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NEW QUALITATIVE PROPERTIES OF SOLUTIONS TO NONLINEAR NONLOCAL CAUCHY PROBLEMS

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ABSTRACT. We introduce new concepts of asymptotically anti-periodic function and semi-Lipschitz continuity. The former is a natural generalization of the well-known anti-periodic function. Then, sufficient conditions, ensuring the existence of asymptotically anti-periodic mild solutions to a Cauchy problem of nonlinear evolution equation with nonlocal initial condition, are established. It is mentioned that one of our main results is proved in the absence of the compactness and Lipschitz continuity of nonlocal item and of the Lipschitz continuity of nonlinearity. Finally, an example is presented as an application.

1. Introduction. As is known, in some cases, the anti-periodic problems, compared with the periodic problems, are more realistic to reflect many physical phenomena in nature, and they have a very strong application background. Please see [**4, 5, 16**] and the references therein for more comments. For this reason, this class of problems has been investigated to a large extent during recent years. In particular, since the work of Okochi [**23**] in 1988 (see also [**24, 25**]), much attention has been attracted by questions of existence of anti-periodic solutions to various anti-periodic problems represented by linear and nonlinear abstract evolution equations. For significant work along this line, we refer to, e.g., [**1, 2, 8, 9, 15, 21, 22, 28**].

To explain the results better we need to introduce some notation and concepts. Let X be a Banach space with norm $\|\cdot\|$. For any

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$a \in \mathbb{R}$, by $C_0([a,+\infty);X)$, we denote the set of continuous functions from $[a,+\infty)$ to X vanishing at infinity. We abbreviate this notation to $C_0(\mathbb{R}^+)$ when $a=0$ and $X=\mathbb{R}^+$. Recall that a continuous function u from $\mathbb{R}$ to X is said to be T-anti-periodic if $u(t+T)=-u(t)$ for all $t \in \mathbb{R}$. By $P_{TA}(\mathbb{R};X)$, denote the set of such functions. It is easy to verify that $C_0([a,+\infty);X)$ and $P_{TA}(\mathbb{R};X)$, endowed with the norms $\|\cdot\|_a = \sup_{t\geq a}\|\cdot(t)\|$ and $\|\cdot\|'_\infty = \sup_{t\in\mathbb{R}}\|\cdot(t)\|$, respectively, are Banach spaces. Let us introduce the following new concept.

Definition 1.1. A function $u:\mathbb{R}^+\to X$ is said to be asymptotically T-anti-periodic if it can be decomposed as

$$u(t)=u_1(t)+u_2(t),\ t\in\mathbb{R}^+,$$

where $u_1\in P_{TA}(\mathbb{R};X)$ and $u_2\in C_0(\mathbb{R}^+;X)$. The set of such functions is denoted by $AA_T(\mathbb{R}^+;X)$.

Remark 1.2. It is noted that the concept of an asymptotically anti-periodic function is a natural generalization of the well-known anti-periodic function and is more complicated than an anti-periodic function. Moreover, little is known about it.

In this work, we are interested in studying the asymptotically T-anti-periodic mild solutions to the Cauchy problem of nonlinear evolution equations with nonlocal initial conditions of form

$$\begin{cases} u'(t)=Au(t)+f(t,u(t)), & t>0,\\ u(0)=H(u), \end{cases} \tag{1.1}$$

where A with the domain $D(A)$ (possibly unbounded) is a closed and densely defined linear operator on X, and $f:\mathbb{R}^+\times X\to X$, $H:AA_T(\mathbb{R}^+;X)\to X$ are given functions to be specified later. As can be seen, H constitutes a nonlocal condition.

Let us point out that a strong motivation for investigating the Cauchy problems of evolution equations with nonlocal initial conditions comes from physics. For example, as presented by Deng **[11]**, $H(u) := \sum_{i=1}^{p} C_i u(s_i)$, where C_i $(i=1,\ldots,p)$ are given constants and $0< s_1<\cdots<s_{p-1}<s_p<+\infty$ $(p\in\mathbb{N})$, is used to describe the diffusion phenomenon of a small amount of gas in a transparent tube. For more information concerning the motivations, relevant developments and the current status of the theory, we refer readers to, e.g., [**6, 7, 18, 19, 27**] and the references therein.

