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A COMPUTATION OF BUCHSBAUM-RIM FUNCTIONS OF TWO VARIABLES IN A SPECIAL CASE

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ABSTRACT. In this paper, we will compute Buchsbaum-Rim functions of two variables associated to a parameter matrix of a special form over a one-dimensional Cohen-Macaulay local ring, and we will determine when the function coincides with the Buchsbaum-Rim polynomial. As a consequence, we have that there exists the case where the function does not coincide with the polynomial function, which should be contrasted with the ordinary Buchsbaum-Rim function of single variable.

1. Introduction. Let $(R, \mathfrak{m})$ be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ of dimension $d > 0$, and let C be a nonzero R-module of finite length. Let $\varphi : R^n \to R^r$ be an R-linear map of free modules with $C = \operatorname{Coker} \varphi$ as the cokernel of φ, and set $M := \operatorname{Im}\varphi \subset F := R^r$. Then one may consider the function,

$$\lambda(p) := \ell_R([\operatorname{Coker} \operatorname{Sym}_R(\varphi)]_p) = \ell_R(S_p/M^p),$$

where S_p (respectively M^p) is a homogeneous component of degree p of $S = \operatorname{Sym}_R(F)$ (respectively $R[M] = \operatorname{Im} \operatorname{Sym}_R(\varphi)$). Buchsbaum-Rim **[3]** first introduced and studied this type of function and proved that $\lambda(p)$ is eventually a polynomial of degree $d + r - 1$, which we call the *Buchsbaum-Rim polynomial.* Then they defined a multiplicity of C as

$$e(C) := (\text{The coefficient of } p^{d+r-1} \text{ in the polynomial}) \times (d + r - 1)!,$$

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which we now call the *Buchsbaum-Rim multiplicity* of C. They also proved that the multiplicity is independent of the choice of φ. The multiplicity $e(C)$ coincides with the ordinary Hilbert-Samuel multiplicity of an ideal I when C is a cyclic module R/I.

Buchsbaum and Rim also introduced the notion of a parameter matrix, which generalizes the notion of a system of parameters. A matrix (a linear map of free modules) φ over R of size $r \times n$ (from R^n to R^r) is said to be a *parameter matrix* for R, if the following three conditions are satisfied:

(i) $\operatorname{Coker} \varphi$ has finite length,
(ii) $d = n - r + 1$,
(iii) $\operatorname{Im}\varphi \subset \mathfrak{m}R^r$.

Then it is known that, if R is Cohen-Macaulay and φ is a parameter matrix, then there exist formulas

$$e(C) = \ell_R(C) = \ell_R(R/\operatorname{Fitt}_0(C))$$

for the Buchsbaum-Rim multiplicity [**2, 3, 6**] and for the Buchsbaum-Rim function [**1**],

$$\lambda(p) = e(C)\binom{p+d+r-2}{d+r-1},$$

for all $p \geq 0$. It is also known [**5**] that, for any $p \geq 0$, the inequality,

$$\lambda(p) \geq e(C)\binom{p+d+r-2}{d+r-1},$$

always holds true for a parameter matrix φ even if R is not Cohen-Macaulay, and the equality for some $p > 0$ characterizes the Cohen-Macaulay property of the ring R.

Kleiman and Thorup [**9, 10**] and Kirby and Rees [**7, 8**] introduced another kind of multiplicity associated to C, which is related to the Buchsbaum-Rim multiplicity (see also [**4**]). They considered the function of two variables,

$$\Lambda(p,q) := \ell_R(S_{p+q}/M^p S_q),$$

and proved that $\Lambda(p,q)$ is eventually a polynomial of total degree $d+r-1$. Then they defined a sequence of multiplicities, for $j =$

$0, 1, \dots, d + r - 1$,

$$e^j(C) := (\text{The coefficient of } p^{d+r-1-j}q^j \text{ in the polynomial}) \\ \times (d + r - 1 - j)!j!,$$

and proved that $e^j(C)$ is independent of the choice of φ. Moreover, they proved that

$$e(C) = e^0(C) \geq e^1(C) \geq \cdots \geq e^{r-1}(C) > e^r(C) = \cdots = e^{d+r-1}(C) = 0,$$

where $r = \mu_R(C)$ is the minimal number of generators of C. Thus, we call $e^j(C)$ the *jth Buchsbaum-Rim multiplicity* of C. Then it is natural to ask the following.

