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DETERMINANTAL AND PERMANENTAL REPRESENTATIONS OF FIBONACCI TYPE NUMBERS AND POLYNOMIALS

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ABSTRACT. In this paper, we compute terms of the matrix $A^{\infty}_{(k)}$, which contains Fibonacci type numbers and polynomials, with the help of determinants and permanents of various Hessenberg matrices. In addition, we show that determinants of these Hessenberg matrices can be obtained by using combinations. The results that we obtain are important, since the matrix $A^{\infty}_{(k)}$ is a general form of Fibonacci type numbers and polynomials, such as k sequences of the generalized order-k Fibonacci and Pell numbers, generalized bivariate Fibonacci p-polynomials, bivariate Fibonacci and Pell p-polynomials, second kind Chebyshev polynomials and bivariate Jacobsthal polynomials, etc.

1. Background and notation. In modern science, there is quite an interest in the theory and applications of Fibonacci numbers, Fibonacci polynomials and their generalizations. Since finding a requested term of these sequences and polynomials by recurrence relation is very difficult, there is a need to find other methods. For this reason, in the past few decades, researchers have done many studies on determinantal and permanental representations of these polynomials and sequences [**1, 6, 8, 9, 11, 12, 13, 24, 27**].

Miles [**22**] defined generalized order-k Fibonacci numbers (GOkF) as:

$$f_{k,n} = \sum_{j=1}^{k} f_{k,n-j} \tag{1.1}$$

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for $n > k \geq 2$, with boundary conditions: $f_{k,1} = f_{k,2} = f_{k,3} = \cdots = f_{k,k-2} = 0$ and $f_{k,k-1} = f_{k,k} = 1$. Er [**2**] defined k sequences of the generalized order-k Fibonacci numbers $(f_{k,n}^{i})$. Kılıç and Taşcı [**10**] defined k sequences of the generalized order-k Pell numbers $(p_{k,n}^{i})$. Kaygısız and Şahin [**4**] defined k sequences of the generalized order-k Van der Laan numbers $(v_{k,n}^{i})$.

The Fibonacci [**14**], Pell [**3**], second kind Chebysev [**26**] and Jacobsthal [**25**] polynomials are defined as:

$$\begin{aligned}
f_{n+1}(x) &= xf_n(x) + f_{n-1}(x), && n \geq 2 \text{ with } f_0(x) = 0,\ f_1(x) = 1,\\
P_{n+1}(x) &= 2xP_n(x) + P_{n-1}(x), && n \geq 2 \text{ with } P_0(x) = 0,\ P_1(x) = x,\\
U_{n+1}(x) &= xU_n(x) - U_{n-1}(x), && n \geq 2 \text{ with } U_0(x) = 1,\ U_1(x) = 2x,\\
J_{n+1}(x) &= J_n(x) + xJ_{n-1}(x), && n \geq 2 \text{ with } J_0(x) = 0,\ J_1(x) = 1,
\end{aligned}$$

respectively.

The generalized bivariate Fibonacci p-polynomials [**25**] are, for $n > p$,

$$F_{p,n}(x,y) = xF_{p,n-1}(x,y) + yF_{p,n-p-1}(x,y), \tag{1.2}$$

with boundary conditions $F_{p,0}(x,y) = 0$, $F_{p,n}(x,y) = x^{n-1}$ for $n = 1, 2, \ldots, p$.

MacHenry [**15**] defined generalized Fibonacci polynomials $(F_{k,n}(t))$, where t_i $(1 \leq i \leq k)$ are constant coefficients of the core polynomial:

$$P(x; t_1, t_2, \ldots, t_k) = x^k - t_1 x^{k-1} - \cdots - t_k,$$

which is denoted by the vector $t = (t_1, t_2, \ldots, t_k)$. $F_{k,n}(t)$ is defined inductively by

$$\begin{aligned}
&F_{k,n}(t) = 0,\ n < 0\\
&F_{k,0}(t) = 1 \qquad\qquad (1.3)\\
&F_{k,n}(t) = t_1 F_{k,n-1}(t) + \cdots + t_k F_{k,n-k}(t).
\end{aligned}$$

In addition, in [**21**], authors obtained $F_{k,n}(t)$ $(n, k \in \mathbb{N},\ n \geq 1)$ as:

$$F_{k,n}(t) = \sum_{a \vdash n} \binom{|a|}{a_{1,\ldots,}a_k} t_1^{a_1} \ldots t_k^{a_k}. \tag{1.4}$$

Throughout this paper, the notations $a \vdash n$ and $|a|$ are used instead of

$$\sum_{j=1}^{k} j a_j = n \quad \text{and} \quad \sum_{j=1}^{k} a_j,$$

respectively.

