THE $SU(2)$-CHARACTER VARIETIES
OF TORUS KNOTS

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ABSTRACT. Let G be the fundamental group of the complement of the torus knot of type (m, n). We study the relationship between $SU(2)$ and $SL(2, \mathbb{C})$-representations of this group, looking at their characters. Using the description of the character variety of G, $X(G)$, we give a geometric description of $Y(G) \subset X(G)$, the set of characters arising from $SU(2)$-representations.

1. Preliminaries and notation. Given a finitely presented group $G = \langle x_1 \cdots x_k | r_1, \ldots, r_s \rangle$, a $SU(2)$-representation is a homomorphism $\rho : G \to SU(2)$. Every representation is completely determined by the image of the generators, the k-tuple $(A_1, \ldots, A_k)$ satisfying the relations $r_j(A_1, \ldots, A_k) = \mathrm{Id}$. Since $SU(2)$ is algebraic, it follows from the definitions that the space of all representations, $R_{SU(2)}(G) = \mathrm{Hom}\,(G, SU(2))$ is a real affine algebraic set.

It is natural to declare a certain equivalence relation between these representations: we say that ρ and ρ' are equivalent if there exists $P \in SU(2)$ such that $\rho'(g) = P^{-1}\rho(g)P$ for all $g \in G$.

We want to consider the moduli space of $SU(2)$-representations, the GIT quotient

$$\mathcal{M}_{SU(2)} = \mathrm{Hom}\,(G, SU(2))//SU(2).$$

There are also analogous definitions for $SL(2, \mathbb{C})$: we can consider $SL(2, \mathbb{C})$-representations of G, which form a set $R_{SL(2,\mathbb{C})}(G)$, consider $SL(2, \mathbb{C})$-equivalence and construct the associated moduli space

$$\mathcal{M}_{SL(2,\mathbb{C})} = \mathrm{Hom}\,(G, SL(2, \mathbb{C}))//SL(2, \mathbb{C}).$$

Note that different conjugacy classes may correspond to the same point when the GIT quotient is made. For example, taking $G = \mathbb{Z}$, the representation defined by

$$\rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

goes in the quotient to the same point as the trivial representation, although they do not belong to the same conjugacy class. This happens because we identify orbits whose closures intersect. Conjugating ρ by $H = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$, we get that

$$H^{-1}\rho H = \begin{pmatrix} 1 & 1/\mu^2 \\ 0 & 1 \end{pmatrix},$$

whose limit, when $\mu \to \infty$, is the trivial representation.

The natural inclusion $SU(2) \hookrightarrow SL(2,\mathbb{C})$ shows that we can regard every $SU(2)$-representation as a $SL(2,\mathbb{C})$-representation. Moreover, if two representations are $SU(2)$-equivalent, then they are also $SL(2,\mathbb{C})$-equivalent. This leads to a map between moduli spaces

$$\mathcal{M}_{SU(2)} \xrightarrow{i_*} \mathcal{M}_{SL(2,\mathbb{C})}.$$

To every representation $\rho \in R_{SL(2,\mathbb{C})}(G)$ we can associate its character χ_ρ, defined as the map $\chi_\rho : G \to \mathbb{C}$, $\chi_\rho(g) = \mathrm{tr}(\rho(g))$. This defines a map $\chi : R_{SL(2,\mathbb{C})}(G) \to \mathbb{C}^G$, where equivalent representations have the same character. Its image $X_{SL(2,\mathbb{C})}(G) = \chi(R_{SL(2,\mathbb{C})}(G))$ is called the character variety of G.

There is an important relation between the $SL(2,\mathbb{C})$-character variety of G and the moduli space $\mathcal{M}_{SL(2,\mathbb{C})}$. It is seen in [1] that:

- $X_{SL(2,\mathbb{C})}(G)$ can be endowed with the structure of algebraic variety.
- The natural map that takes every represenãtion to its character, $\mathcal{M}_{SL(2,\mathbb{C})}(G) \to X_{SL(2,\mathbb{C})}(G)$, is bijective[1]. We specify the nature of this correspondence for the case of $SU(2)$-representations in the next section.

We emphasize that $X_{SL(2,\mathbb{C})}(G)$, as a set, consists of characters of $SL(2,\mathbb{C})$-representations. We can also take the set of characters of

$SU(2)$-representations, and again we will have a map $X_{SU(2)}(G) \overset{i^*}{\to} X_{SL(2,\mathbb{C})}(G)$.

We focus on the case when G is a torus knot group. Consider the torus of revolution $T^2 \subset S^3$. We identify it with $\mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z} = \langle (1,0), (0,1) \rangle$, via the map

$$F : \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow T^2 \subset \mathbb{R}^3 \subset S^3$$
$$(x, y) \longrightarrow ((2 + \cos 2\pi x) \cos 2\pi y, (2 + \cos 2\pi x) \sin 2\pi y, \sin 2\pi x).$$

The image of the line $y = (m/n)x$ defines the torus knot of type (m, n), $K_{m,n} \subset S^3$ for coprime m, n. An important invariant of a knot is the fundamental group of its complement in S^3, here $G_{m,n} = \pi_1(S^3 - K_{m,n})$. These groups admit the following presentation

$$(1.1) \qquad\qquad G_{m,n} = \langle x, y \mid x^m = y^n \rangle.$$

The $SL(2, \mathbb{C})$-character variety of these groups for the case $(m, 2)$ was treated in [6]. A complete description for (m, n) coprime was given in [5], and the general case (m, n) was studied using combinatorial tools in [4]. $SU(2)$-character varieties for knot groups were studied in [3]. For the case $(m, 2)$, the relation between both character varieties has been recently treated in [7].

