

THE DIFFERENTIAL STRUCTURE OF AN ORBIFOLD

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ABSTRACT. We prove that the underlying set of an orbifold equipped with the ring of smooth real-valued functions completely determines the orbifold atlas. Consequently, we obtain an essentially injective functor from orbifolds to differential spaces.

1. Introduction. Consider an (effective) orbifold X ; that is, in particular, a space that locally is the quotient of a smooth manifold by an effective finite Lie group action. The family of all “smooth” functions consists of real-valued functions on X that locally lift to these manifolds as smooth functions invariant under the finite group actions. This family is an example of a (Sikorski) differential structure (see Definition 2.1). The purpose of this paper is to prove the next theorem.

Main theorem. Given an orbifold, its orbifold atlas can be constructed out of invariants of the differential structure.

This result can be tailored to be in the form of a functor from the “category of orbifolds” to differential spaces, which is essentially injective on objects. Of course, the “category of orbifolds” has a number of different definitions, depending on one’s perspective. There is the classical “category” defined by Satake [25] and further developed by Thurston [30] and Haefliger [9]. There are subtle differences between the definitions given by Satake and Haefliger, but we choose not to expand upon these here. (Reference [12] does deal with this subtlety, however). There is also the category of effective proper étale Lie groupoids (with various choices for the arrows), or the corresponding

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2-subcategory of geometric stacks, see, for example, [11, 16, 19, 20, 21, 24]. Choosing to use the weak 2-category of Lie groupoids with bibundles as arrows, we have:

Theorem A. There is a functor F from the weak 2-category of effective proper étale Lie groupoids with bibundles as arrows to differential spaces that is essentially injective on objects.

Here, *essentially injective* means that, given two objects G and H such that

$$F(G) \cong F(H),$$

we have

$$G \simeq H,$$

where in this case \simeq means *Morita equivalent*. It should be noted that this functor is neither full nor faithful (see Example 7.2, which consists of [12, Examples 24, 25]). The other modifications of the category of Lie groupoids (including stacks) mentioned in the references listed above will yield a similar theorem.

In [12], Iglesias-Zemmour, Karshon and Zadka define the notion of a “diffeological orbifold” and show that this agrees with the classical definitions as found in [9, 25]. Using this, we show:

Theorem B. There is a functor G from the weak 2-category of effective proper étale Lie groupoids with bibundles as arrows to diffeological spaces that is essentially injective on objects.

We give two proofs of this. The first uses the fact that G is the restriction of a more general functor from the weak 2-category of Lie groupoids to diffeological spaces introduced in [32, Section 4]. The essential injectivity follows immediately from the work of Iglesias-Zemmour, et al. The second proof of the essential injectivity of Theorem B uses the fact that the functor F in Theorem A factors as $\Phi \circ G$, where Φ is a faithful functor from diffeological spaces to differential spaces sending a diffeological space to its underlying set equipped with the ring of diffeologically smooth functions, see [2] and [31, Chapter 2]. Both Theorem A and Theorem B rely on a known correspondence between effective proper étale Lie groupoids

and orbifolds in the classical sense, see Remark 6.2. For more on the relationship between Lie groupoids and diffeological spaces, see [14, 32].

The central idea behind the proof of the main theorem is as follows. To reconstruct the orbifold, one needs three ingredients: the topology, the orbifold stratification and the order of points at codimension-2 strata (see Definition 5.3). We show that all three of these are invariants of the differential structure of an orbifold. This fact for the topology and, locally, the stratification are more or less already known. We give a local-to-global argument for the stratification in Theorem 4.14 and use codimensions of germs of functions (similar to Milnor numbers) to obtain that the order of a point is an invariant of the differential structure in Proposition 5.8. From here, a method proved by Haefliger and Ngoc Du [10] is used to reconstruct the local isotropy groups, and an argument by induction on the dimension of the orbifold is used to reconstruct the charts.

Differential spaces were introduced by Sikorski in 1967 [27, 28], and the theory was further developed by many since then, oftentimes under different names (see, for example, Schwarz [26], Śniatycki [29] and Aronszajn [1]). When dealing with quotient spaces such as orbifolds, the differential structure is induced by the diffeology, which in turn is induced by the corresponding Lie groupoid/stack. Thus, the differential structure is a fairly weak structure in this setting. It is equivalent to the corresponding Frölicher space structure, see [5].

The fact that Theorem A is true given Theorem B is *a priori* unexpected. Indeed, consider orbifolds of the form $X = \mathbb{R}^n/\Gamma$. As mentioned above, the differential structure on X is induced by the diffeological structure, but this relationship is definitely not one-to-one when looking at general group actions. In fact, the differential structure on $\mathbb{R}^n/O(n)$ is independent of n , while the diffeology is dependent on n , see Example 7.7. What we can conclude from this is that there is something special about the underlying (local) semi-algebraic structure of an orbifold (equipped with its natural differential structure) that allows us to reconstruct the original orbifold atlas.

This paper is broken down as follows. Section 2 reviews the relevant theory of differential spaces. Section 3 reviews the definition of an orbifold, defines its differential structure and develops properties of it.

Section 4 discusses the natural stratification of an orbifold, and here we prove that this stratification is an invariant of the differential structure (Corollary 4.15). In Section 5, we prove that the order of a point is an invariant of the differential structure (Theorem 5.10), reconstruct the isotropy groups (Theorem 5.5) and reconstruct the charts (the proof of the main theorem). Section 6 contains the proof of Theorem A. Section 7 contains both proofs of Theorem B.

Similar unpublished work for orbifolds whose isotropy groups are reflection-free or completely generated by reflections has been done by Moshe Zadka (see [12, introduction]), although this is not available as a preprint, and the author has not seen it.

2. Review of differential spaces. In this section, we review the basics of differential spaces and give relevant examples. For a more detailed presentation of differential spaces, see [29] or [31, subsection 2.2].

Definition 2.1 (Differential space). Let X be a set. A (*Sikorski*) *differential structure* is a family \mathcal{F} of real-valued functions on X satisfying the following two conditions:

- (1) (*Smooth compatibility*). For any positive integer k , functions $f_1, \dots, f_k \in \mathcal{F}$, and $g \in C^\infty(\mathbb{R}^k)$, the composition $g(f_1, \dots, f_k)$ is contained in \mathcal{F} .
- (2) (*Locality*). Equip X with the weakest topology for which each $f \in \mathcal{F}$ is continuous. Let $f: X \rightarrow \mathbb{R}$ be a function such that there exist an open cover $\{U_\alpha\}$ of X and, for each α , a function $g_\alpha \in \mathcal{F}$ satisfying

$$f|_{U_\alpha} = g_\alpha|_{U_\alpha}.$$

Then $f \in \mathcal{F}$.

The topology in the locality condition is called the *functional topology* (or *initial topology*) induced by \mathcal{F} . A set X equipped with a differential structure \mathcal{F} is called a (*Sikorski*) *differential space* and is denoted by (X, \mathcal{F}) .

Definition 2.2 (Functionally smooth map). Let (X, \mathcal{F}_X) and (Y, \mathcal{F}_Y) be two differential spaces. A map

$$F: X \longrightarrow Y$$

is *functionally smooth* if $F^*\mathcal{F}_Y \subseteq \mathcal{F}_X$. The map F is called a *functional diffeomorphism* if it is a bijection and both it and its inverse are smooth.

Remark 2.3. Differential spaces along with functionally smooth maps form a category, which we denote by **DiffSp**. Except where it would be ambiguous, “functional” and “functionally” will be dropped henceforth.

Definition 2.4 (Differential subspace). Let (X, \mathcal{F}) be a differential space, and let $Y \subseteq X$ be any subset. Then Y comes equipped with a differential structure \mathcal{F}_Y induced by \mathcal{F} as follows. A function $f \in \mathcal{F}_Y$ if and only if there is a covering $\{U_\alpha\}$ of Y by open sets of X such that, for each α , there exists $g_\alpha \in \mathcal{F}$ satisfying

$$f|_{U_\alpha \cap Y} = g_\alpha|_{U_\alpha \cap Y}.$$

We call (Y, \mathcal{F}_Y) a *differential subspace* of X . The functional topology on Y induced by \mathcal{F}_Y coincides with the subspace topology on Y , see [31, Lemma 2.28]. If Y is a closed differential subspace of \mathbb{R}^n , then \mathcal{F}_Y is the set of restrictions of smooth functions on \mathbb{R}^n to Y , see [31, Proposition 2.36].

Definition 2.5 (Subcartesian space). A *subcartesian space* is a paracompact, second-countable, Hausdorff differential space $(S, C^\infty(S))$ with an open cover $\{U_\alpha\}$ such that, for each α , there exist $n_\alpha \in \mathbb{N}$ and a diffeomorphism

$$\varphi_\alpha: U_\alpha \longrightarrow \tilde{U}_\alpha \subseteq \mathbb{R}^{n_\alpha}$$

onto a differential subspace \tilde{U}_α of \mathbb{R}^{n_α} .

Example 2.6 (Some semi-algebraic varieties). Let k be a positive integer. Define

$$S_k := \{(x, y) \in \mathbb{R}^2 \mid y^2 - x^k = 0, x \geq 0\}.$$

Then S_k is a closed differential subspace of \mathbb{R}^2 , with a differential structure given by all real-valued functions that extend to a smooth function on \mathbb{R}^2 .

Similarly, define

$$C_k := \{(x, y, z) \mid x^2 + y^2 = z^k, z \geq 0\}.$$

Then C_k is a closed differential subspace of \mathbb{R}^3 , and hence, its differential structure is given by restrictions of smooth functions on \mathbb{R}^3 . We will encounter these spaces again in later examples. \diamond

Definition 2.7 (Quotient differential structure). Let (X, \mathcal{F}) be a differential space, let \sim be an equivalence relation on X , and let

$$\pi : X \longrightarrow X/\sim$$

be the quotient map. Then X/\sim obtains a differential structure \mathcal{F}_\sim , called the *quotient differential structure*, comprising all functions

$$f : X/\sim \longrightarrow \mathbb{R},$$

each of whose pullback by π is in \mathcal{F} . In general, the functional topology generated by \mathcal{F}_\sim is coarser than the quotient topology.