In [**20**], matrices $A_{(k)}^{\infty}$ are defined by using the following matrix:

$$A_{(k)} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ t_k & t_{k-1} & t_{k-2} & \dots & t_1 \end{bmatrix}.$$

They also record the orbit of the k-th row vector of $A_{(k)}$ under the action of $A_{(k)}$, below $A_{(k)}$, and the orbit of the first row of $A_{(k)}$ under the action of $A_{(k)}^{-1}$ on the first row of $A_{(k)}$ is recorded above $A_{(k)}$, and consider the $\infty \times k$ matrix whose row vectors are the elements of the doubly infinite orbit of $A_{(k)}$ acting on any one of them. For $k = 3$, $A_{(k)}^{\infty}$ looks like this:

$$A_{(3)}^{\infty} = \begin{bmatrix} \dots & \dots & \dots \\ S_{(-n,1^2)} & -S_{(-n,1)} & S_{(-n)} \\ \dots & \dots & \dots \\ S_{(-3,1^2)} & -S_{(-3,1)} & S_{(-3)} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ t_3 & t_2 & t_1 \\ \dots & \dots & \dots \\ S_{(n-1,1^2)} & -S_{(n-1,1)} & S_{(n-1)} \\ S_{(n,1^2)} & -S_{(n,1)} & S_{(n)} \\ \dots & \dots & \dots \end{bmatrix}$$

and

$$A_{(k)}^{n} = \begin{bmatrix} (-1)^{k-1} S_{(n-k+1,1^{k-1})} & \cdots & (-1)^{k-j} S_{(n-k+1,1^{k-j})} & \cdots & S_{(n-k+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{k-1} S_{(n,1^{k-1})} & \cdots & (-1)^{k-j} S_{(n,1^{k-j})} & \cdots & S_{(n)} \end{bmatrix}$$

where

$$S_{(n-r,1^r)} = (-1)^r \sum_{j=r+1}^{n} t_j S_{(n-j)}, \quad 0 \le r \le n. \tag{1.5}$$

The right hand column of $A_{(k)}^{\infty}$ contains the generalized Fibonacci polynomials $F_{k,n}(t)$, that is, $F_{k,n}(t) = S_{(n)}$. Also in [**16, 17, 18, 19**], the authors studied generalized Fibonacci and Lucas polynomials and obtained very useful properties of them.

Lemma 1.1. *Let* $(-1)^r S_{(n,1^r)}$ *be the* $(k-r)$*-th column of the matrix* $A_{(k)}^{\infty}$. *Then:*

$$(-1)^r S_{(n+1,1^r)} = t_1(-1)^r S_{(n,1^r)} + \cdots + t_k(-1)^r S_{(n-k+1,1^r)} \tag{1.6}$$

for $n > k \ge 2$ *and* $0 \le r \le k-1$, *with initial conditions*

$$(-1)^r S_{(n,1^r)} = \begin{cases} 1, & n = -r; \\ 0, & \textit{otherwise}. \end{cases}$$

Many researchers studied determinantal and permanental representations of k sequences of the generalized order-k Fibonacci and Lucas numbers. For example, Minc [**23**] defined an $n \times n$ (0,1)-matrix $F(n,k)$ and showed that the permanents of $F(n,k)$ are equal to the generalized order-k Fibonacci numbers (1.1). The authors [**12, 13**] defined two $(0,1)$-matrices and showed that the permanents of these matrices are the generalized Fibonacci (1.1) and Lucas numbers. Öcal et al. [**24**] gave some determinantal and permanental representations of k-generalized Fibonacci and Lucas numbers and obtained Binet's formula for these sequences. Kılıç and Stakhov [**9**] gave permanent representation of Fibonacci and Lucas p-numbers. Kılıç and Taşcı [**11**] studied permanents and determinants of Hessenberg matrices. Yılmaz and Bozkurt [**27**] derived some relationships between Pell sequences, as well as permanents and determinants of a type of Hessenberg matrices. Kaygısız and Şahin [**5, 7**] gave some determinantal and permanental representations of generalized bivariate Lucas p-polynomials and Fibonacci type numbers.

