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PRIME DIVISORS OF IRREDUCIBLE CHARACTER DEGREES

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ABSTRACT. Let G be a finite group. We denote by $\rho(G)$ the set of primes which divide some character degrees of G and by $\sigma(G)$ the largest number of distinct primes which divide a single character degree of G. We show that $|\rho(G)| \leq 2\sigma(G)+1$ when G is an almost simple group. For arbitrary finite groups G, we show that $|\rho(G)| \leq 2\sigma(G)+1$ provided that $\sigma(G) \leq 2$.

1. Introduction. Throughout this paper, all groups are finite, and all characters are complex characters. The set of all complex irreducible characters of G is denoted by $\mathrm{Irr}(G)$, and we let $\mathrm{cd}(G)$ be the set of all complex irreducible character degrees of G. We define $\rho(G)$ to be the set of primes which divide some character degree of G. For $\chi \in \mathrm{Irr}(G)$, let $\pi(\chi)$ be the set of all prime divisors of $\chi(1)$, and let $\sigma(\chi) = |\pi(\chi)|$. Moreover, $\sigma(G)$ is defined to be the maximum value of $\sigma(\chi)$ when χ runs over the set $\mathrm{Irr}(G)$. Huppert's $\rho-\sigma$ conjecture proposed by Huppert in **[7]** states that if G is a solvable group, then $|\rho(G)| \leq 2\sigma(G)$; and, if G is an arbitrary group, then $|\rho(G)| \leq 3\sigma(G)$. For solvable groups, this conjecture has been verified by Manz **[11]** and Gluck **[6]** when $\sigma(G) = 1$ and 2, respectively. In general, it is proved by Manz and Wolf **[13]** that $|\rho(G)| \leq 3\sigma(G)+2$. For arbitrary groups, Manz **[12]** showed that $|\rho(G)| = 3$ if G is nonsolvable and $\sigma(G) = 1$. Recently, it has been proved by Casolo and Dolfi **[3]** that $|\rho(G)| \leq 7\sigma(G)$ for any arbitrary groups G. In **[13]**, Manz and Wolf proposed that, for any group G,

$$|\rho(G)| \leq 2\sigma(G)+1.$$

We call this new conjecture the *strengthened Huppert's $\rho-\sigma$ conjecture.* Obviously, this new conjecture is stronger than the original one. In

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this paper, we first improve the result due to Alvis and Barry in [**1**] by proving the following.

Theorem A. Let G be an almost simple group. Then $|\rho(G)| \leq 2\sigma(G)$ unless $G \cong \mathrm{PSL}_2(2^f)$ with $f \geq 2$ and $|\pi(2^f - 1)| = |\pi(2^f + 1)|$. For the exceptions, we have $|\rho(G)| = 2\sigma(G) + 1$.

This verifies the strengthened Huppert's $\rho - \sigma$ conjecture for almost simple groups. In the next theorem, we verify this new conjecture for groups G with $\sigma(G) \leq 2$.

Theorem B. Let G be a finite group. If $\sigma(G) \leq 2$, then $|\rho(G)| \leq 2\sigma(G) + 1$.

Notice that Theorem B is also a generalization to [**19,** Theorem A].

Notation. For a positive integer n, we denote the set of all prime divisors of n by $\pi(n)$. If G is a group, then we write $\pi(G)$ instead of $\pi(|G|)$ for the set of all prime divisors of the order of G. If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is denoted by $I_G(\theta)$. We write $\mathrm{Irr}(G|\theta)$ for the set of all irreducible constituents of θ^G. Moreover, if $\chi \in \mathrm{Irr}(G)$, then $\mathrm{Irr}(\chi_N)$ is the set of all irreducible constituents of χ when restricted to N. Recall that a group G is said to be an almost simple group with socle S if there exists a nonabelian simple group S such that $S \trianglelefteq G \leq \mathrm{Aut}(S)$. The greatest common divisor of two integers a and b is $\gcd(a, b)$. Denote by $\Phi_k := \Phi_k(q)$ the value of the kth cyclotomic polynomial evaluated at q. Other notation is standard.

2. Proof of Theorem A.

If G is an almost simple group, then G has no normal abelian Sylow subgroup and so, by Ito-Michler's theorem [**14,** Theorem 5.4], $\rho(G) = \pi(G)$. This fact will be used without any further reference.