Problem 1.1. Let $\varphi : R^n \to R^r$ be a parameter matrix with $C = \operatorname{Coker} \varphi$. Suppose that R is Cohen-Macaulay. Then:

(i) does there exist a simple formula for the Buchsbaum-Rim multiplicities $e^j(C)$ for $j = 1, 2, \dots, r - 1$?
(ii) Does the function $\Lambda(p, q)$ coincide with a polynomial function?

In this paper, we will try to calculate the function $\Lambda(p, q)$ and multiplicities $e^j(C)$ in a special case where C is a direct sum of cyclic modules R/Q_i and Q_i is a parameter ideal in a one dimensional Cohen-Macaulay local ring R. In particular, in the case $C = R/Q_1 \oplus R/Q_2$, we will determine when the function $\Lambda(p, q)$ coincides with a polynomial function (Theorem 2.4). As a consequence, we have that there exists the case where the function $\Lambda(p, q)$ does not coincide with a polynomial function. This should be contrasted with a result of Brennan, Ulrich and Vasconcelos [**1**] as stated above: the ordinary Buchsbaum-Rim function $\lambda(p) = \Lambda(p, 0)$ coincides with the Buchsbaum-Rim polynomial for all $p \geq 0$ in the case where R is Cohen-Macaulay and φ is a parameter matrix.

2. Computation in a special case. In what follows, let $(R, \mathfrak{m})$ be a one dimensional Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$. Let $r > 0$ be a fixed positive integer, and let $Q_1, Q_2, \dots, Q_r$ be parameter ideals in R with $Q_i = (x_i)$ for $i = 1, 2, \dots, r$. We set $a_i = \ell_R(R/Q_i) = e(R/Q_i)$ for $i = 1, 2, \dots, r$. Let $\varphi : R^r \to R^r$ be an

R-linear map represented by a parameter matrix,

$$\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{pmatrix},$$

and let $C = \operatorname{Coker} \varphi$, which is a direct sum $R/Q_1 \oplus R/Q_2 \oplus \cdots \oplus R/Q_r$ of cyclic modules. Let $S = \mathrm{Sym}_R(R^r)$ be the symmetric algebra of R^r, and let $N = \mathrm{Im}\varphi$ be the image of φ. With this notation, we want to compute the following:

- the function $\Lambda(p,q) = \ell_R(S_{p+q}/N^p S_q)$ for $p, q \geq 0$,
- the polynomial $\Lambda(p,q) = \ell_R(S_{p+q}/N^p S_q)$ for $p, q \gg 0$,
- the multiplicities $e^j(C)$ for $j = 1, 2, \ldots, r-1$.

In order to calculate the above, we fix a free basis $\{t_1, t_2, \ldots, t_r\}$ for R^r. Then $S = R[t_1, t_2, \ldots, t_r]$ is a polynomial ring and $N = Q_1 t_1 + Q_2 t_2 + \cdots + Q_r t_r \subset S_1 = Rt_1 + Rt_2 + \cdots + Rt_r$. Thus, for any $p, q \geq 0$, the module $N^p S_q$ can be described as follows:

$$\begin{aligned} N^p S_q &= \Bigg(\sum_{\substack{|\boldsymbol{j}|=p \\ \boldsymbol{j} \geq \boldsymbol{0}}} \boldsymbol{Q}^{\boldsymbol{j}} \boldsymbol{t}^{\boldsymbol{j}} \Bigg) \Bigg(\sum_{\substack{|\boldsymbol{k}|=q \\ \boldsymbol{k} \geq \boldsymbol{0}}} R\boldsymbol{t}^{\boldsymbol{k}} \Bigg) \\ &= \sum_{\substack{|\boldsymbol{\ell}|=p+q \\ \boldsymbol{\ell} \geq \boldsymbol{0}}} \Bigg(\sum_{\substack{|\boldsymbol{k}|=q \\ \boldsymbol{0} \leq \boldsymbol{k} \leq \boldsymbol{\ell}}} \boldsymbol{Q}^{\boldsymbol{\ell}-\boldsymbol{k}} \Bigg) \boldsymbol{t}^{\boldsymbol{\ell}} \subset S_{p+q} \\ &= \sum_{\substack{|\boldsymbol{\ell}|=p+q \\ \boldsymbol{\ell} \geq \boldsymbol{0}}} R\boldsymbol{t}^{\boldsymbol{\ell}}. \end{aligned}$$