The main purpose of this paper is to compute terms of the matrix $A_{(k)}^{\infty}$, by using determinant and permanent of some Hessenberg matrices. These results are a general form of determinantal and permanental representations of many types of polynomials and sequences having linear recursions.

2. The determinantal representations. An $n \times n$ matrix $A_n = (a_{ij})$ is called a *lower Hessenberg matrix* if $a_{ij} = 0$ when $j - i > 1$, i.e.,

$$A_n = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{bmatrix}.$$

Lemma 2.1. **[1]** *Let A_n be the $n \times n$ lower Hessenberg matrix for all $n \geq 1$, and define* $\det(A_0) = 1$*. Then,* $\det(A_1) = a_{11}$ *and, for $n \geq 2$,*
(2.1)

$$\det(A_n) = a_{n,n} \det(A_{n-1}) + \sum_{r=1}^{n-1} \left[(-1)^{n-r} a_{n,r} \left(\prod_{j=r}^{n-1} a_{j,j+1} \right) \det(A_{r-1}) \right].$$

Theorem 2.2. *Let $k \geq 2$, $n \geq 1$ and $0 \leq r \leq k-1$ be integers, $(-1)^r S_{(n,1^r)}$ the $(k-r)$-th column of matrix $A_{(k)}^{\infty}$ and $Q_{k,n}^r = (q_{uv})$ an $n \times n$ Hessenberg matrix, given by*:

$$q_{uv} = \begin{cases} i^{|u-v|} \cdot t_{u-v+1}/t_2^{u-v}, & \textit{if } -1 \leq u-v < k \textit{ and } v \neq 1, \\ i^{(u-1)} \cdot t_{r+u}/t_2^{u-1}, & \textit{if } 0 < u < k-r+1 \textit{ and } v = 1, \\ 0, & \textit{otherwise}, \end{cases}$$

i.e.,

$$Q_{k,n}^r = \begin{bmatrix} t_{r+1} & it_2 & 0 & \cdots & 0 \\ \frac{i(t_{r+2})}{t_2} & t_1 & it_2 & \cdots & 0 \\ i^2 \frac{t_{r+3}}{t_2^2} & i & t_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{i^{k-r-1} t_k}{t_2^{k-r-1}} & \frac{i^{k-r-2} t_{k-r-1}}{t_2^{k-r-2}} & \frac{i^{k-r-3}(t_{k-r-2})}{t_2^{k-r-3}} & \cdots & 0 \\ 0 & \frac{i^{k-r-1}(t_{k-r})}{t_2^{k-r-1}} & \frac{i^{k-r-2}(t_{k-r-1})}{t_2^{k-r-2}} & \cdots & 0 \\ & \vdots & \vdots & \ddots & it_2 \\ 0 & 0 & 0 & \cdots & t_1 \end{bmatrix}$$

where $t_0 = 1$ *and* $i = \sqrt{-1}$. *Then,*

$$\det(Q^r_{k,n}) = (-1)^r S_{(n,1^r)}. \tag{2.2}$$

Proof. To prove (2.2), we use mathematical induction on n. The result is true for $n = 1$ by hypothesis.

Assume that it is true for all positive integers less than or equal to n, namely, $\det(Q^r_{k,n}) = (-1)^r S_{(n,1^r)}$. Then, by using Lemma 2.1, we have

$$\begin{aligned}
\det(Q^r_{k,n+1}) &= q_{n+1,n+1} \det(Q^r_{k,n}) \\
&\quad + \sum_{s=1}^{n} \left[(-1)^{n+1-s} q_{n+1,s} \prod_{v=s}^{n} q_{v,v+1} \det(Q^r_{k,s-1}) \right] \\
&= t_1 \det(Q^r_{k,n}) \\
&\quad + \sum_{s=1}^{n-k+1} \left[(-1)^{n+1-s} q_{n+1,s} \prod_{v=s}^{n} q_{v,v+1} \det(Q^r_{k,s-1}) \right] \\
&\quad + \sum_{s=n-k+2}^{n} \left[(-1)^{n+1-s} q_{n+1,s} \prod_{v=s}^{n} q_{v,v+1} \det(Q^r_{k,s-1}) \right] \\
&= t_1 \det(Q^r_{k,n}) \\
&\quad + \sum_{s=n-k+2}^{n} \left[(-1)^{n+1-s} q_{n+1,s} \prod_{v=s}^{n} q_{v,v+1} \det(Q^r_{k,s-1}) \right] \\
&= t_1 \det(Q^r_{k,n}) \\
&\quad + \sum_{s=n-k+2}^{n} \left[(-1)^{n+1-s} \cdot i^{n+1-s} \frac{t_{n-s+2}}{t_2^{n-s+1}} \prod_{v=s}^{n} i t_2 \det(Q^r_{k,s-1}) \right] \\
&= t_1 \det(Q^r_{k,n}) \\
&\quad + \sum_{s=n-k+2}^{n} \left[(-1)^{n+1-s} \cdot i^{n+1-s} \frac{t_{n-s+2}}{t_2^{n-s+1}} \cdot i^{n+1-s} \cdot t_2^{n-s+1} \right. \\
&\qquad \left. \cdot \det(Q^r_{k,s-1}) \right] \\
&= t_1 \det(Q^r_{k,n}) \\
&\quad + \sum_{s=n-k+2}^{n} \left[(-1)^{n+1-s} \cdot i^{n+1-s} t_{n-s+2} \cdot i^{n+1-s} \cdot \det(Q^r_{k,s-1}) \right]
\end{aligned}$$

