to \overline{u}$ in $H^1(\mathbb{R}^3)$ follows. $\square$

4. The proof of Theorem 1.1. Now, we are in a position to show our main result.

Proof of Theorem 1.1. To prove the existence of a ground state solution of (SK), we need to make sure that

$$m \leq m_\infty. \tag{4.1}$$

If equation (4.1) holds, then by Corollary 3.5, it is easy to see that m is achieved by a function u (passing to $|u|$) that is positive and solves (SK). To obtain equation (4.1), we test I with the projection on $\mathcal{N}$, tw_a of the minimizer w_a of I_a on $\mathcal{N}_a$. By virtue of Lemma 2.4, we get $t \geq 1$, which, together with equation (2.1), implies that

$$\begin{aligned} m \leq I(tw_a) &= \frac{1}{2}\|tw_a\|^2 \\ &\quad + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla tw_a|^2\bigg)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|tw_a|^{p+1} \\ &= \frac{1}{4}\|tw_a\|^2 + \bigg(\frac{1}{4} - \frac{1}{p+1}\bigg)\int_{\mathbb{R}^3} a(x)|tw_a|^{p+1} \\ &\leq t^{p+1}\bigg[\frac{1}{4}\|w_a\|^2 + \bigg(\frac{1}{4} - \frac{1}{p+1}\bigg)\int_{\mathbb{R}^3} a(x)|w_a|^{p+1}\bigg] \\ &= t^{p+1}\bigg(\frac{1}{2}\|w_a\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|w_a|^{p+1}\bigg) \\ &= t^{p+1}\bigg(\frac{1}{2} - \frac{1}{p+1}\bigg)\|w_a\|^2 = t^{p+1}m_a. \end{aligned} \tag{4.2}$$

For $w_a \in \mathcal{N}_a$ and $t \geq 1$, we obtain

$$\begin{aligned} 0 &= t^2\|w_a\|^2 + t^4 b\bigg(\int_{\mathbb{R}^3} |\nabla w_a|^2\bigg)^2 - t^{p+1}\int_{\mathbb{R}^3} a(x)|w_a|^{p+1} \\ &\leq t^4\|w_a\|^2 + t^4 b\bigg(\int_{\mathbb{R}^3} |\nabla w_a|^2\bigg)^2 - t^{p+1}\|w_a\|^2 \\ &\leq t^4\|w_a\|^2 + t^4 b\|w_a\|^4 - t^{p+1}\|w_a\|^2, \end{aligned}$$

which implies

$$\begin{aligned} t \leq (1 + b\|w_a\|^2)^{1/(p-3)} &= \bigg(1 + \frac{2(p+1)}{p-1} b m_a\bigg)^{1/(p-3)} \\ &:= (1 + \nu b m_a)^{1/(p-3)}. \end{aligned} \tag{4.3}$$

Using equations (1.2), (4.2) and (4.3), we obtain

$$m \leq I(tw_a) \leq t^{p+1} m_a \leq (1 + \nu b m_a)^{(p+1)/(p-3)} \leq m_\infty.$$

The proof is complete. □

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