ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 46, Number 1, 2016

IMPROVEMENTS TO TURING'S METHOD II

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ABSTRACT. Turing's method uses explicit bounds on $|\int_{t_1}^{t_2} S(t)\,dt|$, where $\pi S(t)$ is the argument of the Riemann zeta-function. This article improves the bound on $|\int_{t_1}^{t_2} S(t)\,dt|$ given in **[8]**.

1. Introduction. Let $\zeta(s)$ be the Riemann zeta-function, and let $N(T)$ denote the number of zeroes of $\zeta(s)$ with $0 < \Re(s) < 1$ and $0 < \Im(s) < T$. One seeks to calculate $N(T)$ as follows.

First, one finds zeroes by locating sign changes of a real-valued function the zeroes of which agree with the non-trivial zeroes of the zeta-function. This gives one a lower bound on the number of zeroes of $\zeta(s)$ with $0 < \Im(s) < T$.

To check whether this initial analysis has omitted some zeroes one employs Turing's method. This was first annunciated by Turing **[11]** in 1953 and has been used extensively since then. Recently, another method has been deployed by Büthe **[2]**.

To apply Turing's method, one needs good explicit bounds on

$$\left| \int_{t_1}^{t_2} S(t)\,dt \right|,$$

for $t_2 > t_1 > 0$, where $\pi S(t)$ is defined to be the argument of $\zeta(\frac{1}{2} + it)$. For a complete definition and a brief history of the problem, see **[8,** Section 1] and **[4,** Chapter 7].

This article improves **[8]** and contains frequent references to the results therein. The main result is

2010 AMS *Mathematics subject classification.* Primary 11M06.
Keywords and phrases. Turing's method, Riemann zeta-function, zeta zeroes.
Received by the editors on March 29, 2014.
DOI:10.1216/RMJ-2016-46-1-325

Theorem 1.1.

$$\left| \int_{t_1}^{t_2} S(t)\,dt \right| \leq 1.698 + 0.183 \log\log t_2 + 0.049 \log t_2, \tag{1.1}$$

for $t_2 > t_1 > 10^5$. If the right-side of (1.1) *is replaced by $a + b \log\log t_2 + c \log t_2$, one may use Table* 1 *in Section* 3 *for more specific values of a, b and c.*

In **[8]**, the main result followed from Lemma 2.8 and Lemma 2.11 which concerned, respectively, obtaining an upper and a lower bound for $\Re \log \zeta(s)$ for $\Re(s) \geq \frac{1}{2}$. This article refines only the upper bound. Theorem 1.1 improves on Theorem 2.2 in **[8]** for all $t_2 \geq 10^5$.

The idea in this article is to use more sophisticated estimates on $\zeta(\sigma + it)$ for $\frac{1}{2} \leq \sigma \leq 1$; these estimates have been given in **[7, 10]**. A bound on $|\zeta(s)|$ is given in Section 2, a proof of Theorem 1 is given in Section 3, and some concluding remarks are provided in Section 4.

2. Bounding $|\zeta(\sigma + it)|$ across the strip $\frac{1}{2} \leq \sigma \leq 1 + \delta$. Using the inequality $\log(1 + x) \leq x$, it is easy to see that

$$\log|Q_0 + \sigma + it| - \log t \leq \frac{1}{2}\left(\frac{\sigma_1 + Q_0}{t_0}\right)^2, \tag{2.1}$$

for $\sigma \leq \sigma_1$ and $t \geq t_0$ and any $Q_0 \geq 0$. With the trivial observations $\log|Q_0 + \sigma + it| \geq \log t$ and $|\arg(Q_0 + \sigma + it)| \leq \frac{\pi}{2}$ at hand, we may apply (2.1) to see that

$$|\log(Q_0 + \sigma + it)| \leq (1 + a_0)\log t, \quad (\sigma \leq \sigma_1), \tag{2.2}$$

where

$$a_0 = a_0(\sigma_1) = \frac{\sigma_1 + Q_0}{2t_0^2 \log t_0} + \frac{\pi}{2\log t_0} + \frac{\pi(\sigma_1 + Q_0)^2}{4t_0 \log^2 t_0}.$$

Suppose that

$$\begin{aligned} |\zeta(\tfrac{1}{2} + it)| &\leq k_1 t^{k_2} (\log t)^{k_3}, \qquad (t \geq t_1), \\ |\zeta(1 + it)| &\leq k_4 \log t^{k_5}, \qquad (t \geq t_2). \end{aligned} \tag{2.3}$$