$$= t_1 \det(Q_{k,n}) + \sum_{s=n-k+2}^{n} t_{n-s+2} \det(Q^r_{k,s-1})$$
$$= t_1 \det(Q^r_{k,n}) + t_2 \det(Q^r_{k,n-1}) + \cdots + t_k \det(Q^r_{k,n-(k-1)}).$$

Thus, from the hypothesis and Lemma 1.1, we obtain

$$\det(Q_{k,n+1}) = t_1(-1)^r S_{(n,1^r)} + \cdots + t_k(-1)^r S_{(n-k+1,1^r)} = (-1)^r S_{(n+1,1^r)}.$$

Therefore, the result is true for all positive integers. □

Example 2.3. We calculate $S_{(4,1^2)}$ for $k = 5$, using Theorem 2.2 as

$$S_{(4,1^2)} = \det \begin{bmatrix} t_3 & it_2 & 0 & 0 \\ (it_4)/t_2 & t_1 & it_2 & 0 \\ -t_5/t_2^2 & i & t_1 & it_2 \\ 0 & -t_3/t_2^2 & i & t_1 \end{bmatrix}$$
$$= t_1^3 t_3 + t_3^2 + 2t_1t_2t_3 + t_1^2 t_4 + t_2 t_4 + t_1 t_5.$$

Corollary 2.4. **[24]**. *Let $k \geq 2$ be an integer and $C_{k,n} = (c_{rs})$ an $n \times n$ Hessenberg matrix, where*

$$c_{rs} = \begin{cases} i^{|r-s|}, & \text{if } -1 \leq r-s < k, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\det(C_{k,n}) = f_{k,k+n-1}$$

where $i = \sqrt{-1}$.

Proof. It is direct from Theorem 2.2 for $t_i = 1$. □

Theorem 2.5. *Let $k \geq 2$, $n \geq 1$ and $0 \leq r \leq k-1$ be integers, $(-1)^r S_{(n,1^r)}$ the $(k-r)$-th column of the matrix $A^\infty_{(k)}$ and $B^r_{k,n} = (b_{ij})$ an $n \times n$ Hessenberg matrix, given by*

$$b_{ij} = \begin{cases} -t_2, & \text{if } j = i+1, \\ \frac{t_{i-j+1}}{t_2^{i-j}}, & \text{if } 0 \leq i-j < k \text{ and } j \neq 1, \\ \frac{t_{r+i}}{t_2^{i-1}}, & \text{if } 0 < i < k-r+1 \text{ and } j = 1, \\ 0, & \text{otherwise,} \end{cases}$$

i.e.,

$$B_{k,n}^{r}=\begin{bmatrix} t_{r+1} & -t_2 & 0 & 0 & \cdots & 0 \\ \frac{t_{r+2}}{t_2} & t_1 & -t_2 & 0 & \cdots & 0 \\ \frac{t_{r+3}}{t_2^2} & 1 & t_1 & -t_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \frac{t_k}{t_2^{k-r-1}} & \frac{t_{k-r-1}}{t_2^{k-r-2}} & \frac{t_{k-r-2}}{t_2^{k-r-3}} & \frac{t_{k-r-3}}{t_2^{k-r-4}} & \cdots & 0 \\ 0 & \frac{t_{k-r}}{t_2^{k-r-1}} & \frac{t_{k-r-1}}{t_2^{k-r-2}} & \frac{t_{k-r-2}}{t_2^{k-r-3}} & \cdots & 0 \\ & \vdots & \vdots & \vdots & \ddots & -t_2 \\ 0 & 0 & 0 & \cdots & \cdots & t_1 \end{bmatrix}$$