In this work, we shall give a systematic theory for the Cauchy problem (1.1). More precisely, we shall first study the completeness of $AA_T(\mathbb{R}^+; X)$ and the composition of anti-periodic functions as well as asymptotically anti-periodic functions. It is well known that the study of the composition of two functions with special properties is important and basic for deep investigations. Then, an entirely different strategy which relies on both the approximating technique in terms of the compact semigroup of strongly continuous operators and the theory of the measure of non-compactness, as well as the fixed point theorem due to Darbo-Sadovskii, is used to obtain the existence of asymptotically anti-periodic mild solutions to the Cauchy problem (1.1) under the hypotheses in which the nonlocal item does not have Lipschitz continuity or the compactness and the nonlinearity do not have Lipschitz continuity. The asymptotically anti-periodic mild solutions to the Cauchy problem (1.1) is also treated under the hypothesis of the nonlocal item and nonlinearity being Lipschitz continuous. As samples of applications, these results will be applied to a partial differential equation with homogeneous Dirichlet boundary condition and nonlocal initial condition.

Hopefully, our results will be helpful in making the up-to-date material in this field accessible and, meanwhile, lay the foundation for future research.

We would like to mention that, in recent papers such as de Andrade, et al., [**10**], Diagana [**12, 13**], Fan et al. [**14**], Liang et al. [**17, 20**] and Xiao et al. [**29**], the problem of the existence of asymptotically almost periodic, weighted pseudo almost periodic, Stepanov-like pseudo almost automorphic, pseudo almost automorphic, and asymptotically almost automorphic solutions for the Cauchy problems of abstract evolution equations has been investigated to a large extent. However, much of the previous research was done under the restriction that the nonlinearity as a whole is Lipschitz continuous or locally Lipschitz continuous, so that the Banach contraction principle becomes one of the key tools in the study of the corresponding problems.

Remark 1.3. (i) As the reader will see, the hypotheses on the nonlocal item and nonlinearity in our results are reasonably weak (see

Theorem 3.9 below). In particular, it is worth mentioning that the nonlinearity does not satisfy Lipschitz continuity or locally Lipschitz continuity with respect to the second variable.

(ii) Let us note that the new strategy as mentioned above plays a key role in the proof of our main results, which enables us to get rid of the compactness and Lipschitz continuity of nonlocal item.

Remark 1.4. It can be easily proved that, if u is anti-periodic with period T, then it is periodic with period $2T$. Hence, from the arguments of our paper, we can also obtain the existence results of asymptotically $2T$-periodic solutions of the Cauchy problem (1.1).

Let us give a short summary of the contents of this paper. In Section 2, some required notation, definitions and lemmas are given. In Section 3, we study the completeness of $AA_T(\mathbb{R}^+;X)$ with the supremum norm and the composition of anti-periodic functions as well as asymptotically anti-periodic functions, which in turn is used to analyze the existence of asymptotically anti-periodic mild solutions to the Cauchy problem (1.1). Finally, we present an example in Section 4 to illustrate our abstract results.

2. Preliminaries. This section is devoted to some preliminaries which are essential tools in the later sections.

Throughout this paper, $C([a,b];X)$ for $-\infty < a < b < +\infty$ is the Banach space of all continuous functions from $[a,b]$ into X with the uniform norm topology, $\mathcal{L}(X)$ stands for the Banach space of all bounded linear operators from X to X endowed with the uniform operator topology denoted by $\|\cdot\|_{\mathcal{L}(X)}$, $A : D(A) \subset X \to X$ is the infinitesimal generator of a compact semigroup of strongly continuous operators $\{T(t)\}_{t\geq 0}$ on X, and $\{T(t)\}_{t\geq 0}$ is uniformly exponentially stable, i.e., there exist constants $\delta > 0$ and $M \geq 1$ such that

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{-\delta t} \quad \text{for all } t \geq 0. \tag{2.1}$$

We recall here the following compact criterions in Banach space $C_0([a,+\infty);X)$. We omit the proof.