Lemma 2.1. *Let S be a sporadic simple group, the Tits group or an alternating group of degree at least* 7. *If G is an almost simple group with socle S, then*

$$|\pi(G)| = |\pi(S)| \leq 2\sigma(G).$$

Proof. Observe first that, if S is one of the simple groups in the lemma, and G is any almost simple group with socle S, then $\pi(G) = \pi(S)$. Since $S \trianglelefteq G$, we see that $\sigma(S) \leq \sigma(G)$. Thus, it suffices to show that $|\pi(S)| \leq 2\sigma(S)$. By using **[4]**, we can easily check that $|\pi(S)| \leq 2\sigma(S)$ when S is a sporadic simple group, the Tits group or an alternating group of degree n with $7 \leq n \leq 14$. Finally, if $S \cong \mathrm{A}_n$ with $n \geq 15$, then the result in **[2]** yields that $|\pi(S)| = \sigma(S)$. This completes the proof. $\square$

For $\epsilon = \pm$, we use the convention that $\mathrm{PSL}_n^{\epsilon}(q)$ is $\mathrm{PSL}_n(q)$ if $\epsilon = +$ and $\mathrm{PSU}_n(q)$ if $\epsilon = -$. Let $q \geq 2$ and $n \geq 3$ be integers with $(n, q) \neq (6, 2)$. A prime ℓ is called a *primitive prime divisor* of $q^n - 1$ if $\ell \mid q^n - 1$ but $\ell \nmid q^m - 1$ for any $m < n$. By Zsigmondy's theorem **[21]**, the primitive prime divisors of $q^n - 1$ always exist. We denote by $\ell_n(q)$ the smallest primitive prime divisor of $q^n - 1$. In Table 1, which is taken from **[10]**, we give the orders of two maximal tori T_i and the corresponding two primitive prime divisors ℓ_i, for $i = 1, 2$, of classical groups.

Let $\mathcal{C}$ be the set consisting of the following simple groups:

$$\begin{array}{lllll}
\mathrm{PSL}_2(q), & \mathrm{PSL}_3(q), & \mathrm{PSU}_3(q), & \mathrm{PSp}_4(q) & \mathrm{PSL}_4(2), \\
\mathrm{PSL}_6(2), & \mathrm{PSL}_7(2), & \mathrm{PSU}_4(2), & \mathrm{PSU}_4(3), & \mathrm{PSU}_6(2), \\
\mathrm{Sp}_4(2)', & \mathrm{Sp}_6(2), & \mathrm{Sp}_8(2), & \Omega_7(3), & \Omega_8^+(2), \\
\Omega_8^-(2), & {}^3\mathrm{D}_4(2), & \mathrm{G}_2(2)', & \mathrm{G}_2(3), & \mathrm{G}_2(4).
\end{array}$$

Lemma 2.2. *Let S be a finite simple group of Lie type in characteristic p which is not the Tits groups nor* $\mathrm{PSL}_2(2^f)$ *with $f \geq 2$. Then $|\pi(S)| \leq 2\sigma(S)$.*

Proof. We consider the following cases.

Case 1. $S \cong \mathrm{PSL}_2(q)$, where $q = p^f \geq 5$ is odd.

Since $\mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4)$, we can assume that $q > 5$. In this case, all character degrees of S divide $q, q-1$ or $q+1$. Observe that

$$\pi(S) = \{p\} \cup \pi(q-1) \cup \pi(q+1), \{p\} \cap \pi(q \pm 1) = \emptyset$$

and

$$\pi(q-1) \cap \pi(q+1) = \{2\}.$$

TABLE 1. Two tori for classical groups.

$G=G(q)$	$\lvert T_1\rvert$	$\lvert T_2\rvert$	ℓ_1	ℓ_2
A_n	$(q^{n+1}-1)/(q-1)$	q^n-1	$\ell_{n+1}(q)$	$\ell_n(q)$
${}^2\mathrm{A}_n$, $(n\equiv 0(4))$	$(q^{n+1}+1)/(q+1)$	q^n-1	$\ell_{2n+2}(q)$	$\ell_n(q)$
${}^2\mathrm{A}_n$, $(n\equiv 1(4))$	$(q^{n+1}-1)/(q+1)$	q^n+1	$\ell_{(n+1)/2}(q)$	$\ell_{2n}(q)$
${}^2\mathrm{A}_n$, $(n\equiv 2(4))$	$(q^{n+1}+1)/(q+1)$	q^n-1	$\ell_{2n+2}(q)$	$\ell_{n/2}(q)$
${}^2\mathrm{A}_n$, $(n\equiv 3(4))$	$(q^{n+1}-1)/(q+1)$	q^n+1	$\ell_{n+1}(q)$	$\ell_{2n}(q)$
$\mathrm{B}_n,\mathrm{C}_n$ $(n\geq 3$ odd$)$	q^n+1	q^n-1	$\ell_{2n}(q)$	$\ell_n(q)$
$\mathrm{B}_n,\mathrm{C}_n$ $(n\geq 2$ even$)$	q^n+1	$(q^{n-1}+1)(q+1)$	$\ell_{2n}(q)$	$\ell_{2n-2}(q)$
D_n, $(n\geq 5$ odd$)$	$(q^{n-1}+1)(q+1)$	q^n-1	$\ell_{2n-2}(q)$	$\ell_n(q)$
D_n, $(n\geq 4$ even$)$	$(q^{n-1}+1)(q+1)$	$(q^{n-1}-1)(q-1)$	$\ell_{2n-2}(q)$	$\ell_{n-1}(q)$
${}^2\mathrm{D}_n$	q^n+1	$(q^{n-1}+1)(q-1)$	$\ell_{2n}(q)$	$\ell_{2n-2}(q)$