where $t_0 = 1$. *Then,*

$$\det(B_{k,n}^{r}) = (-1)^r S_{(n,1^r)}. \tag{2.3}$$

Proof. To prove (2.3), we use mathematical induction on m. The result is true for $m = 1$ by hypothesis. Assume that it is true for all positive integers less than or equal to m, namely, $\det(B_{k,m}^{r}) = (-1)^r S_{(m,1^r)}$. Then, by using Lemma 2.1, we have

$$\begin{aligned}
\det(B_{m+1,k}^{r}) &= b_{m+1,m+1}\det(B_{k,m}^{r}) \\
&\quad + \sum_{s=1}^{m}\left[((-1)^{m+1-s} b_{m+1,s}\prod_{j=s}^{m} b_{j,j+1}\det(B_{k,s-1}^{r}))\right] \\
&= t_1\det(B_{k,m}^{r}) \\
&\quad + \sum_{s=1}^{m-k+1}\left[(-1)^{m+1-s} b_{m+1,s}\prod_{j=s}^{m} b_{j,j+1}\det(B_{k,s-1}^{r})\right] \\
&\quad + \sum_{s=m-k+2}^{m}\left[(-1)^{m+1-s} b_{m+1,s}\prod_{j=s}^{m} b_{j,j+1}\det(B_{k,s-1}^{r})\right] \\
&= t_1\det(B_{k,m}^{r}) \\
&\quad + \sum_{s=m-k+2}^{m}\left[(-1)^{m+1-s}\cdot\frac{t_{m-s+2}}{t_2^{m-s+1}}\prod_{j=s}^{m}(-t_2)\det(B_{k,s-1}^{r})\right] \\
&= t_1\det(B_{k,m}^{r}) \\
&\quad + \sum_{s=m-k+2}^{m}\left[(-1)^{m+1-s}\cdot\frac{t_{m-s+2}}{t_2^{m-s+1}}\cdot(-1)^{m+1-s}t_2^{m-s+1}\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \det(B^r_{k,s-1})\Big] \\
&= t_1 \det(B^r_{k,m}) + \sum_{s=m-k+2}^{m} \Big[t_{m-s+2} \cdot \det(B^r_{k,s-1}) \Big] \\
&= t_1 \det(B^r_{k,m}) + t_2 \det(B^r_{k,m-1}) + \cdots + t_k \det(B^r_{k,m-(k-1)}).
\end{aligned}
$$

Thus, from the hypothesis and Lemma 1.1, we obtain

$\det(B^r_{k,m+1}) = t_1(-1)^r S_{(m,1^r)} + \cdots + t_k(-1)^r S_{(m-k+1,1^r)} = (-1)^r S_{(m+1,1^r)}$.

Therefore, the result is true for all positive integers. □

Example 2.6. We calculate $-S_{(5,1)}$ for $k = 4$, using Theorem 2.5 as

$$
\begin{aligned}
-S_{(5,1)} &= \det \begin{bmatrix} t_2 & -t_2 & 0 & 0 & 0 \\ t_3/t_2 & t_1 & -t_2 & 0 & 0 \\ t_4/t_2^2 & 1 & t_1 & -t_2 & 0 \\ 0 & t_3/t_2^2 & 1 & t_1 & -t_2 \\ 0 & t_4/t_2^3 & t_3/t_2^2 & 1 & t_1 \end{bmatrix} \\
&= t_1^4 t_2 + 3t_1^2 t_2^2 + t_2^3 + t_1^3 t_3 + t_3^2 + 4t_1 t_2 t_3 + t_1^2 t_4 + 2t_2 t_4.
\end{aligned}
$$

Corollary 2.7. [**24**]. *Let $k \geq 2$ be an integer, $f_{k,n}$ the generalized order-k Fibonacci numbers* (1.1) *and $M_{k,n} = (m_{ij})$ an $n \times n$ lower Hessenberg matrix such that*

$$
m_{ij} = \begin{cases} -1, & \textit{if } j = i + 1, \\ 1, & \textit{if } 0 \leq i - j < k, \\ 0, & \textit{otherwise.} \end{cases}
$$

Then,

$$
\det(M_{k,n}) = f_{k,k+n-1}.
$$

Proof. It is direct from Theorem 2.5 for $t_i = 1$. □

3. The permanent representations. Let $A = (a_{i,j})$ be a square matrix of order n over a ring R. The permanent of A is defined by

$$
\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},
$$

where S_n denotes the symmetric group on n letters.