Example 2.8 (Orbit space). Let K be a Lie group acting on a manifold M . Then the quotient differential structure on the orbit space M/K consists of all functions, each of which pulls back to a K -invariant smooth function on M .

Continuing this example, if K is a compact group, or if K acts on M properly, then M/K is in fact a subcartesian space. Indeed, by the local nature of a subcartesian space and the slice theorem [15, 22], it is enough to consider K as a subgroup of $O(n)$ acting on \mathbb{R}^n . By a theorem of Schwarz [26], the Hilbert map

$$\sigma = (\sigma_1, \dots, \sigma_k) : \mathbb{R}^n \longrightarrow \mathbb{R}^k,$$

where $\sigma_1, \dots, \sigma_k$ is a minimal generating set of the ring of K -invariant polynomials, descends to a proper topological embedding of \mathbb{R}^n/K as a closed subset of \mathbb{R}^k . Moreover,

$$\sigma^*(C^\infty(\mathbb{R}^k)) = C^\infty(\mathbb{R}^n)^K,$$

which implies that the quotient differential structure on \mathbb{R}^n/K is equal to the subcartesian structure induced by \mathbb{R}^k . \diamond

3. Orbifolds and their differential structures. We begin this section with the classical definition of an orbifold, based on the presentation in Moerdijk-Pronk [21, Section 1]. We then discuss its natural differential structure.

Definition 3.1 ((Effective) orbifold). Let X be a Hausdorff, paracompact, second-countable topological space. Fix a non-negative integer n .

- (1) An n -dimensional *orbifold chart* on X is a triple (U, Γ, ϕ) where $U \subseteq \mathbb{R}^n$ is an open subset, Γ is a finite group of diffeomorphisms of U , and ϕ is a Γ -invariant map

$$\phi : U \longrightarrow X$$

that induces a homeomorphism $U/\Gamma \rightarrow \phi(U)$.

- (2) An *embedding*

$$\lambda : (U, \Gamma, \phi) \longrightarrow (V, \Delta, \psi)$$

between two charts is smooth

$$\lambda : U \longrightarrow V$$

such that $\psi \circ \lambda = \phi$.

- (3) An n -dimensional *orbifold atlas* on X is a family \mathcal{U} of n -dimensional orbifold charts that cover X and are locally compatible. This last condition means that, for any two charts (U, Γ, ϕ) and (V, Δ, ψ) in \mathcal{U} , there is a family of charts

$$\{(W_\alpha, \Gamma_\alpha, \chi_\alpha)\}$$

with embeddings

$$(W_\alpha, \Gamma_\alpha, \chi_\alpha) \longrightarrow (U, \Gamma, \phi)$$

and

$$(W_\alpha, \Gamma_\alpha, \chi_\alpha) \longrightarrow (V, \Delta, \psi)$$

for each α , and the collection $\{\chi_\alpha(W_\alpha)\}$ forms an open cover of $\phi(U) \cap \psi(V)$.

- (4) An orbifold atlas \mathcal{U} *refines* another orbifold atlas \mathcal{V} if, for any chart in \mathcal{U} , there is an embedding of the chart into a chart of \mathcal{V} . If there exists a common refinement of \mathcal{U} and \mathcal{V} , then we say that the two atlases are *equivalent*. This forms an equivalence relation on

all atlases of X . Each such equivalence class is represented by a maximal atlas.

- (5) An (*effective*) *orbifold* (X, \mathcal{U}) of dimension n is a Hausdorff, paracompact, second-countable space X equipped with a maximal n -dimensional atlas \mathcal{U} .
- (6) Let (X, \mathcal{U}) and (Y, \mathcal{V}) be orbifolds. Then a map

$$F : X \longrightarrow Y$$

is *orbifold smooth* if, for any $x \in X$, there exist charts (U, Γ, ϕ) about x and (V, Δ, ψ) about $F(x)$ such that

$$F(\phi(U)) \subseteq \psi(V),$$

and there exists a smooth map

$$\tilde{F} : U \longrightarrow V$$

such that $\psi \circ \tilde{F} = F \circ \phi$. If F is orbifold smooth and invertible with orbifold smooth inverse, then F is an *orbifold diffeomorphism*.

Remark 3.2.

- (1) The topology on an orbifold is locally compact, and, since it is Hausdorff and second-countable, it follows that the topology is also normal.
- (2) Let X be an orbifold, and

$$\lambda : (W, \Delta, \chi) \longrightarrow (U, \Gamma, \phi)$$

an embedding of charts. Then λ induces a group monomorphism $\bar{\lambda} : \Delta \rightarrow \Gamma$ such that, for any $w \in W$ and $\delta \in \Delta$,

$$\lambda(\delta \cdot w) = \bar{\lambda}(\delta) \cdot \lambda(w).$$

Moreover, if $\gamma \in \Gamma$ such that

$$\lambda(U) \cap \gamma \cdot \lambda(U) \neq \emptyset,$$

then γ is in the image of $\bar{\lambda}$. In particular, for any $w \in W$, λ induces a group isomorphism between the stabilizer of w and that of $\lambda(w)$, see [21, Appendix]. A similar statement appears as Lemma 17 of [12].

Example 3.3 (Reflections and rotations in the plane—Part I). Let D_k be the dihedral group of order $2k$. It is generated by β_1 and β_2 , both of which have order 2, and such that $(\beta_2\beta_1)^k$ is the identity. β_1 acts on $\mathbb{C} \cong \mathbb{R}^2$ by conjugation

$$z \longmapsto \bar{z},$$

and β_2 by

$$z \longmapsto e^{2\pi i/k} \bar{z}.$$

The resulting orbit space \mathbb{R}^2/D_k is an example of an orbifold.

Similarly, let \mathbb{Z}_k be the cyclic group of order k . It is generated by α , which has order k . It acts on $\mathbb{C} \cong \mathbb{R}^2$ by

$$z \longmapsto e^{2\pi i/k} z.$$

We obtain the orbifold $\mathbb{R}^2/\mathbb{Z}_k$. ◇

Theorem 3.4 (A theorem of Leonardo di Vinci—Finite group actions on the plane). *Let $\Gamma \subset O(2)$ be a finite group acting orthogonally on the plane. Then Γ is isomorphic as a group to a dihedral group D_k or to a cyclic group \mathbb{Z}_k .*

Proof. The cyclic and dihedral groups are the only finite Lie subgroups of $O(2)$. See, for example, [33, pages 66, 99] for a reference attributing this discovery to di Vinci. □

Remark 3.5. Due to Theorem 3.4 and the fact that any finite linear group action on the plane can be transformed equivariant-diffeomorphically into an orthogonal group action (one can always construct an invariant metric) we conclude that any two-dimensional orbifold locally looks like \mathbb{R}^2/D_k or $\mathbb{R}^2/\mathbb{Z}_k$ for some k .

Definition 3.6 (Isotropy group). Let X be an orbifold of dimension n , and let $x \in X$. Then an *isotropy group* of X at x is a finite subgroup Γ_x of $\mathrm{GL}(\mathbb{R}^n)$ such that there exists a chart $(\mathbb{R}^n, \Gamma_x, \phi)$ satisfying $\phi(0) = x$.

Remark 3.7. An isotropy group exists at every point $x \in X$ and can be obtained using the slice theorem. It is unique up to conjugation in $\mathrm{GL}(\mathbb{R}^n)$ (see [20, pages 39, 40]). Moreover, we may assume that $\Gamma_x \in O(n)$ if needed.

Definition 3.8 (Differential structure on an orbifold). Let X be an n -dimensional orbifold. Then the *orbifold differential structure* $C^\infty(X)$ on X is given by real-valued functions

$$f : X \longrightarrow \mathbb{R}$$

satisfying the following: given a chart (U, Γ, ϕ) of X , there exists a smooth Γ -invariant function

$$g_U : U \longrightarrow \mathbb{R}$$

such that $g_U = \phi^* f$.

Proposition 3.9 (Properties of the orbifold differential structure). *Let X be an orbifold.*

- (1) *The corresponding functional topology on X equals the orbifold topology.*
- (2) *$C^\infty(X)$ equals the ring of orbifold smooth functions.*
- (3) *$(X, C^\infty(X))$ is subcartesian.*

Proof.

- (1) A basis for the topology on X induced by its orbifold structure is given by the union over all charts (U, Γ, ϕ) of each quotient topology on $\phi(U)$. But by Example 2.8 and Definition 2.4, this is also a basis for the topology induced by $C^\infty(X)$.
- (2) This is immediate from the definitions.
- (3) This is a direct consequence of Example 2.8 and Definition 3.8. \square

Remark 3.10. It follows from Proposition 3.9 that the orbifold differential structure depends only on the natural differential structure of the (local) semi-algebraic variety underlying the orbifold.

Example 3.11 (Reflections and rotations in the plane—Part II). Continuing example Example 3.3, a minimal generating set for the ring of D_k -invariant real polynomials on $\mathbb{C} \cong \mathbb{R}^2$ is given by $\{\delta_1, \delta_2\}$ where $\delta_1(z) = |z|^2$ and $\delta_2(z) = \Re(z^k)$. The resulting orbifold embeds into \mathbb{R}^2 as the semi-algebraic variety

$$R_k := \{(s, t) \mid t^2 \leq s^k, s \geq 0\}.$$

Similarly, a minimal generating set for the ring of \mathbb{Z}_k -invariant real polynomials on $\mathbb{C} \cong \mathbb{R}^2$ is given by $\{\sigma_1, \sigma_2, \sigma_3\}$ where $\sigma_1 = \Re(z^k)$, $\sigma_2 = \Im(z^k)$ and $\sigma_3 = |z|^2$. The resulting orbifold embeds into \mathbb{R}^3 as the semi-algebraic variety

$$C_k := \{(s, t, u) \mid s^2 + t^2 = u^k, u \geq 0\}.$$

This is the same differential subspace C_k introduced in Example 2.6. \diamond

4. The stratification of an orbifold. In this section, we review stratified spaces from the perspective of subcartesian spaces. For more details, see [17] and [29, Chapter 4]. For a general introduction to stratified spaces, see [23]. The main results of this section are Theorem 4.14 and Corollary 4.15. The theorem states that the orbifold stratification is induced by the family of vector fields on the orbifold, which uses the theory of vector fields on subcartesian spaces developed by Śniatycki, see [29]. The corollary uses the fact that the family of vector fields of a subcartesian space is an invariant of the differential structure, and thus, the orbifold stratification is an invariant of the orbifold differential structure.