2. $SU(2)$-character varieties. We recall that $SU(2) \cong S^3$, the isomorphism is given by

$$S^3 \subset \mathbb{C}^2 \longrightarrow SU(2)$$
$$(a, b) \longrightarrow \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}.$$

The correspondence is a ring homomorphism if we look at S^3 as the set of unit quaternions. First of all, we want to point out the following fact, which was already true for $SL(2, \mathbb{C})$.

Proposition 2.1. *The correspondence*

$$\mathcal{M}_{SU(2)}(G) \longrightarrow X_{SU(2)}(G)$$
$$\rho \longrightarrow \chi_\rho$$

that takes a representation to its character is bijective.

Proof. We follow the steps taken in [**1**], this time for $SU(2)$. First of all, every matrix A in $SU(2)$ is normal, hence diagonalizable. Since $\det(A) = 1$, the eigenvalues of A are $\{\lambda, \lambda^{-1}\}$ for some $\lambda \in \mathbb{C}^*$. In particular, $\operatorname{tr}(A)$ completely determines the set of eigenvalues $\{\lambda, \lambda^{-1}\}$.

Now, if ρ is a reducible $SU(2)$-representation, there is a common eigenvector e_1 for all $\rho(g)$ and therefore they are all diagonal with respect to the same basis. If ρ' is a second reducible representation such that $\chi_\rho(g) = \chi_{\rho'}(g)$ for all $g \in G$, this means that they share the same eigenvalues for every $g \in G$. After choosing another basis for ρ' such that $\rho'(g)$ is diagonal for all $g \in G$,

$$\rho(g) = \begin{pmatrix} \lambda(g) & 0 \\ 0 & \lambda^{-1}(g) \end{pmatrix}, \qquad \rho'(g) = \begin{pmatrix} \mu(g) & 0 \\ 0 & \mu^{-1}(g) \end{pmatrix},$$

where either $\lambda(g) = \mu(g)$ or $\lambda(g) = \mu^{-1}(g)$ for every $g \in G$. Interchanging the roles of λ and λ^{-1}, if necessary, there is always $g_1 \in G$ such that $\lambda(g_1) = \mu(g_1)$, so there is $g_1 \in G$ such that $\rho(g_1) = \rho'(g_1)$. We also notice that, if $\rho(g) = \pm \operatorname{Id}$, then $\rho'(g) = \rho(g) = \pm \operatorname{Id}$.

We claim that $\rho(g_2) = \rho'(g_2)$ for all $g_2 \in G$. If not, there exists $g_2 \in G$ such that $\rho(g_2) = \rho'(g_2)^{-1} \neq \pm \operatorname{Id}$. So $\lambda(g_1) = \mu(g_1)$ and $\lambda(g_2) = \mu^{-1}(g_2)$. On the other hand, we know that $\operatorname{tr}(\rho'(g_1 g_2)) = \operatorname{tr}(\rho(g_1 g_2))$, so

$$\mu(g_1)\mu(g_2) + \mu^{-1}(g_1)\mu^{-1}(g_2) = \lambda(g_1)\lambda(g_2) + \lambda^{-1}(g_1)\lambda^{-1}(g_2)$$
$$= \mu(g_1)\mu^{-1}(g_2) + \mu^{-1}(g_1)\mu(g_2).$$

Rearranging the terms,

$$\mu(g_2)(\mu(g_1) - \mu^{-1}(g_1)) = \mu^{-1}(g_2)(\mu(g_1) - \mu^{-1}(g_1)),$$

which implies that $\mu(g_2) = \pm 1$, so that $\rho(g_2) = \pm \operatorname{Id}$, a contradiction. Therefore, $\lambda(g) = \mu(g)$ for all $g \in G$. Hence, there exists $P \in SU(2)$ such that $\rho(g) = P^{-1}\rho(g)P$ for all $g \in G$, i.e., the representations are equivalent.

For the irreducible case, we point out the following fact: if ρ is a irreducible $SU(2)$-representation and $\rho(g) \neq \pm \operatorname{Id}$ for a given $g \in G$, then there exists $h \in G$ such that ρ restricted to the subgroup $H = \langle g, h \rangle$ is again irreducible. To see it, since $\rho(g) \neq \pm \operatorname{Id}$, $\rho(g)$ has two eigenspaces L_1, L_2 associated to the pair of different eigenvalues μ_1, μ_2. Since the representation is irreducible, there are elements h_i

such that L_i is not invariant under $\rho(h_i)$. We can take $h = h_1$ or $h = h_2$ unless L_1 is invariant under $\rho(h_2)$, or L_2 is invariant under $\rho(h_1)$; in this case, we can choose $h = h_1 h_2$.

For a group generated by two elements, $H = \langle g, h \rangle$, the reducibility of a representation is completely determined by $\chi_\rho([g, h])$. It can be seen in the following chain of equivalences:

$$\begin{aligned}
\rho|_H \text{ is reducible } &\Longleftrightarrow \rho(g), \rho(h) \text{ share a common eigenvector} \\
&\Longleftrightarrow \rho(g), \rho(h) \text{ are simultaneously diagonalizable} \\
&\Longleftrightarrow [\rho(g), \rho(h)] = \mathrm{Id} \\
&\Longleftrightarrow \mathrm{tr}[\rho(g), \rho(h)] = 2 \\
&\Longleftrightarrow \chi_\rho([g, h]) = 2.
\end{aligned}$$