Lemma 3.1. [**24**]. *Let A_n be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$, and define* $\mathrm{per}\,(A_0) = 1$. *Then,* $\mathrm{per}\,(A_1) = a_{11}$ *and, for $n \geq 2$,*

$$\mathrm{per}\,(A_n) = a_{n,n}\mathrm{per}\,(A_{n-1}) + \sum_{r=1}^{n-1}(a_{n,r}\prod_{j=r}^{n-1} a_{j,j+1}\mathrm{per}\,(A_{r-1})).$$

Theorem 3.2. *Let $k \geq 2$, $n \geq 1$ and $0 \leq r \leq k-1$ be integers, $(-1)^r S_{(n,1^r)}$ the $(k-r)$-th column of the matrix $A_{(k)}^{\infty}$ and $H_{k,n}^r = (h_{uv})$ an $n \times n$ Hessenberg matrix, given by*

$$h_{uv} = \begin{cases} i^{u-v} \cdot \frac{t_{u-v+1}}{t_2^{u-v}}, & \text{if } -1 \leq u-v < k \text{ and } v \neq 1, \\ i^{u-1} \cdot \frac{t_{r+u}}{t_2^{u-1}}, & \text{if } 0 < u < k-r+1 \text{ and } v = 1, \\ 0, & \text{otherwise}, \end{cases}$$

i.e.,

$$H_{k,n}^r = \begin{bmatrix} t_{r+1} & -it_2 & \cdots & 0 & \\ i\frac{t_{r+2}}{t_2} & t_1 & -it_2 & \cdots & 0 \\ i^2\frac{t_{r+3}}{t_2^2} & i & t_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ i^{k-r-1}\frac{t_k}{t_2^{k-r-1}} & i^{k-r-2}\frac{t_{k-r-1}}{t_2^{k-r-2}} & i^{k-r-3}\frac{t_{k-r-2}}{t_2^{k-r-3}} & \cdots & 0 \\ 0 & i^{k-r-1}\frac{t_{k-r}}{t_2^{k-r-1}} & i^{k-r-2}\frac{t_{k-r-1}}{t_2^{k-r-2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & -it_2 \\ 0 & 0 & 0 & \cdots & t_1 \end{bmatrix},$$

where $t_0 = 1$ and $i = \sqrt{-1}$. Then

$$\mathrm{per}\,(H_{k,n}^r) = (-1)^r S_{(n,1^r)}. \tag{3.1}$$

Proof. Since the proof is similar to the proof of Theorem 2.2, by using Lemma 3.1, we omit the details. □

Corollary 3.3. [**24**]. *Let $k \geq 2$ be an integer, $f_{k,n}$ the generalized order-k Fibonacci numbers and $H_{k,n} = (h_{rs})$ an $n \times n$ lower Hessenberg matrix, given by*

$$h_{rs} = \begin{cases} i^{r-s}, & \text{if } -1 \leq r-s < k, \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$\operatorname{per}(H_{k,n}) = f_{k,k+n-1}.$$

Proof. It is direct from Theorem 3.2 for $t_i = 1$. □

Theorem 3.4. *Let $k \geq 2$, $n \geq 1$ and $0 \leq r \leq k-1$ be integers, $(-1)^r S_{(n,1^r)}$ the $(k-r)$-th column of the matrix $A^{\infty}_{(k)}$ and $L^r_{k,n} = (l_{ij})$ an $n \times n$ Hessenberg matrix, given by*

$$l_{ij} = \begin{cases} -t_2, & \text{if } j = i+1 \text{ and } j \neq 1, \\ \frac{t_{i-j+1}}{t_2^{i-j}}, & \text{if } 0 \leq i-j < k \text{ and } j \neq 1, \\ \frac{t_{r+i}}{t_2^{i-1}}, & \text{if } 0 < i < k-r+1 \text{ and } j = 1, \\ 0, & \text{otherwise}, \end{cases}$$