Definition 4.1 (Smooth stratification). Let S be a subcartesian space. Then a *smooth stratification* of S is a locally finite partition \mathcal{M} of S into locally closed and connected (embedded) submanifolds M , called the *strata* of \mathcal{M} , which satisfy the following *frontier condition*.

Frontier condition: For any M and M' in \mathcal{M} , if $M' \cap \overline{M} \neq \emptyset$, then either $M = M'$ or $M' \subseteq \overline{M} \setminus M$.

Example 4.2 (Orbit-type stratification–Part I). Let K be a Lie group acting properly on a manifold M . Define for any closed subgroup H of K the subset of *orbit-type* (H) by

$$M_{(H)} := \{x \in M \mid \text{there exists } k \in K \text{ such that } \text{Stab}_K(x) = kHk^{-1}\}.$$

Then the collection of all connected components of all (non-empty) subsets $M_{(H)}$ form a smooth stratification of M , called the *orbit-type stratification*, see [7, Theorem 2.7.4]. Moreover, this stratification descends via the quotient map

$$\pi : M \longrightarrow M/K$$

to a smooth stratification on M/K , in which the strata are connected components of $\pi(M_{(H)})$ as H runs over closed subgroups of K such that $M_{(H)}$ is non-empty (see [29, Theorem 4.3.5]). \diamond

Definition 4.3 (Orbifold stratification). Let X be an orbifold. Then X admits a stratification, called the *orbifold stratification* given locally as follows. Let (U, Γ, ϕ) be a chart. Then the orbit-type stratification on U descends to a stratification on U/Γ , and hence, on $\phi(U)$.

Lemma 4.4 (Orbifold stratification is well-defined). *Given an orbifold X , the orbifold stratification is independent of the charts of X , that is, it is well-defined. Moreover, it is a smooth stratification in the sense of Definition 4.1.*

Proof. For any chart (U, Γ, ϕ) of X , the orbit-type stratification on U descends to a smooth stratification on $\phi(U)$, see Example 4.2. Since the conditions of a stratification are local, we can construct a global stratification by piecing together the stratifications on each open set $\phi(U)$ for each chart (U, Γ, ϕ) . We only need to show that this stratification is independent of the chart.

To this end, let n be the dimension of X . Fix two charts (U, Γ, ϕ) and (W, Δ, ψ) such that there is an embedding

$$\lambda : (W, \Delta, \psi) \longrightarrow (U, \Gamma, \phi).$$

We want to show that the strata of $\psi(W)$ match up with those of $\phi(U)$ via the inclusion $\psi(W) \subseteq \phi(U)$. To accomplish this, it is enough to show that λ induces a one-to-one correspondence between the orbit-type strata on W and the connected components of the intersection of orbit-type strata of U with $\lambda(W)$.

By Remark 3.2 (2), there is a group monomorphism

$$\bar{\lambda} : \Delta \longrightarrow \Gamma$$

such that $\lambda(\delta \cdot w) = \bar{\lambda}(\delta) \cdot \lambda(w)$ for all $\delta \in \Delta$, and for any $w \in W$, we have that $\bar{\lambda}$ induces a group isomorphism between $\text{Stab}_{\Delta}(w)$ and $\text{Stab}_{\Gamma}(\lambda(w))$. It follows that λ preserves orbit-types, and since λ is continuous and continuous maps preserve connectedness, we have that λ maps strata into connected components of the orbit-type strata of U that intersect $\lambda(W)$.

Since

$$\lambda^{-1} : \lambda(W) \longrightarrow W$$

is also an embedding, we have that it maps strata of the $\bar{\lambda}(\Delta)$ -action on $\lambda(W)$ into strata of W . Let $u \in U_{(H)} \cap \lambda(W)$ with stabilizer $H \subseteq \Gamma$. By Remark 3.2 (2), H must be a subgroup of $\bar{\lambda}(\Delta)$, and it is the stabilizer of u with respect to the action of $\bar{\lambda}(\Delta)$. We conclude that

$$U_{(H)} \cap \lambda(W) = \lambda(W)_{(H)},$$

and this completes the proof. \square

Remark 4.5. Given an orbifold X with a chart (U, Γ, ϕ) in which U is connected, there is an open, dense stratum of $\phi(U)$, the codimension-0 stratum. The union of all of these yields an open and dense codimension-0 stratum of X , which is a manifold whose dimension equals the dimension of X . Note that the dimension of X is thus a topological invariant of it. Indeed, the topological dimension at almost every $x \in X$ is equal to the dimension of X .

Definition 4.6 (Refinements and minimality). Let S be a subcartesian space, and let \mathcal{M} and \mathcal{M}' be smooth stratifications on it. Then \mathcal{M} is said to *refine* \mathcal{M}' if, for every $M \in \mathcal{M}$, there exists an $M' \in \mathcal{M}'$ such that $M \subseteq M'$. If \mathcal{M} is not a refinement of any other smooth stratification on S , then we say that \mathcal{M} is *minimal*.

Example 4.7 (Orbit-type stratification–Part II). Let K be a non-trivial Lie group acting properly and effectively on a manifold M . Then the orbit-type stratification on M is not generally minimal (as the set of connected components of M itself refines it). On the other hand, the induced stratification on M/K is minimal. This is a result of Bierstone, see [3, 4].

Definition 4.8 (Smooth local triviality). Let S be a subcartesian space, and let \mathcal{M} be a smooth stratification on S . Then S is *smoothly locally trivial* if, for every $M \in \mathcal{M}$ and $x \in M$,

- (1) there is an open neighborhood U of x such that the partition of U into manifolds $N \cap U$ ($N \in \mathcal{M}$) yields a stratification of U ,
- (2) there exists a subcartesian space S' with smooth stratification \mathcal{M}' which contains a singleton set $\{y\} \subseteq \mathcal{M}'$,

(3) there exists a strata-preserving diffeomorphism

$$\varphi : U \longrightarrow (M \cap U) \times S'$$

sending x to (x, y) .

Note that the strata of $(M \cap U) \times S'$ are the sets $(M \cap U) \times M'$ where $M' \in \mathcal{M}'$.

Lemma 4.9. *Let X be an orbifold. Then the orbifold stratification on X is smoothly locally trivial.*

Proof. Since it is enough to prove this locally, we may focus on a chart (U, Γ, ϕ) of X . By Remark 3.7, we may assume that $U = \mathbb{R}^n$, on which Γ acts orthogonally. Thus, we may apply the result [29, Lemma 4.3.6]. \square

Definition 4.10 (Tangent bundles and global derivations). Let S be a subcartesian space.

(1) Given a point $x \in S$, a *derivation* of $C^\infty(S)$ at x is a linear map

$$v : C^\infty(S) \longrightarrow \mathbb{R}$$

that satisfies Leibniz's rule: for all $f, g \in C^\infty(S)$,

$$v(fg) = f(x)v(g) + g(x)v(f).$$

The set of all derivations of $C^\infty(S)$ at x forms a vector space, called the (*Zariski*) *tangent space* of x and is denoted $T_x S$. Define the (*Zariski*) *tangent bundle* TS to be the (disjoint) union

$$TS := \bigcup_{x \in S} T_x S.$$

Denote the canonical projection $TS \rightarrow S$ by τ .

(2) A (*global*) *derivation* of $C^\infty(S)$ is a linear map

$$Y : C^\infty(S) \longrightarrow C^\infty(S)$$

that satisfies Leibniz's rule: for any $f, g \in C^\infty(S)$,

$$Y(fg) = fY(g) + gY(f).$$

Denote the $C^\infty(S)$ -module of all derivations by $\text{Der} C^\infty(S)$.

(3) Fix $Y \in \text{Der}C^\infty(S)$ and $x \in S$. An *integral curve* $\exp(\cdot Y)(x)$ of Y through x is a smooth map from a connected subset $I_x^Y \subseteq \mathbb{R}$ containing 0 to S such that $\exp(0Y)(x) = x$, and for all $f \in C^\infty(S)$ and $t \in I_x^Y$, we have

$$\frac{d}{dt}(f \circ \exp(tY)(x)) = (Yf)(\exp(tY)(x)).$$

An integral curve is *maximal* if I_x^Y is maximal among the domains of all such curves. We adopt the convention that the map

$$c : \{0\} \longrightarrow S : 0 \longmapsto c(0)$$

is an integral curve of every global derivation of $C^\infty(S)$.

Remark 4.11.

- (1) TS is a subcartesian space with its differential structure generated by functions $f \circ \tau$ and df where $f \in C^\infty(S)$ and d is the differential $df(v) := v(f)$. The projection τ is smooth with respect to this differential structure, see [18, page 4] or [29, Proposition 3.3.3].
- (2) Given $x \in S$, the dimension of $T_x S$ is invariant under diffeomorphism: if $\varphi : S \rightarrow R$ is a diffeomorphism of differential spaces, then R is a subcartesian space, and the dimension of $T_{\varphi(x)} R$ is equal to that of $T_x S$. Indeed, it is not hard to show that the pushforward

$$\varphi_* : TS \longrightarrow TR$$

sending $v \in T_x S$ to $\varphi_* v \in T_{\varphi(x)} R$ is a linear isomorphism on each tangent space. (Recall that, for any $f \in C^\infty(R)$, we have $\varphi_* v(f) = v(f \circ \varphi)$.)

- (3) Global derivations of $C^\infty(S)$ are exactly the smooth sections of

$$\tau : TS \longrightarrow S,$$

see [29, Proposition 3.3.5].