Consider the function $h(s) = (s - 1)\zeta(s)$, which is entire. Once we are able to exhibit bounds for $|h(s)|$ using the information in (2.3), we can apply a version of the Phragmén-Lindelöf principle to bound $|\zeta(s)|$. Using [**9,** Lemma 3] and (2.1) and (2.2), we may prove

Lemma 2.1. *Let $h(s) = (s-1)\zeta(s)$, and let δ be a positive real number. Furthermore, let $Q_0 \geq 0$ be a number for which*

$$|h(\tfrac{1}{2}+it)| \leq k_1|Q_0+\tfrac{1}{2}+it|^{k_2+1}(\log|Q_0+\tfrac{1}{2}+it|)^{k_3}$$
$$|h(1+it) \leq k_4|Q_0+1+it|(\log|Q_0+1+it|)^{k_5}$$
$$|h(1+\delta+it)| \leq \zeta(1+\delta)|Q_0+1+\delta+it|,$$

for all t. Then, for $\sigma \in [\frac{1}{2},1]$ and $t \geq t_0$,

$$|\zeta(s)| \leq \alpha_1 k_1^{2(1-\sigma)} k_4^{2(\sigma-1/2)} t^{2k_2(1-\sigma)}(\log t)^{2(k_3(1-\sigma)+k_5(\sigma-1/2))}, \tag{2.4}$$

where

$$\alpha_1 = (1+a_1(1+\delta,Q_0,t_0))^{k_2+1}(1+a_0(1+\delta,Q_0,t_0))^{k_3+k_5},$$

and

$$a_0(\sigma,Q_0,t) = \frac{\sigma+Q_0}{2t^2\log t}+\frac{\pi}{2\log t}+\frac{\pi(\sigma+Q_0)^2}{4t\log^2 t},$$
$$a_1(\sigma,Q_0,t) = \frac{\sigma+Q_0}{t},$$

whereas for $\sigma \in [1,1+\delta]$ and $t \geq t_0$,

$$|\zeta(s)| \leq \alpha_2 k_4^{(1+\delta-\sigma)/\delta}\zeta(1+\delta)^{(\sigma-1)/\delta}(\log t)^{k_5(1+\delta-\sigma)/\delta}, \tag{2.5}$$

where

$$\alpha_2 = (1+a_1(1+\delta,Q_0,t_0))(1+a_0(1+\delta,Q_0,t_0))^{k_5}.$$

Finally, for all $\sigma \in [\frac{1}{2},1+\delta]$ and $t \geq t_0$, we have

$$|\zeta(s)| \leq (1+a_1(1+\delta,Q_0))^{k_2+1}(1+a_0(1+\delta,Q_0))^{k_3+k_5}k_1 t^{k_2}(\log t)^{k_3},$$

provided that

$$t^{k_2}(\log t)^{k_3-k_5} \geq \frac{k_4}{k_1}, \qquad t \geq \exp\left\{\left(\frac{\zeta(1+\delta)}{k_4}\right)^{1/k_5}\right\}. \tag{2.6}$$

Proof. In applying Lemma 3 of **[9]** to $h(s)$, we need to relate $|Q_0+s|$ and $|s-1|$ to t. We simply note that

$$\left|\frac{Q_0+s}{s-1}\right| \leq \frac{|Q_0+s|}{t} \leq 1+\frac{\sigma+Q_0}{t_0}$$
$$= 1+a_1(\sigma,Q_0) \leq 1+a_1(1+\delta,Q_0),$$

in both regions $\sigma \in [\frac{1}{2},1]$ and $\sigma \in [1,1+\delta]$. Since a_0 and a_1 are small for any respectable value of t_0, we throw away some information in the exponents of $1+a_1$ and $1+a_0$. For example, in proving (2.4), we arrive at

$$(1+a_1)^{2k_2(1-\sigma)+1}(1+a_0)^{2k_3(1-\sigma)+2k_5(\sigma-1/2)}.$$

Rather than retain this dependence on σ in the exponents, we simply bound $1-\sigma$ and $\sigma-\frac{1}{2}$ by $\frac{1}{2}$. A similar procedure is applied to prove (2.5).