i.e.,

$$L^r_{k,n} = \begin{bmatrix} t_{r+1} & t_2 & 0 & 0 & \cdots & 0 \\ \frac{t_{r+2}}{t_2} & t_1 & t_2 & 0 & \cdots & 0 \\ \frac{t_{r+3}}{t_2^2} & 1 & t_1 & t_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \frac{t_k}{t_2^{k-r-1}} & \frac{t_{k-r-1}}{t_2^{k-r-2}} & \frac{t_{k-r-2}}{t_2^{k-r-3}} & \frac{t_{k-r-3}}{t_2^{k-r-4}} & \cdots & 0 \\ 0 & \frac{t_{k-r}}{t_2^{k-r-1}} & \frac{t_{k-r-1}}{t_2^{k-r-2}} & \frac{t_{k-r-2}}{t_2^{k-r-3}} & \cdots & 0 \\ & \vdots & \vdots & \vdots & \ddots & t_2 \\ 0 & 0 & 0 & \cdots & \cdots & t_1 \end{bmatrix},$$

where $t_0 = 1$. Then

$$\operatorname{per}(L^r_{k,n}) = (-1)^r S_{(n,1^r)}. \tag{3.2}$$

Proof. This is similar to the proof of Theorem 2.5 by using Lemma 3.1. □

Corollary 3.5. **[23]**. *Let $k \geq 2$ be an integer, $f_{k,n}$ the generalized order-k Fibonacci numbers and $D_{k,n} = (d_{ij})$ an $n \times n$ lower Hessenberg matrix such that*

$$d_{ij} = \begin{cases} 1, & \text{if } -1 \leq i-j < k \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\operatorname{per}(D_{k,n}) = f_{k,k+n-1}.$$

Proof. It is direct from Theorem 3.4 for $t_i = 1$. $\square$

The next two lemmas show that the matrices $A_{(k)}^{\infty}$ are general forms of many types of sequences and polynomials having linear recursions. Some of these have a very wide range of application areas, see papers **[2, 3, 4, 5, 7, 9, 10, 11, 12, 13, 14, 22, 23, 24, 25, 26, 27]**.

Lemma 3.6. *The matrix $A_{(k)}^{\infty}$ involves many types of sequences and polynomials having linear recursions. We obtain some of them from $A_{(k)}^{\infty}$ as follows.*

(i) $(-1)^r S_{(n,1^r)} = f_{k,n}^{r+1}$ *for* $c_i = t_i$ $(1 \le i \le k)$ *and* $0 \le r \le k-1$,
(ii) $(-1)^r S_{(n,1^r)} = p_{k,n}^{r+1}$ *for* $t_1 = 2$ *and* $t_i = 1$ $(2 \le i \le k)$,
(iii) $(-1)^r S_{(n,1^r)} = v_{k,n}^{k-r}$ *for* $t_1 = 0$ *and* $t_i = 1$ $(2 \le i \le k)$,
(iv) $S_{(n+1)} = F_{p,n}(x,y)$ *for* $t_1 = x$, $t_k = y$, $t_i = 0$ $(2 \le i \le k-1)$ *and* $k = (p+1)$.

Lemma 3.7. [25]. *$F_{p,n}(x,y)$ is a general form of many popular sequences and polynomials, such as:*

x	y	p	$F_{p,n}(x,y)$
x	y	1	*bivariate Fibonacci polynomials* $F_n(x,y)$
x	1	p	*Fibonacci* $p-$*polynomials* $F_{p,n}(x)$
x	1	1	*Fibonacci polynomials* $f_n(x)$
1	1	p	*Fibonacci* $p-$*numbers* $F_p(n)$
1	1	1	*Fibonacci numbers* F_n
$2x$	y	p	*bivariate Pell p-polynomials* $F_{p,n}(2x,y)$
$2x$	y	1	*bivariate Pell polynomials* $F_n(2x,y)$
$2x$	1	p	*Pell p-polynomials* $P_{p,n}(x)$
$2x$	1	1	*Pell polynomials* $P_n(x)$
2	1	1	*Pell numbers* P_n
$2x$	-1	1	*second kind Chebysev polynomials* $U_{n-1}(x)$
x	$2y$	p	*bivariate Jacobsthal p-polynomials* $F_{p,n}(x,2y)$
x	$2y$	1	*bivariate Jacobsthal polynomials* $F_n(x,2y)$
1	$2y$	1	*Jacobsthal polynomials* $J_n(y)$
1	2	1	*Jacobsthal numbers* J_n

The following corollaries follow from the preceding two lemmas and theorems concerning determinants and permanents.