Let ρ, ρ' be two $SU(2)$-representations such that $\chi_\rho = \chi_{\rho'}$. By the previous observation, there are $g, h \in G$ such that $\rho|_{\langle g, h \rangle}$ is irreducible, i.e., $\chi_\rho([g, h]) \neq 2$. It follows that, since $\chi_\rho = \chi_{\rho'}$, $\chi_{\rho'}([g, h]) \neq 2$, so $\rho'|_{\langle g, h \rangle}$ is irreducible too. Varying ρ, ρ' in their equivalence classes, we can assume that there are basis B, B' such that

$$\rho(h) = \rho'(h) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

The matrices $\rho(g), \rho'(g)$ will not be diagonal, by irreducibility, and conjugating again by diagonal unitary matrices we can assume that

$$\rho(g) = \begin{pmatrix} a & -b \\ b & \bar{a} \end{pmatrix}, \qquad \rho'(g) = \begin{pmatrix} a' & -b' \\ b' & \bar{a}' \end{pmatrix}$$

for $a, a' \in \mathbb{C}$, $b, b' \in \mathbb{R}^+$. Notice that $b, b' \neq 0$ since $\rho|_{\langle g, h \rangle}$ is irreducible. In general, for any $\alpha \in G$

$$\rho(\alpha) = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix}, \qquad \rho'(\alpha) = \begin{pmatrix} x' & -\bar{y}' \\ y' & \bar{x}' \end{pmatrix}$$

Now, the equations $\chi_\rho(\alpha) = \chi_{\rho'}(\alpha)$, $\chi_\rho(h\alpha) = \chi_{\rho'}(h\alpha)$ imply that

$$x + \bar{x} = x' + \bar{x}' :$$
$$\lambda x + \lambda^{-1} \bar{x} = \lambda x' + \lambda^{-1} \bar{x}'$$

and since $\lambda \neq \pm 1$, we get that $x = x'$.

Substituting $\alpha = g$, we get that $a = a'$, and since $\det(\rho(g)) = \det(\rho'(g)) = 1$, $b = b'$, so $\rho(g) = \rho'(g)$.

Substituting again $g\alpha$ for α, we arrive at the equation $ax - by = ax - by'$, which implies that $y = y'$ and finally that $\rho(\alpha) = \rho'(\alpha)$: we have proved that the representations ρ and ρ', after $SU(2)$-conjugation, are the same, i.e., they are equivalent. $\qquad\square$

Remark 2.2. As a consequence of Proposition 2.1, the moduli space is precisely the set of conjugacy classes of representations, i.e., there are no extra identifications as in the $SL(2, \mathbb{C})$-case.

Corollary 2.3. *We have a commutative diagram:*

$$
\begin{array}{ccc}
\mathcal{M}_{SU(2)}(G) & \xrightarrow{\;1:1\;} & X_{SU(2)}(G) \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
\mathcal{M}_{SL(2,\mathbb{C})}(G) & \xrightarrow{\;1:1\;} & X_{SL(2,\mathbb{C})}(G)
\end{array}
$$

The previous corollary shows that we can equivalently study the relationship between $SU(2)$ and $SL(2, \mathbb{C})$-representations of G from the point of view of their characters or from the point of view of their representations. Looking at the diagram, we also deduce that:

Corollary 2.4. *The natural inclusion* $i_* : \mathcal{M}_{SU(2)}(G) \to \mathcal{M}_{SL(2,\mathbb{C})}(G)$ *is injective.*

3. $SU(2)$-character varieties of torus knots. We focus now on the specific case of the torus knot $G_{m,n}$ of coprime type (m, n). Henceforth, we will often denote $X_{SL(2,\mathbb{C})} = X_{SL(2,\mathbb{C})}(G)$ and omit the group in our notation. In this case

$$R_{SL(2,\mathbb{C})}(G) = \{(A, B) \in SL(2, \mathbb{C}) \mid A^m = B^n\}$$

and

$$R_{SU(2)}(G) = \{(A, B) \in SU(2) \mid A^m = B^n\}.$$

We have a decomposition of $X_{SL(2,\mathbb{C})}$

$$X_{SL(2,\mathbb{C})} = X_{red} \cup X_{irr}$$

where X_{red} is the subset of characters of reducible representations and X_{irr} is the subset of characters of irreducible representations. Inside $X_{SL(2,\mathbb{C})}$, we have $i_*(X_{SU(2)})$, i.e., the set of characters of $SU(2)$-representations. For simplicity, we will denote $Y = i_*(X_{SU(2)})$. Again, Y decomposes in $Y_{red} \cup Y_{irr}$.

Reducible representations.

Proposition 3.1. *There is an isomorphism $Y_{red} \cong [-2, 2] \subset \mathbb{R}$.*

Proof. We will use, from now on, the explicit description of $X_{SL(2,\mathbb{C})}$ given in [**5**]. There is an isomorphism $X_{red} \cong \mathbb{C}$ given by

$$A = \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}, B = \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix} \longrightarrow s = t + t^{-1} \in \mathbb{C}.$$

This is because given a reducible $SL(2, \mathbb{C})$-representation ρ, we can consider the associated split representation $\rho = \rho' + \rho''$, which in a certain basis takes the form

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix},$$

and the equality $A^m = B^n$ implies that $\lambda = t^n$, $\mu = t^m$ for a unique $t \in \mathbb{C}$ (here we use that m, n are coprime). Now, since $A, B \in SU(2)$, t must satisfy that $|t|^2 = 1$, i.e., $t \in S^1 \subset \mathbb{C}$. We also have to take account of the change of order of the basis elements, and therefore $t \sim 1/t$. So the parameter space is isomorphic to $[-2, 2]$ (under the correspondence $t \in S^1 \to s = t + t^{-1} = 2 \operatorname{Re}(t) \in [-2, 2]$). $\qquad \square$

To explicitly describe when a pair (A, B) is reducible, we follow [**5,** 2.2]. First of all, A and B are diagonalizable (recall that $A, B \in SU(2)$), so we can rule out the Jordan type case since it is not possible. So

Proposition 3.2. *In either of the cases*:

- $A^m = B^n \neq \pm \operatorname{Id}$,
- $A = \pm \operatorname{Id}$ *or* $B = \pm \operatorname{Id}$,

the pair (A, B) is reducible.