- (4) Let $Y \in \text{Der}C^\infty(S)$. Then, for any $x \in S$, there exists a unique maximal integral curve $\exp(\cdot Y)(x)$ through x , see [29, Theorem 3.2.1].

Definition 4.12 (Vector fields and their orbits). Let S be a subcartesian space.

- (1) Let D be a subset of $\mathbb{R} \times S$ containing $\{0\} \times S$ such that $D \cap (\mathbb{R} \times \{x\})$ is connected for each $x \in S$. A map

$$\phi : D \longrightarrow S$$

is a *local flow* if D is open, $\phi(0, x) = x$ for each $x \in S$, and $\phi(t, \phi(s, x)) = \phi(t + s, x)$ for all $x \in S$ and $s, t \in \mathbb{R}$ for which both sides are defined.

- (2) A *vector field* on S is a derivation Y of $C^\infty(S)$ such that the map

$$(t, x) \longmapsto \exp(tY)(x),$$

sending (t, x) to the maximal integral curve of Y through x evaluated at t , is a local flow. Denote the set of all vector fields on S by $\text{vect}(S)$.

- (3) Let S be a subcartesian space, and let \mathcal{M} be a smooth stratification of it. Then, the pair (S, \mathcal{M}) is said to *admit local extensions of vector fields* if for any stratum $M \in \mathcal{M}$, any vector field X_M on M , and any $x \in M$, there exist an open neighborhood U of x and a vector field $X \in \text{vect}(S)$ such that $X_M|_{U \cap M} = X|_{U \cap M}$.
- (4) Let S be a subcartesian space. The *orbit* of $\text{vect}(S)$ through a point x , denoted O_x^S , is the set of all points $y \in S$ such that there exist vector fields

$$Y_1, \dots, Y_k$$

and real numbers

$$t_1, \dots, t_k \in \mathbb{R}$$

satisfying

$$y = \exp(t_k Y_k) \circ \dots \circ \exp(t_1 Y_1)(x).$$

Denote by \mathcal{O}_S the set of all orbits $\{O_x^S \mid x \in S\}$.

Remark 4.13. Let S be a subcartesian space.

- (1) Let R be another subcartesian space, and let $F : R \rightarrow S$ be a diffeomorphism. Then F induces a bijection between $\text{vect}(R)$ and $\text{vect}(S)$. Indeed, F induces an isomorphism between the derivations of $C^\infty(R)$ and those of $C^\infty(S)$. If $Z \in \text{vect}(R)$,

then F_*Z is a vector field on S :

$$\left. \frac{d}{dt} \right|_{t=t_0} F \circ \exp(tZ)(x) = F_*Z|_{F(x)}.$$

The reverse direction also holds, and so the result follows.

- (2) $\text{vect}(S)$ is a $C^\infty(S)$ -module, that is, for any $f \in C^\infty(S)$ and any vector field $Y \in \text{vect}(S)$, the derivation fY is a vector field, see [31, Corollary 4.71].
- (3) Let \mathcal{M} be a smoothly locally trivial smooth stratification of S . Then (S, \mathcal{M}) admits local extensions of vector fields, see [17, Theorem 4.5] or [29, Proposition 4.1.5].
- (4) Let \mathcal{M} be a smooth stratification of S . If (S, \mathcal{M}) admits local extensions of vector fields, then the set of orbits \mathcal{O}_S forms a stratification of S , of which \mathcal{M} is a refinement. In particular, if \mathcal{M} is minimal, then $\mathcal{M} = \mathcal{O}_S$, see [17, Theorem 4.6] or [29, Theorem 4.1.6].
- (5) Let \mathcal{O}_S be the set of orbits induced by $\text{vect}(S)$. Then, \mathcal{O}_S is a stratification of S if and only if it is locally finite and each $O \in \mathcal{O}_S$ is locally closed in S , see [17, Theorem 4.3] or [29, Corollary 4.1.3].

Theorem 4.14 (The orbifold stratification is induced by vector fields). *Let X be an orbifold. Then, the orbifold stratification is given by the set of orbits \mathcal{O}_X induced by $\text{vect}(X)$.*

Proof. Let (U, Γ, ϕ) be a chart of X . By Lemma 4.9, the orbifold stratification on $\phi(U)$ is smoothly locally trivial. Hence, it admits local extensions of vector fields by Remark 4.13 (3). Thus, the orbits $\mathcal{O}_{\phi(U)}$ of $\text{vect}(\phi(U))$ form a stratification of $\phi(U)$ which is refined by the orbifold stratification on $\phi(U)$ by Remark 4.13 (4). However, since the orbifold stratification on $\phi(U)$ is minimal (see Example 4.7), we conclude that the stratification by orbits $\mathcal{O}_{\phi(U)}$ is equal to the orbifold stratification.

By Lemma 4.4, we already know that the orbifold stratification is independent of chart. Thus, it remains to show that, for any $x \in \phi(U)$, we have that $\mathcal{O}_x^{\phi(U)}$ is a connected component of $\mathcal{O}_x^X \cap \phi(U)$.

We begin with the inclusion

$$\mathcal{O}_x^{\phi(U)} \subseteq \mathcal{O}_x^X \cap \phi(U).$$

Let $y \in O_x^{\phi(U)}$. Then there exist vector fields

$$Y_1, \dots, Y_k \in \text{vect}(\phi(U)) \quad \text{and} \quad t_1, \dots, t_k \in \mathbb{R}$$

such that

$$(4.1) \quad y = \exp(t_k Y_k) \circ \dots \circ \exp(t_1 Y_1)(x).$$

Fix $i \in \{1, \dots, k\}$. The path

$$c : [0, t_i] \longrightarrow \phi(U)$$

defined by

$$c : s \longmapsto \exp(s Y_i) \circ \exp(t_{i-1} Y_{i-1}) \circ \dots \circ \exp(t_1 Y_1)(x)$$

has a compact image in $\phi(U)$. Now, $\phi(U)$ is open in X , and X is normal, see Remark 3.2 (1). So, we can find an open neighborhood V of $c([0, t_i])$ and an open neighborhood W of the complement of $\phi(U)$ in X that are disjoint. Let

$$b_i : X \longrightarrow \mathbb{R}$$

be a smooth bump function that is equal to 1 on V and has support in the complement of W (it follows from Example 2.8 that $\phi(U) \subset \mathbb{R}^N$ for some N , and so such a b_i can be easily constructed). Then by Remark 4.13 (2), $b_i Y_i \in \text{vect}(X)$. Replacing each vector field Y_i with $b_i Y_i$ in equation (4.1), we obtain that $y \in O_x^X$.

Now consider the partition P of $\phi(U)$ by connected components of $O \cap \phi(U)$ for each $O \in \mathcal{O}_X$. Each element Q of P is an immersed submanifold of $\phi(U)$. Moreover, each element Q of P is a finite union of strata of $\phi(U)$, and since each of these strata is locally closed, we have that Q is locally closed. Since, for each $x \in \phi(U)$, we have

$$O_x^{\phi(U)} \subseteq O_x^X \cap \phi(U),$$

we conclude that P is locally finite. It follows that \mathcal{O}_X is locally finite and its elements are locally closed. By Remark 4.13 (5), \mathcal{O}_X is a smooth stratification of X . Moreover, P is a smooth stratification of $\phi(U)$. Since this stratification is refined by the orbifold stratification of $\phi(U)$, which is minimal, we conclude that $\mathcal{O}_{\phi(U)} = P$. \square

Corollary 4.15 (Invariance of stratification). *The orbifold stratification is an invariant of the orbifold differential structure.*

Proof. This follows from Remark 4.13 (1) and Theorem 4.14. \square

Example 4.16 (Reflections and rotations in the plane—Part III). Continuing Example 3.11, the strata of \mathbb{R}^2/D_k are given by the origin $\{(0, 0)\}$, the two connected components of

$$\{(s, t) \mid t^2 = s^k, s > 0\},$$

and the open dense stratum given by

$$\{(s, t) \mid t^2 > s^k, s > 0\}.$$

Note that the codimension-1 and codimension-2 strata (called the *singular strata*) together form the set S_k of Example 2.6.

Similarly, the strata of $\mathbb{R}^2/\mathbb{Z}_k$ are given by the origin $\{(0, 0, 0)\}$, and the open dense stratum

$$\{(s, t, u) \mid s^2 + t^2 = u^k, u > 0\}. \quad \diamond$$

5. Recovering the charts. We begin with a discussion of orbifold covering spaces, based on [30, Chapter 13]; in particular, we need universal orbifold covering spaces for the proof of the main theorem. Moreover, these motivate orbifold fundamental groups. In [10], Haefliger and Ngoc Du show that the orbifold fundamental group can be obtained using the topology, stratification and orders of points in codimension-2 strata, see Theorem 5.5. In the previous sections, we showed that the topology and stratifications are invariants of the orbifold differential structure, and in Proposition 5.8, we show that the order of a point is also such an invariant. This is important: while the order of a point in an orbifold may show up as the degree of an associated defining-polynomial, see Example 5.6, composition with a diffeomorphism may not yield a polynomial, and so “degree” does not make sense. We then prove the main theorem at the end of the section.

Definition 5.1 (Orbifold covering space). Let X be an orbifold, and fix a base point x_0 in the codimension-0 stratum of X .

(1) An *orbifold covering space* of X is an orbifold \tilde{X} with a base point \tilde{x}_0 in its codimension-0 stratum, and an orbifold smooth “projection” map $p : \tilde{X} \rightarrow X$ which sends \tilde{x}_0 to x_0 . For any $x \in X$, we require that there be a chart (U, Γ, ϕ) of X with $x \in \phi(U)$, and, for each connected

component C_i of $p^{-1}(\phi(U))$, there is a Γ -equivariant diffeomorphism $\Psi_i : C_i \rightarrow U/\Gamma_i$ where $\Gamma_i \subseteq \Gamma$ is a subgroup.

(2) X is called a *good orbifold* if there exists an orbifold covering space that is a smooth manifold; otherwise, it is called a *bad orbifold*.

