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PURE SIMPLICIAL COMPLEXES AND WELL-COVERED GRAPHS

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ABSTRACT. In this paper, we provide some necessary and sufficient conditions to check when a (semi-)perfect graph is well-covered. We show that the checking process of these conditions can be achieved in a polynomial time. Since the comparability graph of a simplicial complex is a perfect graph, our result carries over to simplicial complexes.

1. Introduction. A graph G is said to be well-covered (or unmixed) if every maximal independent set of vertices has the same cardinality. These graphs were introduced by Plummer [**9**] in 1970. Although the recognition problem of well-covered graphs in general is Co-NP-complete ([**14**]), certain classes of well-covered graphs have been characterized. For instance, claw-free well-covered graphs [**12**], well-covered graphs which have girth at least 5 [**2**], (4-cycle, 5-cycle)-free [**3**] or chordal graphs [**11**], are all recognizable in a polynomial time. Excellent surveys of results on well-covered graphs are given in Plummer [**10**] and Hartnell [**5**].

Let G be a graph without loop or multiple edge. Denote the set of vertices of G by $V(G)$ and the set of edges by $E(G)$. A subset A of $V(G)$ is called an *independent set* if there is no edge between any two vertices of A. Denote the cardinality of the largest independent set in G by $\alpha(G)$. A subset C of $V(G)$ is called a clique if any two vertices in C are adjacent.

Let A and B be subsets of $V(G)$. We say A dominates B if, for any vertex v in B, either v is in A or there is at least one vertex in A adjacent to v. A set A is called a vertex cover of G if any edge of G has

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at least one vertex in A. A vertex cover is called minimal if no proper subset of it is a vertex cover.

A subset M of $E(G)$ is called a matching in G if no two edges in M have a common vertex. A matching is called a *perfect matching* if it covers all vertices of G.

Let $[n] = \{1, 2, \ldots, n\}$. A (finite) simplicial complex Δ on n vertices, is a collection of subsets of $[n]$ such that the following conditions hold:

a) $\{i\} \in \Delta$ for every $i \in [n]$,

b) if $E \in \Delta$ and $F \subseteq E$, then $F \in \Delta$.

An element of Δ is called a *face*, and a maximal face with respect to inclusion is called a *facet*. The set of all facets of Δ is denoted by $\mathcal{F}(\Delta)$. The dimension of a face $F \in \Delta$ is defined to be $|F| - 1$, and dimension of Δ is the maximum of dimensions of its faces. A simplicial complex is called *pure* if all of its facets have the same dimension. For more details on simplicial complexes, we refer to **[15]**.

Let G be a graph. The set of all independent sets of vertices of G is a simplicial complex, because any single vertex is independent and any subset of an independent set is again independent. We assume that the empty set is also an independent set. This simplicial complex is called the independence complex of G, and it is denoted by Δ_G. With the above definitions, a graph G is well-covered if and only if the complex Δ_G is pure.

Let Δ be a simplicial complex on the vertex set $[n]$. The comparability graph of Δ is the graph $G(\Delta)$ whose vertices are the nonempty faces of Δ, with two vertices adjacent if their corresponding faces are comparable with the inclusion order. It is known that a simplicial complex Δ is pure if and only if the complement of the graph $G(\Delta)$ is well-covered **[7]**.

2. Well-covered semi-perfect graphs. A graph G is called perfect if the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. It is known that a graph is perfect if and only if its complement graph is perfect **[8]**. Therefore, to define a perfect graph, we may say that, in every induced subgraph H, there are $k = \alpha(H)$ disjoint cliques in H covering all its vertices. The class of perfect graphs includes many important families of graphs such as bipartite, chordal and comparability graphs. In this class, the graph coloring problem, maximum clique problem, and maximum independent set problem can all be solved in polynomial time.