Proof. Let us deal with the first case, when $A^m = B^n \neq \pm \mathrm{Id}$. A is diagonalizable with respect to a basis $\{e_1, e_2\}$ and takes the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. Then

$$B^n = A^m = \begin{pmatrix} \lambda^m & 0 \\ 0 & \lambda^{-m} \end{pmatrix},$$

so B is diagonal in the same basis and the pair is reducible. For the second case, if $A = \alpha \, \mathrm{Id}$, where $\alpha = \pm 1$, then any basis diagonalizing B diagonalizes A; hence, the pair is reducible. The case $B = \alpha \, \mathrm{Id}$ follows in the same way. $\qquad \square$

Irreducible representations. Now we look at the set of irreducible representations, Y_{irr}. Let $(A, B) \in R_{SU(2)}(G)$ be an irreducible pair. Both matrices are diagonalizable and using Proposition 3.2 they must satisfy that $A^m = B^n = \pm \mathrm{Id}$, $A, B \neq \pm \mathrm{Id}$. The eigenvalues $\lambda, \lambda^{-1} \neq \pm 1$ of A satisfy $\lambda^m = \pm 1$, the eigenvalues μ, μ^{-1} of B satisfy $\mu^n = \pm 1$ and $\lambda^m = \mu^n$.

We can associate to A a basis $\{e_1, e_2\}$ under which it diagonalizes, and the same for B, obtaining another basis $\{f_1, f_2\}$. The eigenvalues λ, μ and the eigenvectors e_i, f_i completely determine the representation (A, B). We are interested in $i_*(\mathcal{M}_{SU(2)})$, $SL(2, \mathbb{C})$-equivalence classes of such pairs (A, B), and these are fully described by the projective invariant of the four points $\{e_1, e_2, f_1, f_2\}$, the cross ratio

$$[e_1, e_2, f_1, f_2] \in \mathbb{P}^1 - \{0, 1, \infty\}$$

(we may assume that the four eigenvectors are different since the representation is irreducible, see [5] for details).

Since both $A, B \in SU(2)$, we know that $e_1 \perp e_2$ and $\|e_1\| = \|e_2\| = 1$, so shifting the vectors by a suitable rotation $C \in SU(2)$, we can assume that $e_1 = [1 : 0], e_2 = [0 : 1]$, and therefore $f_1 = [a : b], f_2 = [-\bar{b} : \bar{a}]$, since they are orthogonal too. So the pair (A, B) inside $X_{SL(2,\mathbb{C})}$ is determined by λ, μ satisfying the conditions above and the projective cross ratio

$$r = \left[e_1, e_2, f_1, f_2\right] = \left[0, \infty, \frac{b}{a}, -\frac{\bar{a}}{\bar{b}}\right] = \frac{b\bar{b}}{-a\bar{a}} = \frac{b\bar{b}}{b\bar{b} - 1} = \frac{t}{t - 1}$$

where we have used that $a\bar{a} + b\bar{b} = 1$ and $t = |b|^2, b \in (0,1)$. We also get that r is real and $r \in (-\infty, 0)$.

The converse is also true: if the triple (λ, μ, r), satisfies that $\lambda^m = \mu^n = \pm 1$, $\lambda, \mu \neq \pm 1$ and $r \in (-\infty, 0)$, then $(A, B) \in i_*(\mathcal{M}_{SU(2)})$. To see this, r determines uniquely $t = |b|^2$ since $r(t)$ is invertible for $t \in (0,1)$. Once $|b|$ is fixed, we get that $|a|$ is fixed too, using $|a|^2 = 1 - |b|^2$. We can choose any $(a, b) \in S^1 \times S^1$, and we conclude that (A, B) is $SL(2, \mathbb{C})$-equivalent to a $SU(2)$-representation. To be more precise, it is equivalent to the representation with eigenvalues λ, μ and eigenvectors $[1:0], [0:1], [a:b], [-\bar{b}, \bar{a}]$.

Finally, we have to take account of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action given by the permutation of the eigenvalues:

- Permuting e_1, e_2 takes (λ, μ, r) to $(\lambda^{-1}, \mu, r^{-1})$.
- Permuting f_1, f_2 takes (λ, μ, r) to $(\lambda, \mu^{-1}, r^{-1})$.

Since $\lambda^m = \mu^n = \pm 1$, we get that

$$(3.1) \qquad \lambda = e^{\pi i k/m}, \qquad \mu = e^{\pi i k'/n},$$

where, since $\lambda \sim \lambda^{-1}$, $\mu \sim \mu^{-1}$ and $\lambda \neq \pm 1$, $\mu \neq \pm 1$, we can restrict to the case when $0 < k < m$, $0 < k' < n$. We also notice that $\lambda^m = \mu^n$ implies that $k \equiv k' \pmod 2$. So the irreducible part is made of $(m-1)(n-1)/2$ intervals.

We have just proved

Proposition 3.3.

$Y_{irr} \cong \{(\lambda, \mu, r) : \lambda^m = \mu^n = \pm 1; \lambda, \mu \neq \pm 1; r \in (-\infty, 0)\}/\mathbb{Z}_2 \times \mathbb{Z}_2.$

This real algebraic variety consists of $[(m-1)(n-1)]/2$ open intervals.

To describe the closure of the irreducible orbits, we have to consider the case when $e_1 = f_1$, since this is what happens in the limit (the situation is analogous when $e_2 = f_2$). In this situation $r = 0$, and the representation is equivalent to a reducible representation. Taking into account Lemma 3.1, it corresponds to a certain $t \in S^1$ such that $\lambda = t^n$, $\mu = t^m$. We have another limit case $r = -\infty$, if we allow $e_1 = f_2$. The representation is again reducible and corresponds to another $t' \in S^1$ such that $\lambda = (t')^n$, $\mu^{-1} = (t')^m$.