Here we use the multi-index notation: for a vector $\boldsymbol{i} = (i_1, \ldots, i_r) \in \mathbb{Z}^r_{\geq 0}$, we denote $\boldsymbol{Q}^{\boldsymbol{i}} = Q_1^{i_1} \cdots Q_r^{i_r}$, $\boldsymbol{t}^{\boldsymbol{i}} = t_1^{i_1} \cdots t_r^{i_r}$ and $|\boldsymbol{i}| = i_1 + \cdots + i_r$. For any vector $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_r) \in \mathbb{Z}^r_{\geq 0}$ such that $|\boldsymbol{\ell}| = p + q$, we define the ideal in R as follows:

$$J_{p,q}(\boldsymbol{\ell}) := \sum_{\substack{|\boldsymbol{k}|=q \\ \boldsymbol{0} \leq \boldsymbol{k} \leq \boldsymbol{\ell}}} \boldsymbol{Q}^{\boldsymbol{\ell}-\boldsymbol{k}} = \sum_{\substack{|\boldsymbol{i}|=p \\ \boldsymbol{0} \leq \boldsymbol{i} \leq \boldsymbol{\ell}}} \boldsymbol{Q}^{\boldsymbol{i}}.$$

Then the function $\Lambda(p,q)$ can be described as

$$\Lambda(p,q) = \ell_R(S_{p+q}/N^p S_q) = \sum_{\substack{|\boldsymbol{\ell}|=p+q \\ \boldsymbol{\ell}\geq \mathbf{0}}} \ell_R(R/J_{p,q}(\boldsymbol{\ell})),$$

for any $p,q \geq 0$. Thus, in order to compute the function $\Lambda(p,q)$, it is enough to compute the colength $\ell_R(R/J_{p,q}(\boldsymbol{\ell}))$ of the ideal $J_{p,q}(\boldsymbol{\ell})$.

We first consider the special case where the set of ideals $Q_1, Q_2, \ldots, Q_r$ is an ascending chain. The function $\Lambda(p,q)$ can be explicitly computed as follows in this case.

Proposition 2.1. *Suppose that $Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_r$. Then*

$$\begin{aligned}\Lambda(p,q) = (a_1 + \cdots + a_r)&\binom{p+r-1}{r} \\ &+ \sum_{i=1}^{r-1}(a_{i+1} + \cdots + a_r)\binom{p+r-i-1}{r-i}\binom{q+i-1}{i}\end{aligned}$$

for all $p,q \geq 0$, where $\binom{m}{n} = 0$ if $m < n$. In particular, the function $\Lambda(p,q)$ coincides with a polynomial function and

$$e^j(C) = \begin{cases} a_{j+1} + \cdots + a_r & (j = 0,1,\ldots,r-1) \\ 0 & (j = r) \end{cases}$$

Proof. Let us fix any $p,q \geq 0$. The case $r = 1$ is a well-known result for the Hilbert-Samuel function. The case $q = 0$ is a result of [**1**, Theorem 3.4] on the ordinary Buchsbaum-Rim function $\lambda(p) = \Lambda(p,0)$. So we may assume that $r \geq 2$ and $q \geq 1$. Suppose $Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_r$. Then the ideal $J_{p,q}(\boldsymbol{\ell})$ coincides with the ideal of the product of last p ideals of a sequence of ideals

$$\underbrace{\overbrace{Q_1, \ldots, Q_1}^{\ell_1}, \overbrace{Q_2, \ldots, Q_2}^{\ell_2}, \ldots, \overbrace{Q_r, \ldots, Q_r}^{\ell_r}}_{p+q}.$$

Hence its colength $\ell_R(R/J_{p,q}(\boldsymbol{\ell}))$ is the sum of last p integers of a sequence of integers

$$\underbrace{\overbrace{a_1,\ldots,a_1}^{\ell_1},\overbrace{a_2,\ldots,a_2}^{\ell_2},\ldots,\overbrace{a_r,\ldots,a_r}^{\ell_r}}_{p+q}. \tag{2.1}$$

To compute the sum

$$\sum_{\substack{|\boldsymbol{\ell}|=p+q\\ \boldsymbol{\ell}\geq \mathbf{0}}} \ell_R(R/J_{p,q}(\boldsymbol{\ell})),$$

we divide the sequence (2.1) at the $(p+1)$th integer from the end. If the $(p+1)$th integer from the end is a_i, then the sum of all last p integers of such sequences can be counted by

$$\binom{(q-1)+i-1}{i-1}\Bigg(\sum_{\substack{u_i+\cdots+u_r=p\\ u_i,\ldots,u_r\geq 0}} (u_ia_i+u_{i+1}a_{i+1}+\cdots+u_ra_r)\Bigg).$$