A graph G is called *semi-perfect* if there are $k = \alpha(G)$ disjoint cliques covering all vertices. It is clear that the class of semi-perfect graphs includes strictly the class of perfect graphs. Let G be a semi-perfect graph with $k = \alpha(G)$, and let $Q_1, \ldots, Q_k$ be disjoint cliques such that $V(Q_1) \cup \cdots \cup V(Q_k) = V(G)$. We call such a set of cliques, a basic clique cover of the graph G.

Now, we give some criteria equivalent to the well-covered property of semi-perfect graphs.

Theorem 2.1. *Let G be a semi-perfect graph with a basic clique cover $Q_1, \ldots, Q_k$. Then G is well-covered if and only if, for each i, $1 \leq i \leq k$, the following holds: If $A \subseteq V(G) \setminus Q_i$ dominates Q_i, then A is not an independent set.*

Proof. Assume that G is well-covered. Let i, $1 \leq i \leq k$, be given, and let $A \subseteq V(G) \setminus Q_i$ be a dominating set of Q_i. If A is independent, then there is a maximal independent set B containing A. But, $B \cap Q_i = \varnothing$ because any vertex of Q_i is adjacent to some vertices in $A \subseteq B$. On the other hand, B has at most one element in common with each Q_j, $j \neq i$. Therefore, $|B| < k$, which is a contradiction to well-coveredness of G.

Conversely, let A be a maximal independent set. Then $|A \cap Q_i| \leq 1$ for every i, $1 \leq i \leq k$ and $|A| \leq k$. The claim follows if one shows $|A| = k$. Assume that $|A \cap Q_i| = \varnothing$ for some i, then by the assumption A does not dominate Q_i. Therefore, there exists $v \in Q_i$ which is not adjacent to any vertex of A. By maximality of A, $v \in A$ and hence $|A \cap Q_i| = 1$, which is a contradiction. Therefore, $|A \cap Q_i| = 1$ and the claim follows. □

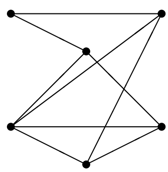
Proposition 2.2. *Let G be an s-partite well-covered graph such that all maximal cliques are of size s. Then all parts have the same cardinality and there is a perfect matching between any two parts.*

Proof. Let $V_1, \ldots, V_s$ be the given partition of V. Let $v \in V_i$ for some $1 \leq i \leq s$. Every vertex of G belongs to some maximal clique, and every maximal clique intersects every part V_j in exactly one vertex.

Therefore, v is adjacent to some vertex in each V_j, $1 \leq j \leq s$, $j \neq i$. Then, V_i is a maximal independent set because for every vertex w outside V_i, there is an edge connecting w to some vertex in V_i. But G is well-covered; therefore, the cardinality of all parts are equal.

Let $1 \leq i < j \leq s$ be two given integers. Let $A \subseteq V_i$ be a nonempty set, and let $N_j(A)$ be the set of all vertices in V_j adjacent to some vertex in A. Suppose that $|N_j(A)| < |A|$. There is no edge joining a vertex of A with a vertex in $V_j \setminus N_j(A)$. Therefore, $A \cup (V_j \setminus N_j(A))$ is an independent set, and its size is strictly greater than the size of V_j. This is a contradiction to well-coveredness of G. Therefore, $|N_j(A)| \geq |A|$ for every nonempty subset A of V_i. Therefore, by the theorem of Hall **[4]**, there is a set of distinct representatives (SDR) for the set $\{N_j(\{v\}) : v \in V_i\}$ which is a perfect matching between V_i and V_j. $\square$

It is not true in general that, in a well-covered s-partite graph G, there is a basic clique cover. For instance, the following graph is 3-partite, well-covered with maximal independent sets of size 2. But, there are no two maximal cliques covering $V(G)$.



Stating many examples motivates the following conjecture.

Conjecture. *Let G be an s-partite well-covered graph with all maximal cliques of size s. Then, G is semi-perfect.*

We now restate a result of Ravindra **[13]** on well-covered bipartite graphs, with a slightly different proof.