Remark 3.4. The explicit description of the set of $SU(2)$-representations allows us to give an alternative proof of Corollary 2.4, which stated that the inclusion $i_* : \mathcal{M}_{SU(2)} \to \mathcal{M}_{SL(2,\mathbb{C})}$ is injective.

Let us see this. Suppose that (A, B) and (A', B') are two $SU(2)$-representations which are mapped to the same point in $\mathcal{M}_{SL(2,\mathbb{C})}$, i.e., which are $SL(2,\mathbb{C})$-equivalent. If we denote by u_1, u_2, u_3, u_4 the set of eigenvectors of (A, B) and by v_1, v_2, v_3, v_4 the set of eigenvectors of (A', B'), we know that

$$[u_1, u_2, u_3, u_4] = [v_1, v_2, v_3, v_4] = r \in (-\infty, 0).$$

Since their cross ratio is the same, we know that there exists $P \in SL(2,\mathbb{C})$ that takes the set $\{u_i\}$ to $\{v_i\}$. Moreover, since P takes the unitary basis $\{u_1, u_2\}$ to the unitary basis $\{v_1, v_2\}$, we get that $P \in SU(2)$, and therefore both representations are $SU(2)$-equivalent.

Topological description. We finally describe Y topologically. We refer to [**5**] for a geometric description of $X_{SL(2,\mathbb{C})}$.

Using Proposition 3.3, Y_{irr} is a collection of real intervals (parametrized by $r \in (-\infty, 0)$) for a finite number of (λ, μ) that satisfy the required conditions. By our last observation, the limit cases when $r = 0, \infty$ (i.e., points in the closure of Y_{irr}) correspond to the points where the closure of Y_{irr} intersects Y_{red}.

As we saw before, each interval has two points in its closure: these are $t_0 \in S^1$ such that $t_0^n = \lambda$, $t_0^m = \mu$ ($r = 0$) and $t_1 \in S^1$ corresponding to $t_1^n = \lambda$, $t_1^m = \mu^{-1}$ ($r = -\infty$). The conditions on λ, μ force that $t_0 \neq t_1$ so that we get different intersection points with Y_{red}.

Y is topologically a closed interval (Y_{red}) with $(m-1)(n-1)/2$ closed intervals (Y_{irr}) attached at $(m-1)(n-1)$ different endpoints (without any intersections among them). The interval $Y_{red} = [-2, 2]$ sits inside $X_{red} \cong \mathbb{C}$, and every real interval in Y_{irr} is inside the corresponding complex line in X_{irr}.

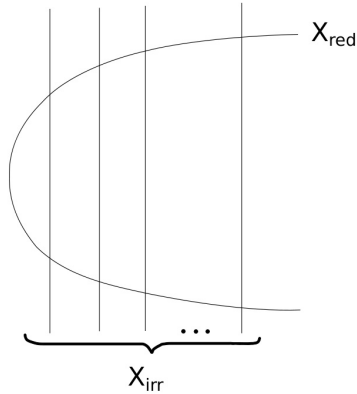The situation is described in the following two pictures:

FIGURE 1. Picture of $X_{SL(2,\mathbb{C})}$, defined over $\mathbb{C}$. The drawn lines are curves isomorphic to $\mathbb{C}$. The closure of each curve in X_{irr} intersects X_{red} at two distinct points.
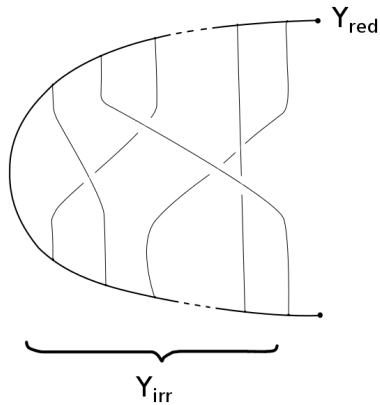

FIGURE 2. Picture of $Y \subset X_{SL(2,\mathbb{C})}$, defined over $\mathbb{R}$. The picture displays the set of real segments which form Y_{irr}.

Note that, in the $SU(2)$-case, since $Y_{red} \cong [-2, 2]$ is a real closed interval, we can look at the particular order of the pairs of intersection points of the closure of Y_{irr} with Y_{red}. This is why the above picture displays Y_{red} as a collection of tangled intervals, in contrast to the $SL(2, \mathbb{C})$-case where no ordering can be defined.

More concretely, each component of Y_{irr} is characterized by a triple (λ, μ, r), where $\lambda = e^{(\pi i k)/m}$, $\mu = e^{(\pi i k')/n}$, $0 < k < m$, $0 < k' < n$ and $k \equiv k' \pmod 2$ (cf., Proposition 3.3). Its closure intersects Y_{red} at two points: the two reducible representations described by the eigenvalues (λ, μ) and (λ, μ^{-1}). There is a unique t_1 such that $t_1^n = \lambda$, $t_1^m = \mu$ and a unique t_2 such that $t_2^n = \lambda$, $t_2^m = \mu^{-1}$. The points $s_i = t_i + t_i^{-1} \in [-2, 2]$ give us the intersection points with $Y_{red} \cong [-2, 2]$. Since both t_i are n-th roots of λ, they will be of the form

$$t_i = e^{[\pi i(k + 2a_i m)]/mn}$$

for certain a_i verifying $0 \le a_i < n$. Solving the equation $t_1^m = \mu$ and $t_2^m = \mu^{-1}$, we get that a_1, a_2 are the unique solutions to the equations:

$$k + 2a_1 m \equiv k' \pmod{2n}$$
$$k + 2a_2 m \equiv 2n - k' \pmod{2n}$$

We finally obtain that the intersection points of the component given by the triple (k, k', r) are the points

$$s_1 = 2\cos\left(\frac{\pi k}{mn} + \frac{2a_1\pi}{n}\right), \qquad s_2 = 2\cos\left(\frac{\pi k}{mn} + \frac{2a_2\pi}{n}\right).$$

The ordering of these sets of pairs of points (one pair for each admissible (k, k')) depends on the type of torus knot group, i.e., on (m, n). As Figure 3 shows, we can obtain all kind of situations depending on the particular choice of (m, n). Looking at $G_{5,6}$, notice that it is not true that we always have pairs of positive and negative endpoints.