Therefore,

$$\begin{aligned}
\Lambda(p,q) &= \sum_{\substack{|\boldsymbol{\ell}|=p+q\\ \boldsymbol{\ell}\geq \mathbf{0}}} \ell_R(R/J_{p,q}(\boldsymbol{\ell}))\\
&= \sum_{i=1}^{r}\binom{(q-1)+i-1}{i-1}\Bigg(\sum_{\substack{u_i+\cdots+u_r=p\\ u_i,\ldots,u_r\geq 0}} (u_ia_i+u_{i+1}a_{i+1}+\cdots+u_ra_r)\Bigg)\\
&= \sum_{i=1}^{r}\binom{q+i-2}{i-1}(a_i+\cdots+a_r)\binom{(r-i+1)+p-1}{p}\frac{p}{r-i+1}\\
&= \sum_{i=1}^{r}(a_i+\cdots+a_r)\binom{q+i-2}{i-1}\binom{r-i+p}{p}\frac{p}{r-i+1}\\
&= \sum_{i=1}^{r}(a_i+\cdots+a_r)\binom{q+i-2}{i-1}\binom{r-i+p}{p-1}\\
&= (a_1+\cdots+a_r)\binom{p+r-1}{r}\\
&\quad + \sum_{i=1}^{r-1}(a_{i+1}+\cdots+a_r)\binom{p+r-i-1}{r-i}\binom{q+i-1}{i}. \qquad \square
\end{aligned}$$

As a direct consequence of Proposition 2.1, we obtain the following.

Corollary 2.2. *Let $(R, \mathfrak{m})$ be a* DVR. *Then, for an arbitrary R-module C of finite length, the function $\Lambda(p,q)$ associated to the module C coincides with a polynomial function. Moreover, we have the formula*

$$e^j(C) = \ell_R(R/\operatorname{Fitt}_j(C)) = e(R/\operatorname{Fitt}_j(C))$$

for any $j = 0, 1, \ldots, r-1$.

Remark 2.3. In [**7,** Theorem 8.1], Kirby and Rees computed the multiplicities $e^j(C)$ in the case where C is a module of finite length and R is a DVR (see also [**8,** Proposition 4.1]). Our results give more detailed information about the function $\Lambda(p,q)$.

The case where the set of ideals $Q_1, Q_2, \ldots, Q_r$ is not an ascending chain is more complicated. However, the case where $r = 2$ can be computed as follows.

Theorem 2.4. *Assume $r = 2$, and put $I := Q_1 + Q_2$. Then:*

(i) *the Buchsbaum-Rim polynomial is*

$$\Lambda(p,q) = (a_1 + a_2)\binom{p+1}{2} + e(R/I)\binom{p}{1}\binom{q}{1} - e_1(I)(p+q) + c$$

for all $p, q \gg 0$, where $e_1(I)$ denotes the first Hilbert coefficient of I and c is a constant. In particular, we have that

$$\begin{cases} e^0(C) = \ell_R(R/\operatorname{Fitt}_0(C)) = \ell_R(R/Q_1Q_2) \\ e^1(C) = e(R/\operatorname{Fitt}_1(C)) = e(R/I) \\ e^2(C) = 0. \end{cases}$$

(ii) *The function $\Lambda(p,q)$ coincides with a polynomial function if and only if the equality $\ell_R(R/I) = e(R/I) - e_1(I)$ holds true. When this is the case,*

$$\Lambda(p,q) = (a_1 + a_2)\binom{p+1}{2} + e(R/I)\binom{p}{1}\binom{q}{1} - e_1(I)(p+q) + e_1(I)$$

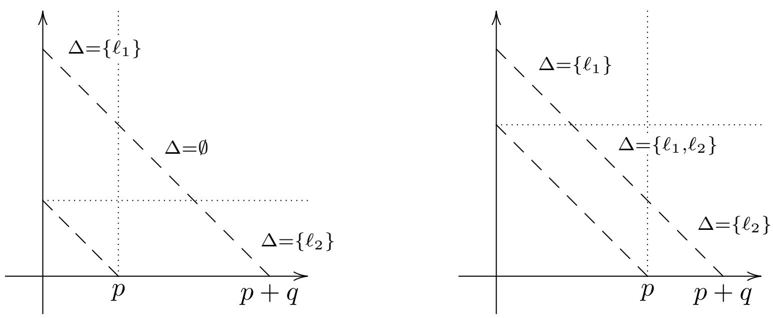
for all $p, q \geq 0$.