Lemma 2.1. *A set $D \subset C_0([a,+\infty);X)$ is relatively compact if*

(1) *D is equicontinuous.*
(2) $\lim_{t\to+\infty} u(t) = 0$ *uniformly for $u \in D$.*
(3) *The set $D(t) := \{u(t); u \in D\}$ is relatively compact in X for every $t \geq a$.*

Lemma 2.2. *A set $B \subset C_0([a,+\infty);X)$ is relatively compact in $C_0([a,+\infty);X)$ if the set $B|_{[a,b]}$ with $a < b$ is relatively compact in $C([a,b];X)$ and $B|_{[b,+\infty)} \subset C_0([b,+\infty);X)$ is relatively compact.*

A continuous function f from $\mathbb{R} \times X$ to X is said to be T-anti-periodic if $f(t+T,-x) = -f(t,x)$ for all $t \in \mathbb{R}$ and $x \in X$. Denote by $P_{TA}(\mathbb{R} \times X; X)$ the set of such functions. Let the notation $C_0(\mathbb{R}^+ \times X; X)$ be the space of functions

$$C_0(\mathbb{R} \times X; X) = \Big\{ f \in C(\mathbb{R} \times X; X); \lim_{t\to+\infty} \|f(t,x)\| = 0 \\ \text{uniformly for } x \text{ in any bounded subset of } X \Big\}.$$

Definition 2.3. A function $f : \mathbb{R}^+ \times X \to X$ is said to be asymptotically T-anti-periodic if it can be decomposed as

$$f(t,x) = f_1(t,x) + f_2(t,x), \quad t \in \mathbb{R}^+,\ x \in X,$$

where $f_1 \in P_{TA}(\mathbb{R} \times X; X)$ and $f_2 \in C_0(\mathbb{R}^+ \times X; X)$. In this case, we write $f \backsim (f_1, f_2)$.

Definition 2.4. An asymptotically T-anti-periodic function $f : \mathbb{R}^+ \times X \to X$ is said to be semi-Lipschitz continuous with the Lipschitz constant L if writing $f \backsim (f_1, f_2)$, there exists a constant $L > 0$ such that

$$\|f_1(t,x) - f_1(t,y)\| \leq L\|x - y\|$$

for all $t \in \mathbb{R}$ and $x, y \in X$. The set of such functions is denoted by $AAL_T(\mathbb{R}^+ \times X; X)$.

Definition 2.5. An asymptotically T-anti-periodic function $f : \mathbb{R}^+ \times X \to X$ is said to be locally semi-Lipschitz continuous if, writing $f \backsim (f_1, f_2)$, there exists a nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f_1(t,x) - f_1(t,y)\| \leq L(r)\|x - y\|.$$

for all $t \in \mathbb{R}$ and $x, y \in X$ satisfying $\|x\|, \|y\| \leq r$.

In the sequel, we briefly state some facts on the Kuratowski measure of non-compactness. Let Θ be the family of bounded sets in the Banach space Y. The Kuratowski measure of non-compactness $\mu : \Theta \to \mathbb{R}^+$ is defined by

$$\mu(B) = \inf\{d > 0; B \text{ admits a finite conver by set of diameter} \leq d\},$$
$$B \in \Theta.$$

The following are some basic properties of $\mu(\cdot)$.

(a) $\mu(B) = 0$ if and only if B is relatively compact in Y.
(b) μ is a seminorm, i.e., $\mu(\lambda B) = |\lambda|\mu(B)$ for constant $\lambda \in \mathbb{R}$ and $\mu(B_1 + B_2) \leq \mu(B_1) + \mu(B_2)$.
(c) $B_1 \subset B_2$ implies $\mu(B_1) \subset \mu(B_2)$.
(d) Let $F : D(F) \subset Y \to Y$ be Lipschitz continuous with Lipschitz constant L_F. Then $\mu(F(B)) \leq L_F\mu(B)$ for any bounded set $B \subset D(F)$.

For a detailed survey on Kuratowski measure of non-compactness, we refer readers to **[3]**.

Let $\Omega \subset Y$ and $F : \Omega \to X$ be continuous. Recall that F is called a μ-contraction on Ω if $\mu(F(B)) \leq k\mu(B)$ for some $0 \leq k < 1$ and any bounded subset $B \subset \Omega$.

This time we present a fixed point theorem concerning μ-contraction mapping.

Lemma 2.6. **[3]**. *Let $\Omega \subset Y$ be a nonempty closed convex set and $F : \Omega \to \Omega$ a μ-contraction. Then F has at least one fixed point in Ω.*

Below, for simplicity, we frequently omit explicit reference to the Banach space Y, on which μ is defined, and simply write "μ" instead of "μ_Y," provided that no ambiguities occur.