To prove the bound in the region $\sigma \in [\frac{1}{2},1+\delta]$, we note that the bounds in (2.4) and (2.5) are decreasing in σ if the inequalities in (2.6) are met. Finally, the bound in (2.4), evaluated at $\sigma=\frac{1}{2}$, exceeds the bound in (2.5), evaluated at $\sigma=1$. This completes the lemma. □

It is worth recording the values of $k_1,\ldots,k_5$, which we do in

Corollary 2.2. *For $\sigma \in [\frac{1}{2},1+\delta]$ and $t \geq t_0$, we have*

$$|\zeta(s)| \leq 0.732(1+a_1(1+\delta,5,t_0))^{7/6}(1+a_0(1+\delta,5,t_0))^2 t^{1/6}\log t,$$

provided that

$$t \geq \max\{1.16, \exp[4\zeta(1+\delta)/3]\}.$$

Proof. In **[10]**, it was shown that $|\zeta(1+it)| \leq \frac{3}{4}\log t$ for $t \geq 3$. In **[7]**, it was shown that $|\zeta(\frac{1}{2}+it)| \leq 0.732|4.678+it|^{1/6}\log|4.678+it|$ for all t. One may therefore choose

$$(k_1,k_2,k_3,k_4,k_5,Q_0) = (0.732, \tfrac{1}{6}, 1, \tfrac{3}{4}, 1, 5) \tag{2.7}$$

in Lemma 2.1, which proves the corollary. □

Although we shall use (2.7) in our computation we proceed with the variables $(k_1,\ldots,Q_0)$ as parameters. We remark that, although

Corollary 2.2 is not used in this article, it is derived at very little additional cost and should prove useful for related problems.

3. Proof of Theorem 1.1. We are now able to proceed to the proof of Theorem 1.1. We need to give an upper bound for

$$I = \Re \int_{1/2+it}^{\infty+it} \log \zeta(s)\, ds.$$

To that end, we shall write

(3.1)
$$I \le \int_{1/2+it}^{1+it} \log|\zeta(s)|\, ds + \int_{1+it}^{1+\delta+it} \log|\zeta(s)|\, ds + \int_{1+\delta}^{\infty} \log|\zeta(\sigma)|\, d\sigma,$$

and apply (2.4) to the first integral in (3.1) and (2.5) to the second. This gives us

Lemma 3.1. *For $t \ge t_0$, we have*

$$I \le A_1 + B_1 \log\log t + C_1 \log t,$$

where

$$\begin{aligned} A_1 = &\int_{1+\delta}^{\infty} \log|\zeta(\sigma)|\, d\sigma + \left(\frac{k_2}{4} + \frac{1}{2} + \delta\right) \log(1 + a_1(1+\delta, Q_0, t_0)) \\ &+ \left(\frac{k_3}{4} + \frac{k_5}{4} + \frac{k_5\delta}{2}\right) \log(1 + a_0(1+\delta, Q_0, t_0)) \\ &+ \left(\frac{1}{4} + \frac{\delta}{2}\right) \log k_4 + \frac{1}{4}\log k_1 + \frac{\delta}{2} \log\zeta(1+\delta), \end{aligned}$$

and

$$B_1 = \left(\frac{k_3 + k_5}{4} + \frac{\delta k_5}{2}\right), \qquad C_1 = \frac{k_2}{4}.$$

The term corresponding to C_1 in [**8,** Lemma 2.8] is $k_2/4 + \delta k_2/2$. Since we are not permitted to take δ too small lest the integral in A_1 become too large, this represents a considerable qualitative saving. This is due entirely to estimating $|\zeta(s)|$, not in one go over $\sigma \in [\frac{1}{2}, 1+\delta]$ as in [**8**], but by using Lemma 2.1. We combine Lemma 3.1 with [**8,** Lemma 2.11] to obtain

Theorem 3.2. *For $t_2 \geq t_1 \geq 10^5$,*

$$\left| \int_{t_1}^{t_2} S(t)\, dt \right| \leq a + b \log\log t_2 + c \log t_2,$$

where

(3.2)

$$\begin{aligned}\pi a = &\int_{1+\delta}^{\infty} \log|\zeta(\sigma)|\, d\sigma + \left(\frac{k_2}{4} + \frac{1}{2} + \delta\right) \log(1 + a_1(1+\delta, Q_0, 10^5)) \\ &+ \frac{1}{4}\log k_1 + \left(\frac{k_3}{4} + \frac{k_5}{4} + \frac{k_5\delta}{2}\right) \log(1 + a_0(1+\delta, Q_0, 10^5)) \\ &+ \left(\frac{1}{4} + \frac{\delta}{2}\right)\log k_4 + \frac{\delta}{2}\log\zeta(1+\delta) + \frac{d^2}{2}\log\pi + 3\times 10^{-4} \\ &+ d^2(\log 4)\left\{-\frac{\zeta'(1/2+d)}{\zeta(1/2+d)} - \frac{1}{2}\log 2\pi + \frac{1}{4}\right\} - \frac{1}{2}\int_{1+2d}^{\infty} \log\zeta(\sigma)\, d\sigma \\ &+ \int_{1/2+d}^{\infty} \log\zeta(\sigma)\, d\sigma - \frac{1}{2}\int_{1+2d}^{1+4d} \log\zeta(\sigma)\, d\sigma + \int_{1/2+d}^{1/2+2d} \log\zeta(\sigma)\, d\sigma,\end{aligned}$$

and

$$\pi b = \left(\frac{k_3 + k_5}{4} + \frac{\delta k_5}{2}\right), \qquad \pi c = \frac{k_2}{4} + \frac{d^2}{2}(\log 4 - 1). \tag{3.3}$$