Hence, we obtain that

$$\begin{aligned}|\pi(S)| &= 1+\sigma(q+1)+\sigma(q-1)-|\pi(q-1)\cap\pi(q+1)|\\ &= \sigma(q+1)+\sigma(q-1)\leq 2\sigma(S).\end{aligned}$$

Case 2. $S\cong \mathrm{PSL}_3^\epsilon(q)$ with $q=p^f$ and $\epsilon=\pm$. As $\mathrm{PSL}_3(2)\cong \mathrm{PSL}_2(7)$ and $\mathrm{PSU}_3(2)$ are not simple, we can assume that $q>2$. The cases when $q=3$ or 4 can be checked directly using **[4]**. So, we can assume that $q\geq 5$. By **[17]**, S possesses irreducible characters χ_i, $i=1,2$, with degree

$$\chi_1(1)=(q-\epsilon 1)^2(q+\epsilon 1)\quad\text{and}\quad \chi_2(1)=q(q^2+\epsilon q+1).$$

Let $d = \gcd(3, q - \epsilon 1)$. Then

$$|S| = \frac{1}{d}q^3(q^2-1)(q^3-\epsilon 1) = \frac{1}{d}q^3(q-\epsilon 1)^2(q+\epsilon 1)(q^2+\epsilon q+1),$$

and so

$$\pi(S) = \pi(\chi_1) \cup \pi(\chi_2).$$

Therefore, $|\pi(S)| \leq 2\sigma(S)$ as wanted.

Case 3. $S \cong \mathrm{PSp}_4(q)$ with $q = p^f > 2$.

By **[5, 18]**, S has two irreducible characters χ_i, $i = 1, 2$, with degrees $\Phi_1^2\Phi_2^2$ and $q\Phi_1\Phi_4$, respectively. Since

$$|S| = \frac{1}{d}q^4\Phi_1^2\Phi_2^2\Phi_4$$

where $d = \gcd(2, q-1)$, we deduce that

$$\pi(S) = \pi(\chi_1) \cup \pi(\chi_2),$$

and thus $|\pi(S)| \leq 2\sigma(S)$.

Case 4. S is one of the remaining simple groups in the list $\mathcal{C}$.

Using **[4]**, it is routine to check that $|\pi(S)| \leq 2\sigma(S)$ in all these cases.

Case 5. S is not in the list $\mathcal{C}$.

We consider the following setup. Let $\mathscr{G}$ be a simple, simply connected algebraic group defined over a field of size q in characteristic p, and let F be a Frobenius map on $\mathscr{G}$ such that $S \cong L/Z$, where $L := \mathscr{G}^F$ and $Z := \mathrm{Z}(L)$. Let the pair $(\mathscr{G}^*, F^*)$ be dual to $(\mathscr{G}, F)$ and let $L^* := \mathscr{G}^{*F^*}$. By Lusztig's theory, the irreducible characters of $\mathscr{G}^F$ are partitioned into rational series $\mathscr{E}(\mathscr{G}^F, (s))$ which are indexed by $(\mathscr{G}^{*F^*})$-conjugacy classes (s) of semisimple elements $s \in \mathscr{G}^{*F^*}$. Furthermore, if $\gcd(|s|, |Z|) = 1$, then every $\chi \in \mathscr{E}(\mathscr{G}^F, (s_i))$ is trivial at Z, and thus $\chi \in \mathrm{Irr}(S) = \mathrm{Irr}(L/Z)$. (See **[15,** page 349**]**). Notice also that $\chi(1)$ is divisible by $|L^* : \mathbf{C}_{L^*}(s)|_{p'}$.

For simple classical groups of Lie type, the restriction on S guarantees that both primitive prime divisors ℓ_i in Table 1 exist. Let $s_i \in \mathscr{G}^{*F^*}$ with $|s_i| = \ell_i$, $i = 1, 2$. Then $\mathbf{C}_{L^*}(s_i) = T_i$ for $i = 1, 2$, where T_i are maximal tori of L^*. Similarly, for each simple exceptional group of Lie type S, by **[15,** Lemma 2.3], one can find two semisimple elements $s_i \in \mathscr{G}^{*F^*}$ with $|s_i| = \ell_i$, $i = 1, 2$. In both cases, we have that $(\ell_i, |Z|) = 1$ for $i = 1, 2$ and, if $a := \gcd(|\mathbf{C}_{L^*}(s_1)|, |\mathbf{C}_{L^*}(s_2)|)$, then either $a = 1$ or, if a prime r divides a, then r also divides $|L^* : \mathbf{C}_{L^*}(s_i)|_{p'}$ for some i. Let $\chi_i \in \mathscr{E}(\mathscr{G}^F, (s_i))$, $i = 1, 2$, be such that $\chi_i(1) = |L^* : \mathbf{C}_{L^*}(s_i)|_{p'}$. Then $\chi_i \in \mathrm{Irr}(S)$ for $i = 1, 2$ and