A natural question is whether the inclusion of the $SU(2)$-character variety Y inside the $SL(2, \mathbb{C})$-character variety is a homotopy equivalence, i.e., if the two varieties have the same homotopy type. The result is in general false if we choose an arbitrary finitely generated group G, but remains true in some cases, for example, $G = \mathbb{Z}^k$ (see [**2, 8**]).

Looking at the explicit description of Y and $X_{SL(2,\mathbb{C})}$ we also obtain in our case

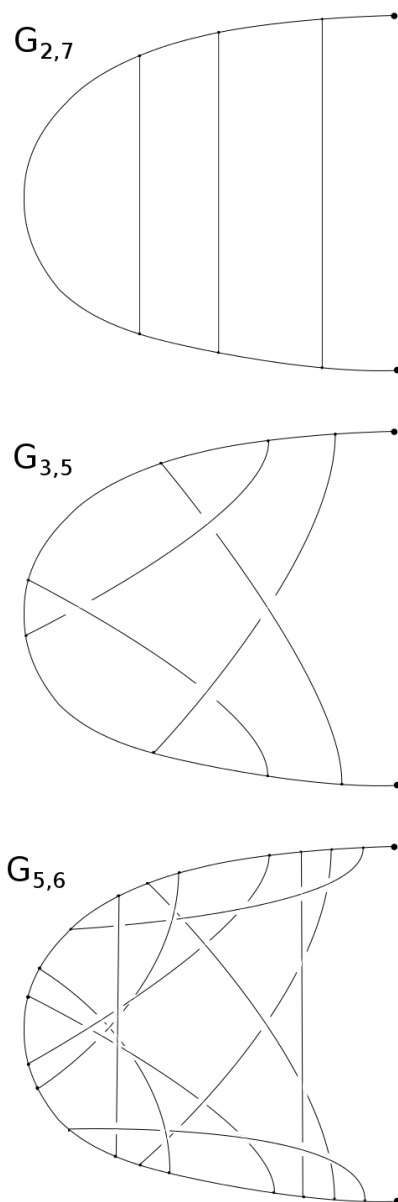**Corollary 3.5.** Y *is a reformation retract of* $X_{SL(2,\mathbb{C})}$.

FIGURE 3. Examples of several character varieties for some $G_{m,n}$.

4. Noncoprime case. If $\gcd(m,n) = d > 1$, then $G_{m,n}$ no longer represents a torus knot, since these are only defined in the coprime case. However, the group $G_{m,n} = \langle x, y \mid x^n = y^m \rangle$ still makes sense, and we can study the representations of this group into $SL(2,\mathbb{C})$ and $SU(2)$ using the method described above. We will denote by a, b the integers that satisfy

$$m = a\,d,$$
$$n = b\,d.$$

As we did before, we focus on $Y = i_*(X_{SU(2)})$, the set of characters of $SU(2)$-representations.

Reducible representations. First of all, we describe what happens in the $SL(2,\mathbb{C})$ case.

Proposition 4.1. *There is an isomorphism*

$$X_{red} \cong \bigsqcup_{i=0}^{\lfloor d/2 \rfloor} X_{red}^i$$

where:

- $X_{red}^i \cong \mathbb{C}^*$ *for* $0 < i < d/2$.
- $X_{red}^i \cong \mathbb{C}$ *for* $i = 0$ *and* $i = d/2$ *if* d *is even.*

Proof. As it is shown in [**5**], an element in X_{red} can be regarded as the character of a split representation, $\rho = \rho' \oplus \rho'^{-1}$. There is a basis such that

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix},$$

where $A^m = B^n$ implies that $\lambda^m = \mu^n$. We deduce that $(\lambda^a)^d = (\mu^b)^d$, so that (λ, μ) belong to one of the components

$$X_{red}^i = \{(\lambda, \mu) \mid \lambda^a = \xi^i \mu^b\} = \{(\lambda, \mu) \mid \lambda^a \mu^{-b} = \xi^i\},$$

where ξ is a primitive d-th root of unity. These components are disjoint, and each one of them is parametrized by $\mathbb{C}^*$. To see this, let us fix a

component, X^i_{red}, and let α be a b-th root of ξ^i. Then

$$\begin{aligned}
X^i_{red} &= \{(\lambda, \mu) | \lambda^a = \xi^i \mu^b\} \\
&= \{(\lambda, \mu) | \lambda^a = \alpha^b \mu^b\} \\
&= \{(\lambda, \nu) | \lambda^a = \nu^b\} \cong \mathbb{C}^*.
\end{aligned}$$

In other words, for each $(\lambda, \mu) \in X^i_{red}$, there is a unique $t \in \mathbb{C}^*$ such that $t^b = \lambda$, $t^a = \alpha\mu$. However, we have to take account of the action given by permuting the two vectors in the basis, which corresponds to the change $(\lambda, \mu) \sim (\lambda^{-1}, \mu^{-1})$. In our decomposition, if $(\lambda, \mu) \in X^i_{red}$, then $(\lambda^{-1}, \mu^{-1}) \in X^{-i}_{red}$. So $t \in X^i_{red}$ is equivalent to $1/t \in X^{-i}_{red}$.