Corollary 2.3. *Let G be a bipartite graph with no isolated vertex. Then, G is well-covered if and only if there is a perfect matching such that for every edge $\{x, y\}$ in this matching, the induced subgraph on $N[\{x, y\}]$ is a complete bipartite graph.*

Proof. Let G be well-covered. By Proposition 2.2, the cardinality of both parts are equal and there is a perfect matching in G. Moreover, the set of all edges in the matching forms a basic clique cover of G. Let $\{x, y\}$ be an edge in the matching. By Theorem 2.1, G is well-covered if and only if every dominating set of $\{x, y\}$ is dependent. The last statement is equivalent to saying that every vertex in $N(\{x\})$ is adjacent to every vertex in $N(\{y\})$, i.e., the induced subgraph on $N[\{x, y\}]$ is a complete bipartite graph. $\square$

3. An algebraic interpretation. In this section, we state an algebraic interpretation of well-coveredness of semi-perfect graphs. First we recall some definitions from commutative algebra.

Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$, and let K be a field. Let $I(G)$ be the ideal in the polynomial ring $K[x_1, \ldots, x_n]$ generated by all monomials $x_i x_j$ provided that v_i and v_j are adjacent in G. The ideal $I(G)$ is called the *edge ideal* of the graph G, and the quotient ring $R(G) = K[x_1, \ldots, x_n]/I(G)$ is said to be the *edge ring* of G. The edge ideal of a graph was introduced by Villarreal **[16]** and has been extensively studied.

Let $\preceq$ be a monomial order on monomials in $K[x_1, \ldots, x_n]$, and let I be an ideal of this ring. We denote by $\overline{f}$ the element $f + I$ in the quotient ring $K[x_1, \ldots, x_n]/I$. It is well known that the set of all elements $\overline{m}$ in $K[x_1, \ldots, x_n]/I$ such that m is a monomial which is not divisible by the initial monomial of any element of I, is a K-basis for the vector space $K[x_1, \ldots, x_n]/I$. By the initial monomial of a polynomial f, we mean the largest monomial in the support of f with respect to a given monomial order. Therefore, a given monomial order on monomials in $K[x_1, \ldots, x_n]$ can be carried to (nonzero) monomials in $K[x_1, \ldots, x_n]/I$. For more details on monomial ideals and orders, see **[6]**.

Lemma 3.1. *Let K be a field, and let $I \subseteq K[x_1, \ldots, x_n]$ be an ideal generated by square-free monomials. Let $\overline{f}$ be a nonzero linear polynomial in $R = K[x_1, \ldots, x_n]/I$. Then, $\overline{f}$ is zero-divisor in R if and only if there is a nonzero square-free monomial $\overline{m} \in R$ such that $\overline{m}\overline{f} = \overline{0}$.*

Proof. One direction of the statement is trivial. For the other direction, first note that, if I is an ideal generated by some square-free monomials, then a polynomial f belongs to I if and only if the square-free part of any monomial of f belongs to I.