$$\pi(S) = \{p\} \cup \pi(\chi_1) \cup \pi(\chi_2).$$

Notice that p is relatively prime to both $\chi_i(1)$ for $i = 1, 2$. So,

$$\begin{aligned} |\pi(S)| &= |\{p\} \cup \pi(\chi_1) \cup \pi(\chi_2)| \\ &= 1 + |\pi(\chi_1)| + |\pi(\chi_2)| - |\pi(\chi_1) \cap \pi(\chi_2)| \\ &= \sigma(\chi_1) + \sigma(\chi_2) - (|\pi(\chi_1) \cap \pi(\chi_2)| - 1) \\ &\leq 2\sigma(S) - (|\pi(\chi_1) \cap \pi(\chi_2)| - 1). \end{aligned}$$

If we can show that $|\pi(\chi_1) \cap \pi(\chi_2)| \geq 1$, then clearly $|\pi(S)| \leq 2\sigma(S)$, and we are done. By way of contradiction, assume that $\pi(\chi_1) \cap \pi(\chi_2)$ is empty. Then $\gcd(\chi_1(1), \chi_2(1)) = 1$, and so

$$\gcd(|L^* : \mathbf{C}_{L^*}(s_1)|_{p'}, |L^* : \mathbf{C}_{L^*}(s_2)|_{p'}) = 1.$$

It follows that $|L^*|_{p'}$ must divide $|\mathbf{C}_{L^*}(s_1)|_{p'} \cdot |\mathbf{C}_{L^*}(s_2)|_{p'}$. However, we can check by using **[15,** Lemma 2.3] and Table 1 that this is not the case. The proof is now complete. □

We now prove Theorem A which we restate here.

Theorem 2.3. *Let G be an almost simple group. Then $|\rho(G)| \leq 2\sigma(G)$ unless $G \cong \mathrm{PSL}_2(2^f)$ with $|\pi(2^f - 1)| = |\pi(2^f + 1)|$. For the exceptions, we have $|\rho(G)| = 2\sigma(G) + 1$.*

Proof. Let G be an almost simple group with simple socle S. Since $S \trianglelefteq G$, we obtain that $\sigma(S) \leq \sigma(G)$.

Case 1. $S \cong \mathrm{PSL}_2(q)$ with $q = 2^f \geq 4$.

It is well known that $|S| = q(q^2 - 1)$, $\gcd(2^f - 1, 2^f + 1) = 1$ and

$$\mathrm{cd}(S) = \{1, q - 1, q, q + 1\}.$$

If $|\pi(q-1)| = |\pi(q+1)|$, then

$$\pi(S) = \{2\} \cup \pi(q-1) \cup \pi(q+1),$$

and thus $|\pi(S)| = 2\sigma(S) + 1$ as $\sigma(S) = |\pi(2^f \pm 1)|$. Assume that $|\pi(q-1)| \neq |\pi(q+1)|$. Then $|\pi(2^f+\delta)| > |\pi(2^f-\delta)|$ for some $\delta \in \{\pm 1\}$. Hence, $\sigma(S) = |\pi(2^f + \delta)|$, and thus

$$\begin{aligned}|\pi(S)| &= |\{2\} \cup \pi(2^f - \delta) \cup \pi(2^f + \delta)| \\ &= 1 + |\pi(2^f - \delta)| + |\pi(2^f + \delta)|.\end{aligned}$$

Since $|\pi(2^f + \delta)| \geq |\pi(2^f - \delta)| + 1$, we obtain that

$$|\rho(S)| \leq 2|\pi(2^f + \delta)| \leq 2\sigma(S).$$

Thus, the result holds when $G = S$.

Assume now that $|G : S|$ is nontrivial. We know that $\mathrm{Aut}(S) = S \cdot \langle\varphi\rangle$, where φ is a field automorphism of S of order f. Thus, $G = S \cdot \langle\psi\rangle$, with $\psi \in \langle\varphi\rangle$. If $f = 2$, then $G \cong \mathrm{A}_5 \cdot 2$, and obviously $|\pi(G)| \leq 2\sigma(G)$. Hence, we can assume that $f > 2$. Clearly, if $f \equiv 2 \pmod 4$ and $G = S \cdot \langle\varphi\rangle$, then $|G : S| > 2$. So by **[20,** Theorem A], G has two irreducible characters $\chi_i \in \mathrm{Irr}(G)$, $i = 1, 2$, with $\chi_1(1) = |G : S|(q-1)$ and $\chi_2(1) = |G : S|(q+1)$. Obviously,

$$\pi(G) = \{2\} \cup \pi(\chi_1) \cup \pi(\chi_2)$$

and

$$\pi(\chi_1) \cap \pi(\chi_2) = \pi(|G : S|) \neq \emptyset.$$

If $|G : S|$ is even, then

$$|\rho(G)| = |\pi(\chi_1) \cup \pi(\chi_2)| \leq |\pi(\chi_1)| + |\pi(\chi_2)| \leq 2\sigma(G).$$