(3) A *universal orbifold covering space* of X is a connected orbifold covering space $p : \tilde{X} \rightarrow X$ such that, if \tilde{X}' is any other orbifold covering space of X with projection $p' : \tilde{X}' \rightarrow X$, then there is a lifting of p via p' to a map $q : \tilde{X} \rightarrow \tilde{X}'$ by which \tilde{X} is an orbifold covering space of \tilde{X}' .

(4) If $p : \tilde{X} \rightarrow X$ is a universal orbifold covering space of X with base point $\tilde{x}_0 \in p^{-1}(x_0)$, then, for any other $y \in p^{-1}(x_0)$, there is a deck transformation taking \tilde{x}_0 to y , that is, an orbifold diffeomorphism $f : \tilde{X} \rightarrow \tilde{X}$ such that $p \circ f = p$ and $f(\tilde{x}_0) = y$. The group of deck transformations of \tilde{X} is called the *orbifold fundamental group* of X . (See [30, Definition 13.2.5].)

Remark 5.2.

- (1) Note that an orbifold covering space of an orbifold X in general is not a covering space in the topological sense.
- (2) If $X = M/\Gamma$ where M is a simply connected manifold on which a finite group Γ acts, then M is the universal orbifold covering space of X . If M is not simply connected, then we can take its (topological) universal covering space as the universal orbifold covering space of X .
- (3) Let X be an orbifold. Then X has a universal orbifold covering space \tilde{X} , which is unique up to orbifold diffeomorphism. Moreover, if X is a good orbifold, then \tilde{X} is a simply connected manifold, see [30, Proposition 13.2.4].

Definition 5.3 (Order of a point). Let X be an orbifold, and let $x \in X$ with isotropy group Γ_x . Then, the *order* of x is equal to the order of the group Γ_x .

Remark 5.4. It follows from Remark 3.7 that the order of a point is well-defined.

Theorem 5.5 (Recovering the groups). *Let X be a connected orbifold. Then, a presentation for the orbifold fundamental group can be constructed using the topology, stratification, and the orders of points in codimension-2 strata.*

Proof. This is proved by Haefliger and Ngoc Du [10], see also [6, subsection 1.3]. We briefly explain the algorithm here. Let X_{reg} be the differential subspace of X consisting of codimension-0 and codimension-1 strata. Fix a base point x in the codimension-0 stratum. Let G be the (topological) fundamental group of X_{reg} with respect to x .

(1) For each codimension-1 stratum S_i , and for each homotopy class μ of paths starting at x and ending on S_i , attach a generator $\beta_{i,\mu}$ to G with relation $\beta_{i,\mu}^2 = 1$.

(2) For each codimension-2 stratum R in the closure of a codimension-1 stratum, for each pair of codimension-1 strata S_i and $S_{i'}$ with R in their closures, and for each pair $\beta_{i,\mu}$ and $\beta_{i',\mu'}$, where $\mu \neq \mu'$, as constructed in (1) above, add the relation $(\beta_{i,\mu}\beta_{i',\mu'})^k = 1$ where $2k$ is the order of any point in R .

(3) For each codimension-2 stratum T_j not in the closure of a codimension-1 stratum, let α_j be an element of G represented by a loop starting at x and around T_j . Then add the relation $\alpha_j^k = 1$ to G where k is the order of any point in T_j .

The resulting group is the orbifold fundamental group of X . \square

Example 5.6 (Reflections and rotations in the plane—Part IV). Consider the orbifold \mathbb{R}^2/D_k . In Example 3.11, we saw that \mathbb{R}^2/D_k embedded into \mathbb{R}^2 as the semi-algebraic variety

$$R_k := \{(s, t) \mid t^2 \leq s^k, s \geq 0\},$$

with its strata listed in Example 4.16. Applying the algorithm in the proof of Theorem 5.5, we have that the orbifold fundamental group is

$$\langle \beta_1, \beta_2 \mid \beta_1^2 = \beta_2^2 = (\beta_1\beta_2)^k = 1 \rangle.$$

But this is exactly D_k .

Similarly, consider the orbifold $\mathbb{R}^2/\mathbb{Z}_k$. In Example 3.11, we saw that $\mathbb{R}^2/\mathbb{Z}_k$ embedded into \mathbb{R}^3 as the semi-algebraic variety

$$C_k := \{(s, t, u) \mid s^2 + t^2 = u^k, u \geq 0\},$$

with its strata listed in Example 4.16. Applying the algorithm in the proof of Theorem 5.5, we have that the orbifold fundamental group is

$$\langle \alpha \mid \alpha^k = 1 \rangle.$$

This is exactly \mathbb{Z}_k . ◇

Definition 5.7 (Codimension of a germ). Let \mathcal{E}_n be the \mathbb{R} -algebra of germs of smooth real-valued functions at $0 \in \mathbb{R}^n$:

$$\mathcal{E}_n = C^\infty(\mathbb{R}^n)/\sim$$

where $f \sim g$ if there exists an open neighborhood of 0 on which $f = g$. (In practice, where it does not cause confusion, we will often identify an element of \mathcal{E}_n with one of its representatives.) Let $f \in \mathcal{E}_n$, and define J_f to be the *Jacobian ideal* of f , which is the ideal of \mathcal{E}_n generated by the germs of partial derivatives of f at 0:

$$J_f = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle.$$

The *codimension* of (the germ of) f at 0, denoted $\text{cod}(f)$, is defined to be the dimension of the quotient algebra \mathcal{E}_n/J_f as a vector space.

Proposition 5.8 (Codimension of a germ is an invariant). *Let $f \in C^\infty(\mathbb{R}^n)$, with $f(0) = a$. Assume that 0 is a critical point of f . Then, the codimension of (the germ of) f at 0 is invariant under diffeomorphism. In particular, $\text{cod}(f)$ is an invariant of the differential structure on the differential subspace $f^{-1}(a) \subseteq \mathbb{R}^n$.*

Proof. The proof that the $\text{cod}(f)$ is invariant under diffeomorphism is an immediate consequence of the chain rule. See [8, Theorem 2.12] for more details.

Next, let φ be a diffeomorphism between $f^{-1}(a)$ and a differential space $(S, C^\infty(S))$. Then, $(S, C^\infty(S))$ is subcartesian. Let $x = \varphi(0) \in S$. Then, there is an open neighborhood U of x in S and a diffeomorphism $\psi : U \rightarrow \tilde{U}$ where \tilde{U} is a differential subspace of \mathbb{R}^m . Without

loss of generality, we may choose m to be the minimal such integer for which the diffeomorphism ψ exists. By [18, Lemma 3.4], this is equal to the dimension of $T_x S$, which is invariant under diffeomorphism by Remark 4.11 (2). Thus, $n \geq m$. If $n > m$, then embed $\tilde{U} \subseteq \mathbb{R}^m$ into \mathbb{R}^n by

$$(x_1, \dots, x_m) \longmapsto (x_1, \dots, x_m, 0, \dots, 0).$$

In either case, we now have a diffeomorphism $\tilde{\varphi}$ from $f^{-1}(a)$ to \tilde{U} which are both differential subspaces of \mathbb{R}^n . Without loss of generality, assume that $\tilde{\varphi}(0) = 0$. By [31, Theorem 6.3], $\tilde{\varphi}$ extends to a diffeomorphism from an open neighborhood of $0 \in \mathbb{R}^n$ to itself. The result now follows. \square

Example 5.9 (Reflections and rotations in the plane—Part V). Continuing Example 5.6, recall that the singular strata of the orbit space \mathbb{R}^2/D_k are given by the relations

$$\begin{aligned} t^2 - s^k &= 0, \\ s &\geq 0. \end{aligned}$$

The codimension of $f(s, t) = t^2 - s^k$ is computed as follows. The partial derivatives are

$$\frac{\partial f}{\partial s}(s, t) = -ks^{k-1} \quad \text{and} \quad \frac{\partial f}{\partial t}(s, t) = 2t.$$

It follows that \mathcal{E}_2/J_f is generated by

$$s, s^2, \dots, s^{k-2},$$

and so $\text{cod}(f) = k - 2 + 1 = k - 1$ (where we add 1 to account for the constant functions). Note that $|D_k| = 2((\text{cod})(f) + 1)$.

Similarly, recall that $\mathbb{R}^2/\mathbb{Z}_k$ is given by the relations

$$\begin{aligned} s^2 + t^2 - u^k &= 0 \\ u &\geq 0. \end{aligned}$$

The codimension of $f(s, t, u) = s^2 + t^2 - u^k$ is computed as follows.

The partial derivatives are

$$\begin{aligned}\frac{\partial f}{\partial s}(s, t, u) &= 2s, \\ \frac{\partial f}{\partial t}(s, t, u) &= 2t,\end{aligned}$$

and

$$\frac{\partial f}{\partial u}(s, t, u) = -ku^{k-1}.$$

It follows that \mathcal{E}_3/J_f is generated by

$$u, u^2, \dots, u^{k-2}$$

and so $\text{cod}(f) = k - 2 + 1 = k - 1$. Note that $|\mathbb{Z}_k| = \text{cod}(f) + 1$. \diamond

Theorem 5.10 (Order of a point is an invariant). *Let X be an orbifold, and let $x \in X$. If x is in a codimension-2 stratum, then the order of x is an invariant of the orbifold differential structure. Consequently, the order of any point of X is an invariant of the orbifold differential structure.*

Proof. Recall that the orbifold stratification is an invariant of the differential structure by Corollary 4.15. Let n be the dimension of X , let Γ_x be an isotropy group of x , and let M be the stratum containing x . By Lemma 4.9, there is an open neighborhood U of x , a smooth stratified subcartesian space S' with a one-point stratum $\{y\}$, and a strata-preserving diffeomorphism $U \rightarrow (M \cap U) \times S'$ sending x to (x, y) . Let $(\mathbb{R}^n, \Gamma_x, \phi)$ be a chart at x such that $\phi(0) = x$, in which Γ_x acts orthogonally. Without loss of generality, assume that $U = \phi(\mathbb{R}^n)$. Let $E = (\mathbb{R}^n)^{\Gamma_x}$ be the linear subspace of Γ_x -fixed points and F an orthogonal complement to E . Then, since $\phi(0) = x$ and 0 is a fixed point, we have that $\phi(E) = M \cap U$. Since Γ_x acts trivially on E , we have that

$$E \cong \mathbb{R}^{n-2},$$

and so $F \cong \mathbb{R}^2$, on which Γ_x acts with unique fixed point 0 . Hence, by Theorem 3.4, Γ_x is a dihedral group (if M is in the closure of a codimension-1 stratum) or a cyclic group (if M is not in the closure of a codimension-1 stratum). By Example 5.9 and Proposition 5.8, the

order of Γ_x can be obtained from invariants of the orbifold differential structure.