Let $\overline{f}$ be a zero-divisor linear form in R, then, there is a nonzero polynomial $\overline{g}$ in R such that $\overline{f}\overline{g} = \overline{0}$, that is, $fg \in I$. We may re-index the variables such that $\overline{f} = \overline{x}_1 + \overline{a}_2\overline{x}_2 + \cdots + \overline{a}_s\overline{x}_s$, $a_j \in K$. Let $\prec$ be the lexicographic order on monomials in $K[x_1, \ldots, x_n]$ with respect to the order on variables $x_1 \succ x_2 \succ \cdots \succ x_n$. Let $\overline{g} = \overline{b}_1\overline{m}_1 + \overline{b}_2\overline{m}_2 + \cdots + \overline{b}_t\overline{m}_t$ be the decomposition of $\overline{g}$ to nonzero monomials such that $m_1 \succ m_2 \succ \cdots \succ m_t$ and $b_i \in K$. Then, in $\overline{f}\overline{g}$, the monomial $\overline{x}_1\overline{m}_1$ is strictly greater than all other monomials. Therefore, $\overline{x}_1\overline{m}_1$ must be zero in R. The ideal I is square-free and $x_1 m_1 \in I$; therefore, we may assume that $x_1 \nmid m_1$. Otherwise, if $x_1 \mid m_1$ then $m_1 \in I$ and $\overline{m}_1 = \overline{0}$ which is a contradiction. The order is lexicographic and hence $x_1 \nmid m_i$ for all $1 \leq i \leq t$. On the other hand, $fg - b_1 x_1 m_1 \in I$, and its greatest monomial is $x_1 m_2$ and then $x_1 m_2 \in I$. Continuing this process, we have $x_1 m_i \in I$ for all $1 \leq i \leq t$ and therefore, $x_1 g \in I$. The polynomial $(f - x_1)g$ belongs to the ideal I, and its greatest monomial is $x_2 m_1$ which must be in I. Similarly, $x_2 m_i \in I$ for all $1 \leq i \leq t$. Finally, we get $x_i m_j \in I$ for each $1 \leq i \leq s$ and $1 \leq j \leq t$. This means that $m_i f \in I$ for each $1 \leq i \leq t$. Specially $m_1 f \in I$, and because I is square-free and f is linear, we may take m_1 to be square-free. $\square$

Note that, in the above lemma, the assumption that I is square-free is essential. For example, let $I = \langle x_1^3, x_2^3 \rangle \subset K[x_1, x_2]$. Then, $(x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2) \in I$, that is, $\overline{x}_1 - \overline{x}_2$ is a zero-divisor in $K[x_1, x_2]/I$. But, there is no nonzero square-free monomial annihilating $\overline{x}_1 - \overline{x}_2$.

Now, we state the main theorem of this section.

Theorem 3.2. *Let G be a semi-perfect graph, and let $Q_1, \ldots, Q_k$ be a basic clique cover of G, where $k = \alpha(G)$. Consider*

$$\theta_i = \sum_{v_j \in Q_i} x_j, \quad i = 1, \ldots, k.$$

Then, G is well-covered if and only if for every $i = 1, \ldots, k$, the linear form $\overline{\theta}_i$ is a non-zero-divisor in the ring $R(G)$.

Proof. Let $\overline{\theta}_i$ be zero-divisor in $R(G)$ for some $1 \leq i \leq k$. Then, by Lemma 3.1, there is a nonzero square-free monomial $\overline{m}$ in $R(G)$ such that $\overline{m}\overline{\theta}_i = \overline{0}$, or equivalently, $m\theta_i \in I(G)$. The ideal $I(G)$ is a monomial ideal; thus, for each v_j in Q_i, we have $mx_j \in I(G)$. Let $m = x_{i_1} \cdots x_{i_r}$ and $A = \{v_{i_1}, \ldots, v_{i_r}\}$. Then, $mx_j \in I(G)$ means that there is a vertex v_{i_l} in A such that v_{i_l} is adjacent to v_j, or equivalently, the set A is a dominating set of Q_i. On the other hand, if v_j is in $A \cap Q_i$, then $\overline{x}_j\overline{\theta} = \overline{x}_j^2$ in $R(G)$, and there is a v_{i_l} in A adjacent to v_j. Therefore, $\overline{m} = \overline{0}$ which is a contradiction. Therefore, $A \subseteq V(G) \setminus V(Q_i)$. Note that A is independent if and only if $\overline{m}$ is not zero in $R(G)$. Now, Theorem 2.1 implies that if $\overline{\theta}_i$ is a zero-divisor in $R(G)$ for some $1 \leq i \leq k$, then G is not well-covered.