For $0 \le i \le d - 1$, we have two possibilities. If $i \not\equiv -i \pmod{d}$, then X^i_{red} and X^{-i}_{red} get identified. If $i \equiv -i \pmod{d}$, then $t \sim t^{-1} \in X^i_{red} \cong \mathbb{C}$, and thus $X^i_{red}/\sim \cong \mathbb{C}^*/_{a\sim a^{-1}} \cong \mathbb{C}$.

When d is even, there are two $i \in \mathbb{Z}/d\mathbb{Z}$ such that $i \equiv -i \pmod{d}$, so we get two copies of $\mathbb{C}$ in Y_{red}. When d is odd, we get just one, since there is only one solution ($i \equiv 0$). The remaining copies of X^i_{red} get identified pairwise: $X^i_{red} \sim X^{-i}_{red}$. $\qquad\square$

Now, for the case of $SU(2)$-representations, we have

Proposition 4.2. *There is an isomorphism*

$$Y_{red} \cong \bigsqcup_{i=0}^{\lfloor d/2 \rfloor} Y^i_{red},$$

where:

- $Y^i_{red} \cong S^1$ *for* $0 < i < d/2$.
- $Y^i_{red} \cong [-2, 2]$ *for* $i = 0$, $i = d/2$ *if* d *is even.*

Proof. If (A, B) is a reducible $SU(2)$-representation, both are diagonalizable with respect to a certain basis, and therefore

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}.$$

The equality $A^m = B^n$ gives us that $\lambda^m = \mu^n$. So the pair (λ, μ) belongs to a certain component X^i_{red}. Since it is a $SU(2)$-representation,

the eigenvalues λ and μ satisfy that $|\lambda| = |\mu| = 1$. This implies that $(\lambda, \mu) \in S^1 \subset \mathbb{C}^* \cong X^i_{red}$: we define $Y^i_{red} := S^1 \subset X^i_{red}$.

We have to take into account the equivalence relation in X_{red} given by the permutation of the eigenvectors. If $i \not\equiv -i \pmod{d}$, then $Y^i_{red} \cong Y^{-i}_{red}$. If $i \equiv -i \pmod{d}$, then $Y^i_{red} \cong S^1/_{a \sim a^{-1}} \cong [-2, 2]$. This gives the desired result. $\qquad\square$

Irreducible representations. We start by describing what happens in the $SU(2)$ case.

Proposition 4.3. *We have an isomorphism*

$$Y_{irr} \cong \{(\lambda, \mu, r) : \lambda^m = \mu^n = \pm 1;$$
$$\lambda, \mu \neq \pm 1, \ r \in (-\infty, 0)\}/\mathbb{Z}_2 \times \mathbb{Z}_2.$$

This real algebraic variety consists of:

- $(m-1)(n-1) + 1/2$ *open intervals if* m, n *are both even,*
- $(m-1)(n-1)/2$ *open intervals in any other case.*

Proof. By Proposition 3.2, a representation (A, B) is reducible unless $A^m = B^n = \pm \text{Id}$, $A, B \neq \pm \text{Id}$. So the set of irreducible representations can be described using the same tools as before: the set of equivalence classes of irreducible representations is a collection of intervals $r \in (-\infty, 0)$ parametrized by pairs (k, k') satisfying

(4.1) $0 < k < m, \qquad 0 < k' < n, \qquad k \equiv k' \pmod{2}.$

We compute the number of such pairs, separating in three different cases according to the parity of m and n:

Suppose m, n are both even. If $k \equiv k' \equiv 0 \pmod{2}$, then $k \in \{2, 4, \ldots, m-2\}$, $k' \in \{2, 4, \ldots n-2\}$, so there are $[(m-2)(n-2)]/4$ such pairs. If $k \equiv k' \equiv 1 \pmod{2}$, $k \in \{1, 3, \ldots m-1\}$, $k' \in \{1, 3, \ldots, n-1\}$, we have $mn/4$ pairs. The sum is $[(m-2)(n-2)]/4 + mn/4 = [(m-1)(n-1) + 1]/2$.

Suppose m is even and n is odd (the case m odd and n even is similar). Then, if $k \equiv k' \equiv 0 \pmod{2}$, $k \in \{2, 4, \ldots, m-2\}$, $k' \in \{2, 4, \ldots n-1\}$, we get $(m-2)(n-1)/4$ such pairs. If $k \equiv k' \equiv 1 \pmod{2}$, $k \in \{1, 3, \ldots m-1\}$, $k' \in \{1, 3, \ldots, n-2\}$, and

there are $m(n-1)/4$ such pairs. We get in total $m(n-1)/4 + (m-2)(n-1)/4 = (m-1)(n-1)/2$.

Finally, suppose both m, n odd. If $k \equiv k' \equiv 0 \pmod 2$, $k \in \{2, 4, \ldots, m-1\}$, $k' \in \{2, 4, \ldots n-1\}$, and we get $(m-1)(n-1)/4$ such pairs. If $k \equiv k' \equiv 1 \pmod 2$, $k \in \{1, 3, \ldots m-2\}$, $k' \in \{1, 3, \ldots, n-2\}$, there are $(m-1)(n-1)/4$ such pairs. We get $(m-1)(n-1)/2$ pairs in total.