Definition 2.7. By a mild solution of the Cauchy problem (1.1), we mean a function $u \in C(\mathbb{R}^+; X)$ satisfying the integral equation

$$u(t) = T(t)H(u) + \int_0^t T(t-s)f(s, u(s))\,\mathrm{d}s, \quad t \geq 0.$$

3. Main results. We begin in this section by deriving some properties of asymptotically anti-periodic functions.

Lemma 3.1. *$AA_T(\mathbb{R}^+; X)$, endowed with the supremum norm $\|\cdot\|_\infty = \sup_{t\in\mathbb{R}^+} \|\cdot(t)\|$, is a Banach space.*

Proof. Note that $AA_T(\mathbb{R}^+; X)$, endowed with the norm $\|\cdot\|_\infty$, is a normed linear space.

Given $u \in AA_T(\mathbb{R}^+; X)$, there exist $u_1 \in P_{TA}(\mathbb{R}; X)$ and $u_2 \in C_0(\mathbb{R}^+; X)$ such that

$$u(t) = u_1(t) + u_2(t), \quad t \in \mathbb{R}^+.$$

We claim that

$$\{u_1; t \in \mathbb{R}\} \subset \overline{\{u(t); t \in \mathbb{R}^+\}}. \tag{3.1}$$

In fact, if this is not the case, then there exist some $t_0 \in \mathbb{R}$ and $\epsilon > 0$ such that

$$\|u_1(t_0) - u(t)\| \geq \epsilon \quad \text{for all } t \in \mathbb{R}^+.$$

Since $u_1 \in P_{TA}(\mathbb{R}; X)$ implies that $u_1(t_0 + 2nT) = u_1(t_0)$ for all $n \in \mathbb{N}$, one has

$$\|u_2(t_0 + 2nT)\| = \|u_1(t_0) - u(t_0 + 2nT)\| \geq \epsilon$$

for all $n \geq -t_0/2T$. This contradicts the fact that u_2 vanishes at infinity.

Now, letting $\{u_n\}_{n=1}^{+\infty}$ be a Cauchy sequence in $AA_T(\mathbb{R}^+; X)$, it follows that

$$u_n(t) = u_{n1}(t) + u_{n2}(t), \quad t \in \mathbb{R}^+,\ n = 1, 2, \ldots, \tag{3.2}$$

where $u_{n1} \in P_{TA}(\mathbb{R}; X)$ and $u_{n2} \in C_0(\mathbb{R}^+; X)$. From (3.1), it follows readily that $\{u_{n1}\}_{n=1}^{+\infty}$ is a Cauchy sequence in $P_{TA}(\mathbb{R}; X)$ which, together with (3.2), implies that $\{u_{n2}\}_{n=1}^{+\infty}$ is also a Cauchy sequence in $C_0(\mathbb{R}^+; X)$. Hence, from the completeness of $P_{TA}(\mathbb{R}; X)$

and $C_0(\mathbb{R}^+;X)$, it follows that there exist $u_{01} \in P_{TA}(\mathbb{R};X)$ and $u_{02} \in C_0(\mathbb{R}^+;X)$ such that

$$\begin{aligned} u_{n1} &\longrightarrow u_{01} \quad \text{in } P_{TA}(\mathbb{R};X), \\ u_{n2} &\longrightarrow u_{02} \quad \text{in } C_0(\mathbb{R}^+;X) \end{aligned}$$

as $n \longrightarrow \infty$. Consequently,

$$u_n \longrightarrow u_{01} + u_{02}, \quad \text{as } n \to \infty$$

in the norm $\|\cdot\|_\infty$ and $u_{01} + u_{02} \in AA_T(\mathbb{R}^+;X)$. Thus, $AA_T(\mathbb{R}^+;X)$ is complete. The conclusion follows. $\square$

Remark 3.2. Let $u \in AA_T(\mathbb{R}^+;X)$. Notice, in particular, that the decomposition of u is unique. Indeed, if there exist $u_1, u_1' \in P_{TA}(\mathbb{R};X)$ and $u_2, u_2' \in C_0(\mathbb{R}^+;X)$ such that

$$u(t) = u_1(t) + u_2(t) = u_1'(t) + u_2'(t), \quad t \in \mathbb{R}^+,$$

then we have that, for fixed $t \in \mathbb{R}$,

$$u_1(t) - u_1'(t) = u_2'(t+2nT) - u_2(t+2nT), \quad n \in \mathbb{N}$$
$$\text{with } n \geq -\frac{t}{2T}$$

in view of $u_1(t) = u_1(t+2nT)$ and $u_1'(t) = u_1'(t+2nT)$. Taking the limit as $n \to +\infty$, it is shown that $u_1(t) = u_1'(t)$, $t \in \mathbb{R}$, as required.