For the second statement, recall that the orbifold fundamental group of any orbifold can be obtained from the topology, stratification and orders of points of codimension-2 strata by Theorem 5.5. By Proposition 3.9 (1), Corollary 4.15, and what was proved above, we have that the orbifold fundamental group can be obtained from invariants of the orbifold differential structure. From Remark 5.2 (2), if $(\mathbb{R}^n, \Gamma_x, \phi)$ is a chart of an orbifold X in which $x = \phi(0)$ and Γ_x is its isotropy group (which always exists by Remark 3.7), then the orbifold fundamental group of $\phi(U)$ is isomorphic to Γ_x . Thus, $|\Gamma_x|$ can be obtained from invariants of the orbifold differential structure. \square

Definition 5.11 (Link at a point). Let X be an orbifold of dimension n , and let $x \in X$. Let Γ_x be an isotropy group of x with associated chart $(\mathbb{R}^n, \Gamma_x, \phi)$. Without loss of generality, assume that $\Gamma_x \subset O(n)$. Then \mathbb{S}^{n-1} is a Γ_x -invariant set. Define the *link* at x to be the image $S := \phi(\mathbb{S}^{n-1})$.

Lemma 5.12. *Let X be an orbifold of dimension n , and let $x \in X$ with isotropy group Γ_x . Let $(\mathbb{R}^n, \Gamma_x, \phi)$ be a chart with $\phi(0) = x$ and such that $\Gamma_x \subset O(n)$. Then, there is a diffeomorphism Φ_S from the link S at x as a differential subspace of X to $\mathbb{S}^{n-1}/\Gamma_x$, where the action of Γ_x on \mathbb{S}^{n-1} is the restriction of that on \mathbb{R}^n in the chart. Moreover, S has a smooth stratification given by the connected components of $S \cap M$ where M runs over strata of X , and Φ_S preserves this with respect to the orbifold stratification on $\mathbb{S}^{n-1}/\Gamma_x$. Finally, Φ_S preserves the orders of points contained in the codimension-2 strata of S .*

Proof. The existence of the diffeomorphism Φ_S follows from the definition of a chart and the following commutative diagram.

$$\begin{array}{ccc} \mathbb{S}^{n-1} & \longrightarrow & \mathbb{R}^n \\ \phi|_{\mathbb{S}^{n-1}} \downarrow & & \downarrow \phi \\ S & \longrightarrow & \phi(\mathbb{R}^n) \end{array}$$

Let M be a stratum of X , and let $y \in C \subseteq S \cap M$ where C is a connected component of $S \cap M$. Then, there exists a subgroup H of Γ

such that $y \in \phi(\mathbb{R}_{(H)}^n)$. Note that $y \neq \phi(0)$. Also, $\mathbb{R}_{(H)}^n$ is a cone, that is, it is closed under scalar multiplication by non-zero real numbers. Thus, $y \in \phi(\mathbb{S}_{(H)}^{n-1})$, and since $y \in C$ is arbitrary, we have

$$C \subseteq \phi(\mathbb{S}_{(H)}^{n-1}).$$

For the opposite inclusion, fix $y \in \mathbb{S}^{n-1}/\Gamma_x$, and let H be a subgroup of Γ_x such that $y \in \phi(\mathbb{S}_{(H)}^{n-1})$. Then, similar to the previous argument, $y \in \phi(\mathbb{R}_{(H)}^n)$. Thus, there is a stratum M of the orbifold stratification on X such that, if C is the connected component of the stratum $\phi(\mathbb{S}_{(H)}^{n-1})$ containing y , then $C \subseteq S \cap M$. Finally, the fact that Φ_S preserves the orders of points follows immediately from the definitions. \square

Proof of Main theorem. First, recall that the dimension of X is a topological invariant, see Remark 4.5. Moreover, this topology, the orbifold stratification and the order of points in codimension-2 strata are all invariants of the differential structure $C^\infty(X)$ by Proposition 3.9 (1), Corollary 4.15 and Theorem 5.10, respectively.

If the dimension of X is 0, then X is a countable set of points with the discrete topology, and the orbifold atlas is trivial.

Now, assume that X has dimension 1. Then there are no codimension-2 strata, and applying Theorem 5.5 to X yields the following isotropy groups Γ_x at each point $x \in X$:

- (i) If x is in the open dense stratum, then $\Gamma_x = \{1\}$.
- (ii) If x is a codimension-1 stratum, then $\Gamma_x = \mathbb{Z}_2$.

Thus, we can construct the following charts.

- (1) If x is in a codimension-0 stratum, then there is an open neighborhood U of x such that

$$U \cong \mathbb{R} \cong \mathbb{R}/\{1\},$$

that is, U is diffeomorphic to a connected open interval of \mathbb{R} . We thus take a chart $(\mathbb{R}, \{1\}, \phi)$ where ϕ is the diffeomorphism from \mathbb{R} onto U .

- (2) If x is equal to a codimension-1 stratum, then there is only one non-trivial action of \mathbb{Z}_2 on \mathbb{R} given by $\pm 1 \cdot u = \pm u$. So, we must

take as a chart near x the triple

$$(U, \Gamma_x, \phi) = (\mathbb{R}, \mathbb{Z}_2, \phi)$$

where $\phi: \mathbb{R} \rightarrow \phi(U)$ is the quotient map of this \mathbb{Z}_2 -action.

This completes the one-dimensional case.

Next, assume that the dimension of X is 2. Applying Theorem 5.5 to X yields the following four possible isotropy groups Γ_x at each point $x \in X$:

- (i) If x is in a codimension-0 stratum, then $\Gamma_x = \{1\}$.
- (ii) If x is in a codimension-1 stratum, then $\Gamma_x = \mathbb{Z}_2$.
- (iii) If x is equal to a codimension-2 stratum that is in the closure of a codimension-1 stratum, and the order of x is $2k$, then $\Gamma_x = D_k$.
- (iv) If x is equal to a codimension-2 stratum that is not in the closure of a codimension-1 stratum, and the order of x is k , then $\Gamma_x = \mathbb{Z}_k$.

We construct the following charts.

- (1) If x is in a codimension-0 stratum, then there is an open neighborhood U of x such that $U \cong \mathbb{R}^2 \cong \mathbb{R}^2/\{1\}$. Similar to what was done for the one-dimensional case, take $(\mathbb{R}^2, \{1\}, \phi)$ to be a chart.
- (2) If x is in a codimension-1 stratum, then there is an open neighborhood U of x such that $U \cong \mathbb{R}^2/\mathbb{Z}_2$ where \mathbb{Z}_2 acts by reflection through some line passing through the origin.
- (3) If x is equal to a codimension-2 stratum that is in the closure of a codimension-1 stratum, and the order of x is $2k$, then we take as a chart near x the triple $(\mathbb{R}^2, D_k, \phi)$ where D_k acts on $\mathbb{R}^2 \cong \mathbb{C}$ by reflections, see Example 3.3.
- (4) If x is equal to a codimension-2 stratum that is not in the closure of a codimension-1 stratum, and the order of x is k , then we take as a chart near x the triple $(\mathbb{R}^2, \mathbb{Z}_k, \phi)$ where \mathbb{Z}_k acts on $\mathbb{R}^2 \cong \mathbb{C}$ by rotations, see Example 3.3.

By Theorem 3.4 and Remark 3.5, this exhausts all the possible scenarios in the two-dimensional case.

Now, as an induction hypothesis, assume that we can reconstruct an atlas for any orbifold of dimension n . Let X be an orbifold of dimension $n+1$. Fix $x \in X$, and let Γ_x be its isotropy group at x . Our goal is to reconstruct a chart $(\mathbb{R}^{n+1}, \Gamma_x, \phi)$ about x such that $\phi(0) = x$.

Let S be the link at x . By Lemma 5.12, there is a strata-preserving diffeomorphism that preserves the order of points on codimension-2 strata from S to \mathbb{S}^n/Γ_x for some action of Γ_x on \mathbb{S}^n . By our induction hypothesis, we now have enough information on S to obtain an orbifold atlas on S .

Now, S is a good orbifold. Thus, by Remark 5.2 (3), there is a simply-connected manifold that serves as a universal orbifold covering space for S , and this is unique up to equivariant diffeomorphism. Hence (safely assuming that $n \geq 2$), \mathbb{S}^n is the universal orbifold covering space for S , with the action of Γ_x given by deck transformations. Extend this action to the unique orthogonal action of Γ_x on \mathbb{R}^{n+1} such that

$$\gamma \cdot x := \begin{cases} 0 & \text{if } x = 0, \\ |x|(\gamma \cdot x/|x|) & \text{if } x \neq 0. \end{cases}$$

This finishes the construction of the chart $(\mathbb{R}^{n+1}, \Gamma_x, \phi)$. Since $x \in X$ is arbitrary, we are done. \square

6. Proof of Theorem A. The purpose of this section is to express the main theorem in terms of a functor. We choose to use the weak 2-category of effective proper étale Lie groupoids with bibundles as arrows and isomorphisms of bibundles as 2-arrows. We give the definition of these objects, arrows and 2-arrows, but the reader should consult [16] for a more detailed exposition. Similar categories have been developed, toward which the main theorem could be tailored, which we shall not do here. These categories include that of Hilsun-Skandalis [11], as well as the calculus of fractions developed by Pronk [24].

Set $G = (G_1 \rightrightarrows G_0)$ as our notation for a Lie groupoid, with $s : G_1 \rightarrow G_0$ and $t : G_1 \rightarrow G_0$ the source and target maps, respectively.