If $|G : S|$ is odd, then

$$\begin{aligned}|\rho(G)| &= |\{2\} \cup \pi(\chi_1) \cup \pi(\chi_2)| \\ &= 1 + |\pi(\chi_1)| + |\pi(\chi_2)| - |\pi(\chi_1) \cap \pi(\chi_2)| \\ &= \sigma(\chi_1) + \sigma(\chi_2) - (|\pi(|G : S|)| - 1) \\ &\leq \sigma(\chi_1) + \sigma(\chi_2) \\ &\leq 2\sigma(G).\end{aligned}$$

Case 2. S is a sporadic simple group, the Tits group or an alternating group of degree at least 7.

By Lemma 2.1, we obtain that $|\rho(G)| \leq 2\sigma(G)$.

Case 3. S is a finite simple group of Lie type in characteristic p and S is not the Tits group nor $\mathrm{PSL}_2(2^f)$ with $f \geq 2$.

Subcase 3a. $\pi(G) = \pi(S)$.

By Lemma 2.2, we have that $|\pi(S)| \leq 2\sigma(S)$. Thus,

$$|\rho(G)| = |\pi(S)| \leq 2\sigma(S) \leq 2\sigma(G).$$

Subcase 3b. $\pi := \pi(G) - \pi(S)$ is nonempty.

Let A be the subgroup of the group of coprime outer automorphisms of S induced by the action of G on S. By [**15,** Lemma 2.10], A is cyclic and central in $\mathrm{Out}(S)$. Moreover, A is generated by a fixed field automorphism $\gamma \in \mathrm{Out}(S)$. It follows that the group $S \cdot A$ is normal in G and $\pi(S \cdot A) = \pi(G)$. Thus we can assume that $G = S \cdot A$ with $A = \langle \gamma \rangle$ and γ a field automorphism of S. Furthermore, $\pi(\gamma) = \pi$. Replacing A by a normal subgroup if necessary, we can also assume that $|A| = |\gamma|$ is the product of all distinct primes in π.

As in the proof of Lemma 2.2, let $\mathscr{G}$ be a simple, simply connected algebraic group defined over a field of size $q = p^f$ in characteristic p, and let F be a Frobenius map of $\mathscr{G}$ such that $S \cong L/Z$, where $L := \mathscr{G}^F$ and $Z := \mathrm{Z}(L)$. Let the pair $(\mathscr{G}^*, F^*)$ be dual to $(\mathscr{G}, F)$, and let $L^* := {\mathscr{G}^*}^{F^*}$. As $\pi \subseteq \pi(f)$, where $\pi = \pi(G) - \pi(S)$, it is easy to check that both the primitive prime divisors in [**15,** Lemmas 2.3 and 2.4] exist, and thus one can find two semisimple elements $s_i \in {\mathscr{G}^*}^{F^*}$ with $|s_i| = \ell_i$ such that $(\ell_i, |Z|) = 1$ for $i = 1, 2$. Arguing as in the proof of Lemma 2.2, we obtain that

$$\pi(S) = \{p\} \cup \pi(\chi_1) \cup \pi(\chi_2),$$

where $\chi_i \in \mathscr{E}(\mathscr{G}^F, (s_i))$ such that $\chi_i(1) = |L^* : \mathbf{C}_{L^*}(s_i)|_{p'}$ and χ_i can be considered as characters of S for $i = 1, 2$.

We next claim that the inertia group for both χ_i, $i = 1, 2$, in G is exactly S. It suffices to show that no field automorphism of S of prime order can fix χ_i for $i = 1, 2$. Let τ be a field automorphism of S of prime order s. We can extend τ to an automorphism of $\mathscr{G}^F$ and $\mathscr{G}^{*F^*}$, which we also denote by τ. Notice that $\mathbf{C}_{\mathscr{G}^{*F^*}}(\tau)$ is a finite group of Lie type of the same type as that of $\mathscr{G}^{*F^*}$ but defined over a field of size $q^{1/s}$. Now it is straightforward to check that both ℓ_i, $i = 1, 2$, are relatively prime to $|\mathbf{C}_{\mathscr{G}^{*F^*}}(\tau)|$. Hence, $\mathscr{G}^{*F^*}$-conjugacy classes (s_i) of s_i in $\mathscr{G}^{*F^*}$ are not τ-invariant for $i = 1, 2$ (see [**15,** Proposition 2.6]). Then $\tau(s_i)$ and s_i are not $\mathscr{G}^{*F^*}$-conjugate for $i = 1, 2$, and thus $\chi_i \in \mathscr{E}(\mathscr{G}^F, (s_i))$, $i = 1, 2$, are not τ-invariant (see [**15,** Theorem 2.7]). Therefore, we obtain that $\chi_i^G \in \mathrm{Irr}(G)$ for $i = 1, 2$; hence, $\chi_i^G(1) = |G : S|\chi_i(1) \in \mathrm{cd}(G)$. Since

$$\pi(S) = \{p\} \cup \pi(\chi_1) \cup \pi(\chi_2) \text{ and } \pi(G) = \pi(S) \cup \pi(|G : S|),$$

we obtain that

$$\begin{aligned}\pi(G) &= \{p\} \cup \pi(|G : S|\chi_1(1)) \cup \pi(|G : S|\chi_2(1)) \\ &= \{p\} \cup \pi(\chi_1^G) \cup \pi(\chi_2^G).\end{aligned}$$