We have obtained a decomposition

$$Y_{irr} = \bigsqcup_{k,k'} Y_{irr}^{(k,k')},$$

where every $Y_{irr}^{(k,k')}$ is an open interval isomorphic to $(-\infty, 0)$.   □

For the case of $SL(2, \mathbb{C})$-representations, we have the following

Proposition 4.4. *The component $X_{irr} \subset X_{SL(2,\mathbb{C})}$ is described as*

$$X_{irr} = \bigsqcup_{k,k'} X_{irr}^{(k,k')},$$

where k, k' satisfy (4.1), and $X_{irr}^{(k,k')} = \mathbb{P}^1 - \{0, 1, \infty\}$. This complex algebraic variety consists of $[(m-1)(n-1)+1]/2$ components if m, n are both even, of $(m-1)(n-1)/2$ components if one of m, n is odd. Moreover $Y_{irr}^{(k,k')} = (-\infty, 0) \subset X_{irr}^{(k,k')}$ in the natural way.

The limit cases $r = 0$, $r = -\infty$ correspond to the closure of the irreducible components, and these points are exactly where $\overline{Y}_{irr}$ intersects Y_{red}. The triples $(\lambda, \mu, 0)$, $(\lambda, \mu, -\infty)$ correspond to the reducible representations with eigenvalues (λ, μ) and (λ, μ^{-1}). Since $\lambda, \mu \neq \pm 1$, we get two different intersection points. Note that the pattern of intersections for $\overline{X}_{irr}$ and X_{red} is the same, but the components are complex algebraic varieties now.

To understand the way the closure of the components of Y_{irr} intersect Y_{red}, we have the following:

Proposition 4.5. *The closure of $Y_{irr}^{(k,k')}$ is a closed interval that joins $Y_{red}^{i_0}$ with $Y_{red}^{i_1}$, where*

$$i_0 = \frac{k - k'}{2}, \qquad i_1 = \frac{k + k'}{2} \quad (\text{mod } d).$$

Proof. Set $D = 2d\,ab$, and consider ω a primitive D-th root of unity. Then $\xi := \omega^{D/d} = \omega^{2ab}$ is a primitive d-th root of unity. The irreducible component $Y_{irr}^{(k,k')}$ is the interval (λ, μ, r), $r \in (-\infty, 0)$, where

$$\lambda = (\omega^b)^k, \qquad \mu = (\omega^a)^{k'},$$

and k, k' are subject to the conditions (4.1), see equation (3.1). The points in the closure of $Y_{irr}^{(k,k')}$ correspond to the reducible representations with eigenvalues (λ, μ) and (λ, μ^{-1}). Clearly, $(\lambda, \mu) \in X_{red}^{i_0}$, since

$$\lambda^a \mu^{-b} = \omega^{kab} \omega^{-k'ab} = \omega^{2ab(k-k')/2} = \omega^{i_0 2ab} = \xi^{i_0},$$

and $(\lambda, \mu^{-1}) \in X_{red}^{i_1}$, since

$$\lambda^a \mu^b = \omega^{kab} \omega^{k'ab} = \xi^{i_1}. \qquad \square$$

Proposition 4.5 gives a clear rule to depict $Y = Y_{irr} \cup Y_{red}$ for every pair (m, n). Actually, Y is a collection of intervals attached on their endpoints to Y_{red}, which consists of several disjoint copies of S^1 and $[-2, 2]$. Note that the pattern of intersections for the irreducible components of $X_{SL(2,\mathbb{C})} = X_{irr} \cup X_{red}$ is the same as that of Y.

When m, n are coprime, we recover our previous pictures.

Corollary 4.6. *For any two different components $Y_{red}^{i_0}, Y_{red}^{i_1} \subset Y_{red}$, there is a pair (k, k') such that $\overline{Y}_{irr}^{(k,k')}$ joins them.*

In particular, Y is a connected topological space.

Proof. We can assume $0 \leq i_0 < i_1 \leq d/2$. Then $0 < k = d + i_0 - i_1 < d \leq m$ and $0 < k' = d - i_0 - i_1 < d \leq n$ both satisfy that $k \equiv k' \pmod{2}$ and $(k - k')/2 = i_0$, $(k + k')/2 = i_1$. $\qquad \square$

Remark 4.7. It can be checked that there is no component $\overline{Y}_{irr}^{(k,k')}$ which joins $Y_{red}^{i_0}$ to itself when $m = n$, or when one of m, n divides the

other, and we are dealing with $i_0 = 0$ or $i_0 = d/2$ (the latter only if d is even).

Actually, such a component would correspond to a pair (k, k') such that $(k - k')/2 \equiv \pm i_0 \pmod{d}$ and $(k + k')/2 \equiv \pm i_0 \pmod{d}$. Accounting for all possibilities of signs, we have either $k \equiv \pm 2i_0$, $k' \equiv 0$ $\pmod{d}$, or $k \equiv 0$, $k' \equiv \pm 2i_0 \pmod{d}$. This has solutions unless $m > n = d$, $i_0 = 0, d/2$; $n > m = d$, $i_0 = 0, d/2$; or $m = n = d$, any i_0.

Finally, as it happened in the coprime case,

Corollary 4.8. *Y is a reformation retract of $X_{SL(2,\mathbb{C})}$.*

Proof. We see, looking at Propositions 4.1 and 4.2, that each component of Y_{red}, which is either isomorphic to $[-2, 2]$ or S^1, is a deformation retract of its corresponding component in X_{red} (isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$, respectively). Besides, the closure of each component in Y_{irr}, isomorphic to $[0, \infty]$, is again a deformation retract of the closure of its corresponding component in X_{irr} (isomorphic to $\mathbb{C}$). Using the gluing lemma, we can construct a global homotopy to show that Y is a deformation retract of X, as desired. $\square$

ENDNOTES

1. This map is an isomorphism of algebraic varieties for the torus knot groups $G = G_{m,n}$ in (1.1). This is shown in [**5**] directly as a consequence of the description of the character variety.

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