Corollary 3.8. *By using Lemma* 3.6, *we rewrite equations* (2.2), (2.3), (3.1) *and* (3.2):

(i) *for* $0 \leq r \leq k-1$ *and* $t_i = c_i$ $(1 \leq i \leq k)$, *we obtain*

$$\det(Q^r_{k,n}) = \det(B^r_{k,n}) = \operatorname{per}(H^r_{k,n}) = \operatorname{per}(L^r_{k,n}) = f^{r+1}_{k,n},$$

(ii) *for* $0 \leq r \leq k-1$, $t_1 = 2$ *and* $t_i = 1$ *for* $2 \leq i \leq k$, *we obtain*

$$\det(Q^r_{k,n}) = \det(B^r_{k,n}) = \operatorname{per}(H^r_{k,n}) = \operatorname{per}(L^r_{k,n}) = p^{r+1}_{k,n},$$

(iii) *for* $0 \leq r \leq k-1$, $t_1 = 0$ *and* $t_i = 1$ *for* $2 \leq i \leq k$, *we obtain*

$$\det(Q^r_{k,n}) = \det(B^r_{k,n}) = \operatorname{per}(H^r_{k,n}) = \operatorname{per}(L^r_{k,n}) = v^{k-r}_{k,n},$$

(iv) *for* $t_1 = x$, $t_k = y$, $t_i = 0$ *for* $2 \leq i \leq k-1$ *and* $k = (p+1)$, *we obtain*

$$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(Q^0_{k,n}) = \operatorname{per}(B^0_{k,n}) = F_{p,n-1}(x,y).$$

Corollary 3.9. *By using Lemma* 3.7, *we rewrite equations* (2.2), (2.3), (3.1) *and* (3.2) *for* $t_1 = x$, $t_k = y$, $t_i = 0$ $(2 \leq i \leq k-1)$ *and* $k = p+1$. *We obtain the following table*:

x	y	p	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = F_{p,n-1}(x,y)$,
x	y	1	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = F_{n-1}(x,y)$,
x	1	p	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = F_{p,n-1}(x)$,
x	1	1	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = f_{n-1}(x)$,
1	1	p	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = F_p(n-1)$,
1	1	1	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = F_{n-1}$,
$2x$	y	p	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = F_{p,n-1}(2x,y)$,
$2x$	y	1	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = F_{n-1}(2x,y)$,
$2x$	1	p	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = P_{p,n-1}(x)$,
$2x$	1	1	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = P_{n-1}(x)$,
2	1	1	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = P_{n-1}$,
$2x$	-1	1	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = U_n(x)$,
x	$2y$	p	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = F_{p,n-1}(x,2y)$,
x	$2y$	1	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = F_{n-1}(x,2y)$,
1	$2y$	1	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = J_{n-1}(y)$,
1	2	1	$\det(Q^0_{k,n}) = \det(B^0_{k,n}) = \operatorname{per}(H^0_{k,n}) = \operatorname{per}(L^0_{k,n}) = J_{n-1}$.

Now we show that determinants of Hessenberg matrices can be obtained by using combinations.

Corollary 3.10. *Let $k \geq 2$, $n \geq 1$ and $0 \leq r \leq k-1$ be integers, $(-1)^r S_{(n,1^r)}$ the $(k-r)$-th column of the matrix $A_{(k)}^{\infty}$. Then,*

$$\det(Q_{k,n}^r) = \det(B_{k,n}^r) = \operatorname{per}(H_{k,n}^r) = \operatorname{per}(L_{k,n}^r)$$
$$= (-1)^r S_{(n,1^r)} = \sum_{j=r+1}^{k} t_j \left[\sum_{a \vdash n-j+r} \binom{|a|}{a_{1,\ldots,}a_k} t_1^{a_1} \ldots t_k^{a_k} \right].$$

Proof. It is direct from equations (2.2), (2.3), (3.1) and (3.2) by using equations (1.4) and (1.5). □

4. Conclusions. In this paper, we showed how extensive are the generalized Fibonacci polynomials defined by MacHenry, and the results obtained by many researchers before are, in fact, special cases of generalized Fibonacci polynomials. In addition, we obtained any term of Fibonacci polynomials by using determinants and permanents of matrices, which are easier to calculate. Moreover, we showed how to calculate any term of the matrices $A_{(k)}^{\infty}$, and consequently any term of sequences and polynomials mentioned above.

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