Moreover, $p \nmid |G : S|\chi_i(1) = \chi_i^G(1)$ for $i = 1, 2$, and

$$|\pi(\chi_1^G) \cap \pi(\chi_2^G)| \geq 1.$$

Therefore,

$$\begin{aligned}|\pi(G)| &= 1 + \sigma(\chi_1^G) + \sigma(\chi_2^G) - |\pi(\chi_1^G) \cap \pi(\chi_2^G)| \\ &\leq 2\sigma(G) - (|\pi(\chi_1^G) \cap \pi(\chi_2^G)| - 1) \\ &\leq 2\sigma(G).\end{aligned}$$

The proof is now complete. $\square$

The next results will be needed in the proof of Theorem B.

Lemma 2.4. *Let S be a nonabelian simple group. If $\sigma(S) \leq 2$, then S is one of the following groups.*

(i) $S \cong \mathrm{PSL}_2(2^f)$ *with* $|\pi(2^f \pm 1)| \leq 2$, *and so* $|\pi(S)| \leq 5$.
(ii) $S \cong \mathrm{PSL}_2(q)$ *with* $q > 5$ *odd and* $|\pi(q \pm 1)| \leq 2$ *and so* $|\pi(S)| \leq 4$.
(iii) $S \in \{\mathrm{M}_{11}, \mathrm{A}_7, {}^2\mathrm{B}_2(8), {}^2\mathrm{B}_2(32), \mathrm{PSL}_3^{\pm}(3), \mathrm{PSL}_3^{\pm}(4), \mathrm{PSL}_3(8)\}$ *and* $|\pi(S)| = 4$.

Proof. As S is a nonabelian simple group, we have that $|\pi(S)| \geq 3$. If $S \cong \mathrm{PSL}_2(q)$ with $q \geq 4$, then the lemma follows easily as the character degree set of S is known. Now assume that $S \not\cong \mathrm{PSL}_2(q)$. Then Lemmas 2.2 and 2.1 imply that $|\pi(S)| \leq 2\sigma(S)$. So, $3 \leq |\pi(S)| \leq 4$. By checking the list of nonabelian simple groups with at most four prime divisors in [**8**], we deduce that only those nonabelian simple groups appearing in (iii) above satisfy the hypotheses of the lemma. $\square$

Lemma 2.5. *Let G be an almost simple group with simple socle S. If $\sigma(G) \leq 2$, then $\pi(G) = \pi(S)$, where S is one of the simple groups in Lemma 2.4.*

Proof. Since $\sigma(S) \leq \sigma(G) \leq 2$, we deduce that S is isomorphic to one of the simple groups in the conclusion of Lemma 2.4. If $|\pi(S)| = 3$, then S is one of the simple groups in [**8**, Table 1], and we can check that $\pi(G) = \pi(S)$ in these cases. Thus, we assume that $|\pi(S)| \geq 4$. Now, if $G = S$, then we have nothing to prove. So, we assume that $G \neq S$. In particular, $G \not\cong \mathrm{PSL}_2(2^f)$ with $f \geq 2$. Then $|\pi(G)| \leq 2\sigma(G) \leq 4$ by Theorem A, and thus $4 \leq |\pi(S)| \leq |\pi(G)| \leq 4$, which forces $|\pi(S)| = |\pi(G)|$ and, hence, $\pi(G) = \pi(S)$ as wanted. $\square$

3. Proof of Theorem B. The following two lemmas are obvious.

Lemma 3.1. *Let A and B be groups such that $|\rho(A)| \geq 3$ and $|\rho(B)| \geq 3$. If*

$$\sigma(A \times B) \leq 2,$$

then $\sigma(A) = 1 = \sigma(B)$.

Lemma 3.2. *Let N be a normal subgroup of a group G. If $\rho(G/N) = \pi(G/N)$, then*

$$\rho(G) - \rho(G/N) \subseteq \rho(N).$$

Recall that the solvable radical of a group G is the largest normal solvable subgroup of G.

Lemma 3.3. *Let G be a nonsolvable group, and let N be the solvable radical of G. Suppose that $\sigma(G) \leq 2$ and $|\rho(G)| \geq 5$. Then G/N is an almost simple group.*

Proof. We first claim that, if M/N is a chief factor of G, then M/N is a nonabelian simple group.