Conversely, if G is not well-covered then, again by Theorem 2.1, there is an independent set $\{v_{i_1}, \ldots, v_{i_r}\} \subseteq V(G) \setminus V(Q_i)$ which dominates Q_i for some $1 \leq i \leq k$. In this case, $\overline{m} = \overline{x}_{i_1} \cdots \overline{x}_{i_r}$ is a nonzero monomial in $R(G)$ such that $\overline{m}\overline{\theta}_i = \overline{0}$, and hence $\overline{\theta}_i$ is zero-divisor. This completes the proof. $\square$

Let G be a semi-perfect graph. Then, by Theorem 3.2, G is well-covered if and only if every form θ_i is non-zero-divisor in the ring $R(G)$. On the other hand, the set of all zero-divisors of $R(G)$ is the union of all minimal primes of the ideal $I(G)$. The minimal primes of $I(G)$ correspond to the minimal vertex covers of G. Therefore, checking well-coveredness of the graph G is equivalent to checking that, for each i, $1 \leq i \leq k$, the set of vertices of Q_i is part of a minimal vertex cover of G. But, this is a simple task: it is enough to check that the set of vertices of Q_i is a minimal vertex cover of the subgraph of G induced by $N(Q_i)$, which can be done in polynomial time. Summing up, we have proved the following.

Corollary 3.3. *The well-coveredness of a semi-perfect graph can be checked in polynomial time.*

By the argument at the end of the introduction, an arbitrary graph G is well-covered if and only if the complement of the corresponding graph $G(\Delta_G)$ is well-covered. The graph $G(\Delta_G)$ and its complement are perfect and, therefore, well-coveredness of them can be checked in polynomial time. But, this does not solve the hardness of the problem of checking well-coveredness of a graph, because passing from G to $G(\Delta_G)$ cannot be performed in polynomial time. In fact, the graph $G(\Delta_G)$ has a huge number of vertices compared to those of G.

REFERENCES

1. F. Brenti and V. Welker, *f-vectors of barycentric subdivisions*, Math. Z. **259** (2008), 849–865.

2. A. Finbow, B. Hartnell and R. Nowakowski, *A characterization of well-covered graphs of girth* 5 *or greater*, J. Combin. Theor. **57** (1993), 44–68.

3. ______, *A characterization of well-covered graphs that contain neither* 4- *nor* 5-*cycles*, J. Graph Theor. **18** (1994), 713–721.

4. P. Hall, *On representatives of subsets*, J. Lond. Math. Soc. **10** (1935), 26–30.

5. B.L. Hartnell, *Well-covered graphs*, J. Combin. Math. Combin. Comp. **29** (1999), 107–115.

6. M. Kreuzer and L. Robbiano, *Computational commutative algebra* I, Springer-Verlag, 2000.

7. M. Kubitzke and V. Welker, *The multiplicity conjecture for barycentric subdivisions*, Comm. Alg. **36** (2008), 4223–4248.

8. L. Lovász, *A characterization of perfect graphs*, J. Combin. Theor. **13** (1972), 95–98.

9. M.D. Plummer, *Some covering concepts in graphs*, J. Combin. Theor. **8** (1970), 91–98.

10. ______, *Well-covered graphs*: *A survey*, Quaest. Math. **16** (1993), 253–287.

11. E. Prisner, J. Topp and P.D. Vestergaard, *Well-covered simplicial, chordal and circular arc graphs*, J. Graph Theor. **21** (1996), 113–119.

12. B. Randerath and L. Volkmann, *A characterization of well-covered block-cactus graphs*, Austral. J. Combin. **9** (1994), 307–314.

13. G. Ravindra, *Well covered graphs*, J. Combin. Inf. Syst. Sci. **2** (1977), 20–21.

14. R.S. Sankaranarayana and L.K. Stewart, *Complexity results for well-covered graphs*, Networks **22** (1992), 247–262.

15. R. Stanley, *Combinatorics and commutative algebra*, 2nd ed., Progress in Mathematics, Birkhauser, 1996.

16. R.H. Villarreal, *Cohen-Macaulay graphs*, Manuscr. Math. **66** (1990), 277–293.

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