(iii) *The function* $\Lambda(p,q)$ *coincides with the following simple polynomial function*

$$\Lambda(p,q) = (a_1 + a_2)\binom{p+1}{2} + e(R/I)\binom{p}{1}\binom{q}{1}$$

if and only if there is an inclusion between Q_1 *and* Q_2.

Before detailing the proof, we will state a lemma which is an explicit description of the function $\Lambda(p,q)$. In order to do this, we will introduce some notation. Let $p, q \geq 0$, and let $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \mathbb{Z}^2_{\geq 0}$ be such that $|\boldsymbol{\ell}| = p + q$. Let $\delta = \delta(\boldsymbol{\ell})$ be the number of elements of the set $\Delta = \Delta(\boldsymbol{\ell}) = \{\ell_i \mid \ell_i < p\}$.



$p \leq q$. $p > q$.

Let $h_n = \ell_R(R/I^n)$ be the Hilbert-Samuel function of the ideal I. With this notation, we have the following.

Lemma 2.5.

(i)

$$J_{p,q}(\boldsymbol{\ell}) = \begin{cases} I^p & \textit{if } \delta = 0 \\ Q_j^{p-\ell_i} I^{\ell_i} & \textit{if } \delta = 1, \Delta = \{\ell_i\}, i \neq j, \\ Q_1^{p-\ell_2} Q_2^{p-\ell_1} I^q & \textit{if } \delta = 2. \end{cases}$$

(ii)

$$\ell_R(R/J_{p,q}(\boldsymbol{\ell})) = \begin{cases} h_p & \textit{if } \delta = 0, \\ a_j(p-\ell_i) + h_{\ell_i} & \textit{if } \delta = 1, \Delta = \{\ell_i\}, \\ & \qquad i \neq j, \\ a_1(p-\ell_2) + a_2(p-\ell_1) + h_q & \textit{if } \delta = 2, \end{cases}$$

(iii)

$$\Lambda(p,q) = \begin{cases} (a_1+a_2)\binom{p+1}{2} + 2(h_1+\cdots+h_{p-1}) + (q-p+1)h_p & \textit{if } p \leq q, \\ (a_1+a_2)\binom{p+1}{2} + 2(h_1+\cdots+h_q) + (p-q-1)h_q & \textit{if } p > q. \end{cases}$$

Proof. Assertion (i) is easy and implies assertion (ii). Assertion (iii) follows from assertion (ii). Indeed, if $p \leq q$, then $0 \leq \delta \leq 1$, and we have that

$$\begin{aligned}
\Lambda(p,q) &= \sum_{\substack{\ell_1+\ell_2=p+q \\ \ell_1,\ell_2 \geq 0}} \ell_R(R/J_{p,q}(\ell_1,\ell_2)) \\
&= \sum_{\ell_1=0}^{p-1} (a_2(p-\ell_1) + h_{\ell_1}) + \sum_{\ell_1=p}^{q} h_p + \sum_{\ell_1=q+1}^{p+q} (a_1(p-\ell_2) + h_{\ell_2}) \\
&= \sum_{\ell_1=0}^{p-1} (a_2(p-\ell_1)+h_{\ell_1}) + (q-p+1)h_p + \sum_{\ell_2=0}^{p-1} (a_1(p-\ell_2)+h_{\ell_2}) \\
&= (a_1+a_2)(1+2+\cdots+p) + 2(h_1+\cdots+h_{p-1}) + (q-p+1)h_p \\
&= (a_1+a_2)\binom{p+1}{2} + 2(h_1+\cdots+h_{p-1}) + (q-p+1)h_p.
\end{aligned}$$

The case where $p > q$ is similar. $\square$

Proof of Theorem 2.4. Let p_0 be the postulation number of I, that is, $h_p = e(R/I)p - e_1(I)$ for all $p > p_0$ and $h_{p_0} \neq e(R/I)p_0 - e_1(I)$. To compute the Buchsbaum-Rim polynomial, we may assume that $p_0 < p \leq q$. Then, by Lemma 2.5 (iii),

$$\begin{aligned}
\Lambda(p,q) = (a_1+a_2)\binom{p+1}{2} \\
+ 2(h_1 + \cdots + h_{p_0} + h_{p_0+1} + \cdots + h_{p-1}) + (q-p+1)h_p
\end{aligned}$$

$$= (a_1 + a_2)\binom{p+1}{2} + e(R/I)\binom{p}{1}\binom{q}{1} - e_1(I)(p+q) + c,$$

where $c = 2(h_1 + \cdots + h_{p_0}) - e(R/I)(p_0+1)p_0 + e_1(I)(2p_0+1)$. This proves assertion (i).