Let M be a normal subgroup of G such that M/N is a chief factor of G. Since N is the largest normal solvable subgroup of G, we deduce that M/N is nonsolvable so that $M/N \cong S^k$ for some integer $k \geq 1$ and some nonabelian simple group S. Let $C/N = \mathbf{C}_{G/N}(M/N)$. Then G/C embeds into $\mathrm{Aut}(S^k)$.

Assume first that $k \geq 3$. Since $|\rho(S)| = |\pi(S)| \geq 3$, there exist three distinct prime divisors r_i, $1 \leq i \leq 3$, and three characters $\psi_i \in \mathrm{Irr}(S)$ for $1 \leq i \leq 3$ with $r_i \mid \psi_i(1)$. Let

$$\varphi = \psi_1 \times \psi_2 \times \psi_3 \times 1 \times \cdots \times 1 \in \mathrm{Irr}(S^k).$$

Then $\sigma(\varphi) \geq 3$, which is a contradiction since

$$\sigma(S^k) = \sigma(M/N) \leq \sigma(M) \leq \sigma(G) \leq 2.$$

Thus $k \leq 2$.

Now assume that $k = 2$. Let $B/C = (G/C) \cap \mathrm{Aut}(S)^2$. Then G/B is a nontrivial subgroup of the symmetric group of degree 2, and thus $|G : B| = 2$. Since $S^2 \cong MC/C \trianglelefteq B/C \trianglelefteq G/C$ and $\sigma(G) \leq 2$, we deduce that $\sigma(S^2) \leq 2$, and thus $\sigma(S) = 1$, by Lemma 3.1. By [**12,** Satz 8], we know that S is isomorphic to either $\mathrm{PSL}_2(4)$ or $\mathrm{PSL}_2(8)$. In both cases, we obtain that $\pi(\mathrm{Aut}(S)) = \pi(S)$; hence, $\pi(B/C) = \pi(S)$. Moreover, as $|G : B| = 2$, we deduce that $\pi(G/C) = \pi(S)$. As G/C has no nontrivial normal abelian Sylow subgroups, Ito-Michler's theorem yields that $\rho(G/C) = \pi(G/C) = \pi(S)$. Since $|\pi(G/C)| = |\pi(S)| = 3$ and $|\rho(G)| \geq 5$, there exists $r \in \rho(G) - \pi(G/C)$. Then $r > 2$ and $r \in \rho(C)$ by Lemma 3.2. Let $\theta \in \mathrm{Irr}(C)$ be such that $r \mid \theta(1)$. Let L be a normal subgroup of MC such that $L/C \cong S$. Notice that $MC/C \cong S^2$. By applying [**19,** Lemma 4.2], θ extends to $\theta_0 \in \mathrm{Irr}(L)$. By Gallagher's theorem [**9,** Corollary 6.17], $\theta_0\mu \in \mathrm{Irr}(L)$ for all $\mu \in \mathrm{Irr}(L/C)$. Let $\mu_0 \in \mathrm{Irr}(L/C)$ with $2 \mid \mu_0(1)$, and let $\varphi = \theta_0\mu_0 \in \mathrm{Irr}(L)$. Then $\pi(\varphi(1)) = \{2, r\}$ with $r > 2$. As $MC/L \cong S$, we can apply [**19,** Lemma 4.2] again to obtain that φ extends to $\varphi_0 \in \mathrm{Irr}(MC)$ and then, by applying Gallagher's theorem, $\varphi_0\mu \in \mathrm{Irr}(MC)$ for all $\mu \in \mathrm{Irr}(MC/L)$. Clearly, $MC/L \cong S$ has an irreducible character $\tau \in \mathrm{Irr}(MC/L)$ with $s \mid \tau(1)$, where $s \notin \{2, r\}$. We now have that $\varphi_0\tau \in \mathrm{Irr}(MC)$. But then this is a contradiction as $\pi(\varphi_0(1)\tau(1)) = \{2, s, r\}$. This contradiction shows that $k = 1$, as wanted.

Let M/N be a chief factor of G, and let $C/N = \mathbf{C}_{G/N}(M/N)$. We claim that $C = N$ and thus G/N is an almost simple group as required. By the claim above, we know that $M/N \cong S$ for some nonabelian simple group S. Hence, G/C is an almost simple group with socle $MC/C \cong M/N$. Suppose, by contradiction, that $C \neq N$. Now let L/N be a chief factor of G with $N \leq L \leq C$. By the claim above, we deduce that L/N is isomorphic to some nonabelian simple group. In particular, $|\rho(C/N)| \geq |\pi(L/N)| \geq 3$. We have that $MC/N \cong C/N \times M/N$. Since $\sigma(MC/N) \leq \sigma(MC) \leq \sigma(G) \leq 2$, we deduce that $\sigma(C/N \times M/N) \leq 2$ and thus by Lemma 3.1, $\sigma(C/N) = 1 = \sigma(M/N)$. By **[12]**, we have $C/N \cong T \times A$, where A is abelian, T is a nonabelian simple group and $S, T \in \{\mathrm{PSL}_2(4), \mathrm{PSL}_2(8)\}$. Since $C \trianglelefteq G$ and the solvable radical W of C is characteristic in C, we obtain that $W \trianglelefteq G$, and thus $W \leq N$ as N is the largest normal solvable subgroup of G. Clearly, $N \leq W$ as N is also a solvable normal subgroup of C, so $W = N$. Therefore, C/N has no nontrivial normal abelian subgroup. Thus, $A = 1$, and hence $C/N \cong T$. Since $\pi(G/C) = \pi(M/N)$ and G/N has no normal abelian Sylow subgroup, we obtain that