Definition 6.1 (Effective proper étale Lie groupoid).

(1) Let G be a Lie groupoid. Then G is *étale* if the source and target maps are local diffeomorphisms.

(2) Let M be a manifold. Denote the topological groupoid with objects the set of points of M and arrows the space of germs of (local) diffeomorphisms equipped with the sheaf topology by $\Gamma(M)$. This is an étale groupoid in the topological sense, i.e., the source and target

maps are local homeomorphisms. It attains a smooth structure via these local homeomorphisms.

(3) Let G be an étale Lie groupoid, and let $\Gamma(G_0)$ be the groupoid of germs associated to G_0 . Then, for each arrow $(g : x \rightarrow y) \in G_1$, there exist an open neighborhood U of g and a diffeomorphism

$$\phi_g = t|_U \circ (s|_U)^{-1}.$$

The germ of ϕ_g is an element of $\Gamma(G_0)$, and we have a smooth functor $\gamma : G \rightarrow \Gamma(G_0)$, sending g to ϕ_g , and objects to themselves. G is *effective* if γ is faithful.

(4) A Lie groupoid G is *proper* if the smooth functor $(s, t) : G_1 \rightarrow G_0 \times G_0$ is a proper map between manifolds.

Remark 6.2. Any effective proper étale Lie groupoid is Morita equivalent to the effective groupoid associated to an orbifold constructed using pseudogroups. This construction yields a one-to-one correspondence between orbifolds in the classical sense and Morita equivalence classes of effective proper étale Lie groupoids, see [20] for definitions, details, and a proof ([20, Theorem 5.32]). The important point for our purposes is that, given one of these groupoids $G_1 \rightrightarrows G_0$, the orbit space G_0/G_1 is the underlying set of the orbifold.

Definition 6.3 (Bibundle).

(1) Let $G = (G_1 \rightrightarrows G_0)$ and $H = (H_1 \rightrightarrows H_0)$ be Lie groupoids. Then, a *bibundle* $P : G \rightarrow H$ is a manifold P equipped with a left groupoid action of G with anchor map $a_L : P \rightarrow G_0$ and a right groupoid action of H with anchor map $a_R : P \rightarrow H_0$ such that the following are satisfied.

- (a) The two actions commute.
- (b) $a_L : P \rightarrow G_0$ is a principal (right) H -bundle.
- (c) a_R is G -invariant.

$$\begin{array}{ccccc} G_1 & & P & & H_1 \\ \Downarrow & \nearrow a_L & & \searrow a_R & \Downarrow \\ G_0 & & & & H_0 \end{array}$$

(2) Let G and H be Lie groupoids, and let $P : G \rightarrow H$ and $Q : G \rightarrow H$ be bibundles between them. An *isomorphism of bibundles* $\alpha : P \rightarrow Q$ is a diffeomorphism that is $(G\text{-}H)$ -equivariant, that is, $\alpha(h \cdot p \cdot g) = h \cdot \alpha(p) \cdot g$.

(3) Let $G = (G_1 \rightrightarrows G_0)$ and $H = (H_1 \rightrightarrows H_0)$ be Lie groupoids, and let $P : G \rightarrow H$ be a bibundle between them. P is *invertible* if its right anchor map $a_R : P \rightarrow H_0$ makes P into a principal (left) G -bundle, defined similarly to a principal (right) bundle. In this case, we can construct a bibundle $P^{-1} : H \rightarrow G$ by switching the anchor maps, inverting the left G -action into a right G -action, and doing the opposite for the H -action. Then, $P \circ P^{-1}$ is isomorphic to the bibundle corresponding to the identity map on H , and $P^{-1} \circ P$ is isomorphic to the bibundle representing the identity map on G . In the case that G and H admit an invertible bibundle between them, they are called *Morita equivalent* groupoids.

Definition 6.4 (Weak 2-category **Orb**). Lie groupoids with bibundles as arrows and isomorphisms of bibundles as 2-arrows form a weak 2-category, see [16] for more details. Effective proper étale Lie groupoids form a full (weak) subcategory and will be denoted by **Orb**. Many view this (or slight modifications to this definition) to be “the” category of effective orbifolds.

Lemma 6.5. *Let $G_1 \rightrightarrows G_0$ and $H_1 \rightrightarrows H_0$ be two effective proper étale Lie groupoids, and let P be a bibundle between them. Then P descends to a unique smooth map $\bar{P} : G_0/G_1 \rightarrow H_0/H_1$ such that the following diagram commutes.*

$$\begin{array}{ccccc}
 G_1 & & P & & H_1 \\
 \Downarrow & \nearrow a_L & & \searrow a_R & \Downarrow \\
 G_0 & & & & H_0 \\
 \downarrow \pi_G & & & & \downarrow \pi_H \\
 G_0/G_1 & \xrightarrow{\quad \bar{P} \quad} & & & H_0/H_1
 \end{array}$$

Moreover, if P is a Morita equivalence, then \overline{P} is a diffeomorphism. Finally, if Q is another bibundle between $G_1 \rightrightarrows G_0$ and $H_1 \rightrightarrows H_0$ that is isomorphic to P , then $\overline{P} = \overline{Q}$.

Proof. Fix $x \in G_0$ and denote by $[x]$ the point $\pi_G(x)$. Then define

$$\overline{P}([x]) := \pi_H \circ a_R \circ \sigma(x)$$

where σ is a smooth local section of a_L about x . Such a local section exists since a_L is a surjective submersion, by definition of a principal H -bundle.

We claim that \overline{P} is independent of the chosen local section, as well as the representative x . Indeed, let $y \in G_0$ be another representative of $[x]$. Then there exists $g \in G_1$ such that $s(g) = x$ and $t(g) = y$. So, $a_L(g \cdot \sigma(x)) = y$, and hence, $g \cdot \sigma(x) \in a_L^{-1}(y)$. Let σ' be a local section of a_L about y . Since $a_L : P \rightarrow G_0$ is a principal H -bundle, there exists $h \in H_1$ such that $(g \cdot \sigma(x)) \cdot h = \sigma'(y)$. Since the G - and H -actions on P commute and a_R is G -invariant, it follows that $a_R(\sigma'(y)) = s(h)$. Since $a_R(\sigma(x)) = t(h)$, we have

$$\pi_H(a_R(\sigma(x))) = \pi_H(a_R(\sigma'(y))).$$

To show uniqueness, consider $p \in P$. In order for the above diagram to commute, we require that $\pi_G(a_L(p))$ be sent to $\pi_H(a_R(p))$. This defines a unique map, which is equal to \overline{P} .

Denote the quotient differential structures on G_0/G_1 and H_0/H_1 by $C^\infty(G_0/G_1)$ and $C^\infty(H_0/H_1)$, respectively. Denote the spaces of smooth invariant functions on G_0 and H_0 by $C^\infty(G_0)^{G_1}$ and $C^\infty(H_0)^{H_1}$, respectively. Fix $f \in C^\infty(H_0/H_1)$. Then, there exists $\tilde{f} \in C^\infty(H_0)^{H_1}$ such that $\tilde{f} = \pi_H^* f$. By definition of a right H -action, $a_R^* \tilde{f}$ is H -invariant on P . Since $a_L : P \rightarrow G_0$ is a principal H -bundle, $a_R^* \tilde{f}$ descends to a smooth function $\tilde{f}' \in C^\infty(G_0)$:

$$a_L^* \tilde{f}' = a_R^* \tilde{f}.$$

By definition of a left G -action, and using the fact that a_R is G -invariant, we obtain that $\tilde{f}' \in C^\infty(G_0)^{G_1}$. Therefore, \tilde{f}' descends to a function $f' \in C^\infty(G_0/G_1)$, and $f' = \overline{P}^* f$.

Next, P is a Morita equivalence if and only if P is invertible, that is, $a_R : P \rightarrow H_0$ is a principal G -bundle. It follows immediately that, in this case, \overline{P} is a diffeomorphism.

Finally, the fact that isomorphic bibundles P and Q descend to the same smooth map $\overline{P} = \overline{Q}$ between orbit spaces immediately arises from the uniqueness of \overline{P} and the fact that α is $(G\text{-}H)$ -equivariant. \square

Proof of Theorem A. We define a functor $F : \mathbf{Orb} \rightarrow \mathbf{DiffSp}$ as follows. Let $G_1 \rightrightarrows G_0$ be an effective proper étale Lie groupoid. Then, $F(G_1 \rightrightarrows G_0)$ is the orbit space G_0/G_1 equipped with the quotient differential structure. Let $G_1 \rightrightarrows G_0$ and $H_1 \rightrightarrows H_0$ be two effective proper étale Lie groupoids, and let P be a bibundle between them. Then, define $F(P)$ to be \overline{P} as defined in Lemma 6.5. F trivialises 2-arrows by Lemma 6.5.

To show that F is a functor, note that, if $P : (G_1 \rightrightarrows G_0) \rightarrow (G_1 \rightrightarrows G_0)$ is the identity bibundle, then \overline{P} is the identity map on G_0/G_1 . We also need to show that F respects composition. Let $G_1 \rightrightarrows G_0$, $H_1 \rightrightarrows H_0$ and $K_1 \rightrightarrows K_0$ be effective proper étale Lie groupoids, and let $P : (G_1 \rightrightarrows G_0) \rightarrow (H_1 \rightrightarrows H_0)$ and $Q : (H_1 \rightrightarrows H_0) \rightarrow (K_1 \rightrightarrows K_0)$ be bibundles. The composition of P and Q is defined to be the quotient

$$Q \circ P := (P \times_{H_0} Q)/H_1$$

where $P \times_{H_0} Q$ is the fibred product with respect to anchor maps $a_R^P : P \rightarrow H_0$ and $a_L^Q : Q \rightarrow H_0$, on which $H_1 \rightrightarrows H_0$ acts diagonally. Note that $F(Q \circ P) = \overline{Q \circ P}$ is the unique map causing the following diagram to commute.

$$\begin{array}{ccc}
 & P \times_{H_0} Q & \\
 \text{pr}_1 \swarrow & & \searrow \text{pr}_2 \\
 P & & Q \\
 \downarrow \pi_G \circ a_L^P & & \downarrow \pi_K \circ a_R^Q \\
 G_0/G_1 & \xrightarrow{\overline{Q \circ P}} & K_0/K_1
 \end{array}$$

To show that $\overline{Q \circ P} = \overline{Q} \circ \overline{P}$, it is enough to show that, for any $(p, q) \in P \times_{H_0} Q$, we have

$$\overline{Q} \circ \overline{P}(\pi_G \circ a_L^P(p)) = \pi_K \circ a_R^Q(q).$$

But this reduces to showing that

$$\pi_H \circ a_R^P(p) = \pi_H \circ a_L^Q(q),$$

and this is automatic by definition of $P \times_{H_0} Q$. We have shown that F is a functor.