Suppose that the function $\Lambda(p,q)$ coincides with the polynomial function. Then, by substituting $p = 1$ in the polynomial, $\Lambda(1,q) = (e(R/I) - e_1(I))q + (a_1 + a_2 - e_1(I) + c)$ for any $q \geq 0$. On the other hand, by Lemma 2.5 (iii), $\Lambda(1,q) = h_1 q + (a_1 + a_2)$ for any $q \geq 1$. By comparing the coefficients of q, we have $h_1 = e(R/I) - e_1(I)$.

Conversely, suppose that $h_1 = e(R/I) - e_1(I)$. Then it is known by [**11,** Theorem 1.5 and Corollary 1.6] that the Hilbert-Samuel function h_n coincides with the polynomial function for all $n > 0$. Hence, the function $\Lambda(p,q)$ also coincides with the polynomial function with the following form

$$\Lambda(p,q) = (a_1 + a_2)\binom{p+1}{2} + e(R/I)\binom{p}{1}\binom{q}{1} - e_1(I)(p+q) + e_1(I),$$

by Lemma 2.5 (iii). Thus, we have assertion (ii).

For assertion (iii), if the function $\Lambda(p,q)$ coincides with the following simple polynomial function

$$\Lambda(p,q) = (a_1 + a_2)\binom{p+1}{2} + e(R/I)\binom{p}{1}\binom{q}{1},$$

then $e_1(I) = 0$ and $h_1 = e(R/I)$. This implies that I is a parameter ideal for R, and hence, $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$. The other implication follows from Proposition 2.1. $\square$

Consequently, there exists the case where the Buchsbaum-Rim function $\Lambda(p,q)$ does not coincide with a polynomial function even if the ring R is Cohen-Macaulay and the module has a parameter matrix. This should be contrasted with a result due to Brennan, Ulrich and Vasconcelos [**1,** Theorem 3.4] on the classical Buchsbaum-Rim function $\lambda(p) = \Lambda(p,0)$ associated to a parameter matrix.

REFERENCES

1. J. Brennan, B. Ulrich and W.V. Vasconcelos, *The Buchsbaum-Rim polynomial of a module*, J. Algebra **241** (2001), 379–392.

2. W. Bruns and U. Vetter, *Length formulas for the local cohomology of exterior powers*, Math. Z. **191** (1986), 145–158.

3. D.A. Buchsbaum and D.S. Rim, *A generalized Koszul complex*, II, *Depth and multiplicity*, Trans. Amer. Math. Soc. **111** (1964), 197–224.

4. R. Callejas-Bedregal and V.H. Jorge Pérez, *Mixed multiplicities and the minimal number of generator of modules*, J. Pure Appl. Alg. **214** (2010), 1642–1653.

5. F. Hayasaka and E. Hyry, *On the Buchsbaum-Rim function of a parameter module*, J. Algebra **327** (2011), 307–315.

6. D. Kirby, *On the Buchsbaum-Rim multiplicity associated with a matrix*, J. Lond. Math. Soc. **32** (1985), 57–61.

7. D. Kirby and D. Rees, *Multiplicities in graded rings*, I, *The general theory*, Contemp. Math. **159**, American Mathematical Society, Providence, RI, 1994.

8. ———, *Multiplicities in graded rings*, II, *Integral equivalence and the Buchsbaum-Rim multiplicity*, Math. Proc. Cambr. Philos. Soc. **119** (1996), 425–445.

9. S. Kleiman and A. Thorup, *A geometric theory of the Buchsbaum-Rim multiplicity*, J. Algebra **167** (1994), 168–231.

10. ———, *Mixed Buchsbaum-Rim multiplicities*, Amer. J. Math. **118** (1996), 529–569.

11. J. Lipman, *Stable ideals and Arf rings*, Amer. J. Math. **93** (1971), 649–685.

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