$$\rho(G/N) = \pi(G/N) = \pi(C/N) \cup \pi(M/N) = \pi(S) \cup \pi(T).$$

It follows that

$$|\rho(G/N)| = |\pi(S) \cup \pi(T)| \leq |\pi(\mathrm{PSL}_2(4)) \cup \pi(\mathrm{PSL}_2(8))| = 4.$$

Hence, $\rho(G) - \rho(G/N)$ is nonempty. Now let $r \in \rho(G) - \rho(G/N)$. As $\{2, 3\} \subseteq \rho(G/N)$, we obtain that $r \notin \{2, 3\}$. By Lemma 3.2, $r \in \rho(N)$, and hence $r \mid \theta(1)$ for some $\theta \in \mathrm{Irr}(N)$. Since $\sigma(M) \leq \sigma(G) \leq 2$ and $M/N \cong S$, by **[19,** Lemma 4.2], we deduce that θ extends to $\theta_0 \in \mathrm{Irr}(M)$. Now let $\lambda \in \mathrm{Irr}(M/N)$ with $2 \mid \lambda(1)$. By Gallagher's theorem, $\varphi = \theta_0\lambda \in \mathrm{Irr}(M)$ with $\pi(\varphi(1)) = \{2, r\}$. Notice that $r \geq 5$ since $r \notin \{2, 3\}$. Now let $K = MC \trianglelefteq G$. Then $K/M \cong T$ and $\sigma(K) \leq 2$. Applying the same argument as above, we deduce that φ extends to $\varphi_0 \in \mathrm{Irr}(K)$. Clearly, $K/M \cong T$ has an irreducible character μ with $3 \mid \mu(1)$ and thus, by Gallagher's theorem again, $\psi = \varphi_0\mu \in \mathrm{Irr}(K)$ and obviously $\sigma(\psi) \geq 3$, which is a contradiction. $\square$

We are now ready to prove Theorem B, which we state here.

Theorem 3.4. *Let G be a group. If $\sigma(G) \leq 2$, then $|\rho(G)| \leq 2\sigma(G)+1$.*

Proof. Let G be a counterexample to the theorem with minimal order. Then $\sigma(G) \leq 2$, but $|\rho(G)| > 2\sigma(G)+1$. If G is solvable or G is nonsolvable with $\sigma(G)=1$, then

$$|\rho(G)| \leq 2\sigma(G)+1$$

by **[6, 11, 12]**, which is a contradiction. Thus, we can assume that G is nonsolvable, $\sigma(G) = 2$ and $|\rho(G)| \geq 6$. Let N be the solvable radical of G. By Lemma 3.3, G/N is an almost simple group with simple socle M/N. Since $\sigma(M/N) \leq \sigma(G/N) \leq \sigma(G) = 2$, we deduce from Lemmas 2.5 and 2.4 that

$$|\pi(G/N)| = |\pi(M/N)| \leq 5.$$

As $|\rho(G)| \geq 6$, we have that $\rho(G) - \rho(G/N)$ is nonempty and let $r \in \rho(G) - \rho(G/N)$. By Lemma 3.2, $r \mid \theta(1)$ for some $\theta \in \mathrm{Irr}(N)$. Since $\sigma(M) \leq 2$, by applying [**19,** Lemma 4.2], we deduce that θ extends to $\theta_0 \in \mathrm{Irr}(M)$. Using Gallagher's theorem, we must have that $\sigma(M/N) = 1$, and hence $M/N \cong \mathrm{PSL}_2(4)$ or $\mathrm{PSL}_2(8)$. Thus, $|\pi(G/N)| = |\pi(M/N)| = 3$; hence, $|\tau| \geq 3$ with $\tau = \rho(G) - \rho(G/N)$. By Lemma 3.2, we have that $\tau \subseteq \rho(N)$ and, since N is solvable, by applying Pálfy's condition [**16,** Theorem], there exists $\psi \in \mathrm{Irr}(N)$ such that $\psi(1)$ is divisible by two distinct primes in τ. Now, by applying [**19,** Lemma 4.2] again, we obtain a contradiction. This contradiction shows that $|\rho(G)| \leq 2\sigma(G)+1$, as wanted. □

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