Now, let $G_1 \rightrightarrows G_0$ and $H_1 \rightrightarrows H_0$ be effective proper étale Lie groupoids. Then, $X_G := G_0/G_1$ and $X_H := H_0/H_1$ are naturally equipped with orbifold atlases. Assume that $(X_G, C^\infty(X_G))$ and $(X_H, C^\infty(X_H))$ are diffeomorphic as differential spaces. Without loss of generality, we may identify the underlying sets via this diffeomorphism. By the main theorem, the orbifold atlases for X_G and X_H can be reconstructed from $C^\infty(X_G)$ and $C^\infty(X_H)$, and these orbifold atlases are equivalent since they are constructed out of the same invariants of differential spaces. We conclude that $G_1 \rightrightarrows G_0$ and $H_1 \rightrightarrows H_0$ are Morita equivalent, that is, isomorphic in **Orb**. \square

Remark 6.6.

- (1) Note that, while composition of bibundles is only weakly associative, by Lemma 6.5, F carries this weak associativity to true associativity.
- (2) F is neither full nor faithful, see Example 7.2.
- (3) In the proof above, we did not use the fact that the Lie groupoids are effective proper étale to show that F is a functor; this was only used to show that F is essentially injective. Indeed, F is a restriction of a functor from the weak 2-category of Lie groupoids to differential spaces.

7. Proof of Theorem B. This section is designed for readers with some familiarity with the category **Diffeol** of diffeological spaces. The main resource on diffeology is the book by Iglesias-Zemmour [13], although for purposes of this section regarding diffeological orbifolds, the required details appear in [12]. The purpose of this section is as follows. Iglesias-Zemmour, et al. [12] proved that there is a one-to-one correspondence between orbifolds in the classical sense and diffeological orbifolds. We tailor this result into a functor $G : \mathbf{Orb} \rightarrow \mathbf{Diffeol}$ that

is essentially injective on objects, which is Theorem B. We give two proofs that this functor is essentially injective. The first comes directly from the result of Iglesias-Zemmour, et al. For the second, we introduce a functor

$$\Phi : \mathbf{Diffeol} \longrightarrow \mathbf{DiffSp}$$

that sends a diffeological space to its underlying set equipped with the ring of diffeologically smooth real-valued functions. We show that $F = \Phi \circ G$. By the main theorem, F is essentially injective, and so it follows that G is as well. The functor Φ is studied in [31, Chapter 2], as well as [2], and more details about it can be found there.

Definition 7.1 (Diffeological orbifold). A *diffeological orbifold* is a diffeological space that is locally diffeologically diffeomorphic to quotient diffeological spaces of the form \mathbb{R}^n/Γ , where $\Gamma \subset \mathrm{GL}(\mathbb{R}^n)$ is a finite subgroup, see [12, Definition 6].

Proof of Theorem B. Similar to the functor $F : \mathbf{Orb} \rightarrow \mathbf{DiffSp}$ defined in Theorem A, G is the restriction of a functor from the weak 2-category of Lie groupoids with bibundles as arrows and isomorphisms of bibundles as 2-arrows to diffeological spaces. See [32, Section 4] for details on this functor between Lie groupoids and diffeological spaces.

The fact that G is essentially injective follows from the result of Iglesias-Zemmour, et al., see [12, Proposition 38, Theorems 39, 46], and from Remark 6.2. \square

The functor G is neither faithful nor full, as the next example illustrates.

Example 7.2. These examples are due to Iglesias-Zemmour, et al., and appear as [12, Examples 24, 25]. Let $\rho_n : \mathbb{R} \rightarrow [0, 1]$ be a smooth function with non-empty support inside

$$\left[\frac{1}{n+1}, \frac{1}{n} \right].$$

Let $\sigma = (\sigma_1, \sigma_2, \dots) \in \{-1, 1\}^{\mathbb{N}}$, and define $f_\sigma : \mathbb{R} \rightarrow \mathbb{R}$ to be the smooth function

$$f_\sigma(x) := \begin{cases} \sigma_n e^{-1/x} \rho_n(x) & \text{if } 1/(n+1) < x \leq 1/n \text{ with } n \in \mathbb{N}, \\ 0 & \text{if } x > 1 \text{ or } x \leq 0. \end{cases}$$

For any σ , the function f_σ descends to the same diffeologically smooth function $f : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}_2$ (where \mathbb{Z}_2 acts by reflection). Thus, the functor G is not faithful.

Next, set $r = \sqrt{x^2 + y^2}$ for $(x, y) \in \mathbb{R}^2$. Define $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ to be the smooth function

$$g(x, y) := \begin{cases} e^{-r} \rho_n(r)(r, 0) & \text{if } 1/(n+1) < r \leq 1/n \text{ and } n \text{ is even,} \\ e^{-r} \rho_n(r)(x, y) & \text{if } 1/(n+1) < r \leq 1/n \text{ and } n \text{ is odd,} \\ 0 & \text{if } r > 1 \text{ or } r = 0. \end{cases}$$

Then, for any integer $k \geq 2$, the function g descends to a diffeologically smooth function

$$\bar{g} : \mathbb{R}^2/\mathbb{Z}_k \longrightarrow \mathbb{R}^2/\mathbb{Z}_k$$

(where \mathbb{Z}_k acts by rotation). While \bar{g} has a smooth lift $\mathbb{R}^2 \rightarrow \mathbb{R}^2$, this lift is h_n -equivariant when restricted to the annulus

$$\frac{1}{n+1} < r \leq \frac{1}{n},$$

where $h_n : \mathbb{Z}_k \rightarrow \mathbb{Z}_k$ is a group homomorphism. In particular, if n is even, then h_n is the trivial homomorphism; whereas if n is odd, then h_n must be the identity. Thus, there certainly is no functor, nor even a bibundle, between the groupoid

$$\mathbb{Z}_k \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$$

and itself that corresponds to f . Thus, G is not full.

Definition 7.3 (The functor Φ). Let (X, \mathcal{D}) be a diffeological space. Define $\Phi\mathcal{D}$ by

$$\Phi\mathcal{D} := \{f : X \rightarrow \mathbb{R} \mid f \circ p \text{ is smooth for all } p \in \mathcal{D}\}.$$

Then, $\Phi\mathcal{D}$ is a differential structure on X , see [31, Lemma 2.42]. This extends to a functor $\Phi : \mathbf{Diffeol} \rightarrow \mathbf{DiffSp}$ which sends diffeologically smooth maps to themselves (see the proof of [31, Theorem 2.48]). Note that $\Phi\mathcal{D}$ is just the ring of diffeologically smooth functions of (X, \mathcal{D}) .

Proposition 7.4 ($F = \Phi \circ G$). *The functor $F : \mathbf{Orb} \rightarrow \mathbf{DiffSp}$ is equal to the composition $\Phi \circ G$.*

Proof. We only need to show that, given a diffeological orbifold (X, \mathcal{D}) , $\Phi\mathcal{D}$ is equal to the orbifold differential structure on X . First, we note that the topology induced by the diffeology on X is equal to the standard orbifold topology, see [13, Article 2.12], which, in turn, is equal to the functional topology induced by $C^\infty(X)$, see Proposition 3.9 (1).

Now, from [12, Proposition 38, Theorems 39, 46], we have that the local diffeomorphisms defining the diffeological orbifold structure are exactly the charts of the corresponding orbifold in the sense of Definition 3.1. Let $f \in \Phi\mathcal{D}$. Then, locally where (X, \mathcal{D}) is diffeologically diffeomorphic to \mathbb{R}^n/Γ , it follows from the definition of a quotient diffeology that f will restrict and lift to a Γ -invariant function on \mathbb{R}^n . But, this is exactly the pullback of f via an orbifold chart. Thus, $f \in C^\infty(X)$. In the reverse direction, if $f \in C^\infty(X)$, then locally at a chart of the form $(\mathbb{R}^n, \Gamma, \phi)$, which exists at every point by Remark 3.7, we have that f restricts and lifts to a Γ -invariant function on \mathbb{R}^n . Hence, since the quotient map is a plot of \mathcal{D} , it descends to a (local) diffeologically smooth function, that is, a function in $\Phi\mathcal{D}$. Since smoothness is a local property, the result follows. \square

Corollary 7.5. *G is an essentially injective functor.*

Proof. This is immediate from Proposition 7.4 and the fact that F is an essentially injective functor, due to the main theorem. \square

Remark 7.6.

- (1) In general, the functor $\Phi : \mathbf{Diffeol} \rightarrow \mathbf{DiffSp}$ is not injective on objects, as the next example illustrates. Also, while it is faithful, it is not full (see, for example, [31, end of Example 2.67]). It remains an open question whether or not Φ restricted to diffeological orbifolds is full.
- (2) Since G is neither faithful nor full, see Example 7.2, it follows from Proposition 7.4 that F is neither faithful nor full.

Example 7.7 (Rotations of \mathbb{R}^n). Let $O(n)$ act on \mathbb{R}^n by rotations about the origin. Then the quotient diffeology \mathcal{D}_n on $\mathbb{R}^n/O(n)$ depends on n (see [13, Exercise 50] with solution at the back of the book). The corresponding quotient differential structure which is equal to $\Phi\mathcal{D}_n$, however, is equal to $C^\infty([0, \infty))$, the subspace differential structure of $[0, \infty) \subset \mathbb{R}$.

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