3.1. Computation. Before we commence an analysis of the coefficients appearing in Theorem 3.2 we make the following observation. One may replace the values of (k_1, k_2, k_3) in (2.7) by

$$(k_1, k_2, k_3) = \left(\frac{4}{(2\pi)^{1/4}}, \frac{1}{4}, 0\right), \tag{3.4}$$

which appear in [**5,** Lemma 2]. The values in (3.4) are obtained using the approximate functional equation of $\zeta(s)$: the values in (2.7) are obtained using exponential sums. The value $k_2 = \frac{1}{4}$ follows from convexity theorems. We call (3.4) the convexity result and (2.7) the sub-convexity result.

As in [**8,** Theorem 2.12], no term in either (3.2) or (3.3) depends on both δ and d. We can run two one-dimensional optimizations on each of a, b and c. In Table 1, we compare the results obtained from the convexity result (C), the sub-convexity result (SC), and the coefficients in [**8,** Theorem 2.2], when $t_1 \approx T$. The values of δ, d, a, b and c

correspond to the sub-convexity result. We find that the sub-convexity result overtakes the convexity result when $T \geq 2.85 \times 10^{10}$, which is, just barely, beneath the height to which the Riemann hypothesis has been verified—see [**6**].

The values used in Theorem 1.1 are taken from the row $T = 10^{10}$. It should be stressed that all of the results in this table are valid for $t_1 \geq 10^5$. The stated values of a, b and c are those that are close to the best values obtainable by this method when $t_1 \approx T$.

TABLE 1. Comparison of bounds for $|\int_{t_1}^{t_2} S(t)\,dt| \leq a + b \log\log t_2 + c \log t_2$.

T	Theorem 2.2	C	SC	d	δ	a	b	c
10^5	2.747	2.629	2.658	0.883	0.279	1.457	0.204	0.062
10^6	2.883	2.800	2.827	0.845	0.237	1.520	0.197	0.058
10^7	3.018	2.959	2.982	0.817	0.206	1.573	0.192	0.055
10^8	3.154	3.110	3.128	0.795	0.182	1.620	0.189	0.053
10^9	3.290	3.255	3.266	0.777	0.163	1.661	0.186	0.051
10^{10}	3.426	3.395	3.398	0.762	0.148	1.698	0.183	0.049
10^{11}	3.562	3.530	3.526	0.749	0.135	1.733	0.181	0.048
10^{12}	3.698	3.663	3.649	0.738	0.124	1.764	0.179	0.047
10^{13}	3.834	3.792	3.770	0.729	0.115	1.792	0.178	0.046
10^{14}	3.969	3.919	3.887	0.720	0.107	1.820	0.177	0.046
10^{15}	4.105	4.044	4.002	0.713	0.100	1.844	0.176	0.045

4. Conclusion. It seems difficult to improve substantially on Theorem 1.1. Given that the improvements obtained in this paper are only modest, and since further improvements would require a lot of effort in estimating $\zeta(\frac{1}{2} + it)$ or $\zeta(1 + it)$, it seems hopeless to try to improve this part of the argument.

One could try one's luck at reducing the term $\log 4$ that appears in both (3.2) and (3.3). This comes from [**1,** Lemma 4.4]. Reducing this would have a more profound influence on bounding $|\int_{t_1}^{t_2} S(t)\,dt|$ than better bounds for $|\zeta(s)|$.

Finally, it is worth considering [**8,** Theorems 3.3 and 4.3], which relate to Dirichlet L-functions and Dedekind zeta-functions. Both of these could be improved, in line with this article, were one in possession of explicit estimates on the lines $\sigma = \frac{1}{2}$ and $\sigma = 1$. One such estimate, bounding $|L(1+it,\chi)|$ for $L(s)$ a Dirichlet L-function, appears in [**3**]. It is possible that this could be used to obtain an improvement to [**8,** Theorem 3.3].

Acknowledgments. I should like to thank Jan Büthe for detailed discussions on this problem.

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