Remark 3.3. Let $f \backsim (f_1, f_2)$. Noticing that, for $x, y \in X$,

$$f_1(t+2nT,x) - f_1(t+2nT,y) = f_1(t,x) - f_1(t,y)$$
$$\text{for all } n \in \mathbb{N},$$

and

$$\lim_{t\to+\infty} \|f_2(t,x) - f_2(t,y)\| = 0,$$

it follows, by a similar argument as used in Lemma 3.1, that

$$\{f_1(t,x) - f_1(t,y); t \in \mathbb{R}\} \subset \overline{\{f(t,x) - f(t,y); t \in \mathbb{R}^+\}}$$
$$\text{for } x, y \in X.$$

From this, one sees easily that

$$\sup_{t\in\mathbb{R}} \|f_1(t,x) - f_1(t,y)\| \leq \sup_{t\in\mathbb{R}^+} \|f(t,x) - f(t,y)\| \quad \text{for } x, y \in X,$$

which in particular implies that f belongs to $AAL_T(\mathbb{R}^+ \times X; X)$ when f is Lipschitz continuous with respect to the second variable uniformly for $t \in \mathbb{R}^+$ (with the Lipschitz constant L).

Lemma 3.4. *Assume that $f \backsim (f_1, f_2)$ is semi-Lipschitz continuous with the Lipschitz constant L. Then $F_1(\cdot) := f_1(\cdot, u(\cdot))$ and $F(\cdot) := f(\cdot, u(\cdot))$ belong to $AA_T(\mathbb{R}^+; X)$ for each $u \in AA_T(\mathbb{R}^+; X)$.*

Proof. Since $u \in AA_T(\mathbb{R}^+; X)$, one has

$$u(t) = u_1(t) + u_2(t), \quad t \in \mathbb{R}^+,$$

where $u_1 \in P_{TA}(\mathbb{R}; X)$ and $u_2 \in C_0(\mathbb{R}^+; X)$. This, together with $f(t+T, -x) = -f(t, x)$, for all $t \in \mathbb{R}$ and $x \in X$, gives that $f_1(\cdot, u_1(\cdot))$ is T-anti-periodic. Therefore, to show that $F_1(\cdot)$ belongs to $AA_T(\mathbb{R}^+; X)$, it remains to prove that $F_1(\cdot) - f_1(\cdot, u_1(\cdot))$ vanishes at infinity. In fact, this can been seen from

$$\lim_{t \to +\infty} \|F_1(t) - f_1(t, u_1(t))\| \leq L \lim_{t \to +\infty} \|u_2(t)\| \longrightarrow 0$$

in view of f being semi-Lipschitz continuous and u_2 vanishing at infinity.

Noting that $f_2 \in C_0(\mathbb{R}^+ \times X; X)$, we have, in view of the boundedness of u, that $f_2(\cdot, u(\cdot)) \in C_0(\mathbb{R}^+; X)$. Since $F(\cdot) = F_1(\cdot) + f_2(\cdot, u(\cdot))$, we deduce that $F(\cdot)$ belongs to $AA_T(\mathbb{R}^+; X)$. This completes the proof. $\square$

Lemma 3.5. *Given $u_0 \in X$, $u_1 \in P_{TA}(\mathbb{R}; X)$, $u_2 \in C_0(\mathbb{R}^+; X)$, and $v \in AA_T(\mathbb{R}^+; X)$. Write*

$$\begin{aligned}
G_1(t) &:= \int_{-\infty}^{t} T(t-s)u_1(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \\
G_2(t) &:= T(t)u_0 - \int_{-\infty}^{0} T(t-s)u_1(s)\,\mathrm{d}s \\
&\qquad + \int_0^t T(t-s)u_2(s)\,\mathrm{d}s, \quad t \in \mathbb{R}^+, \\
G_3(t) &:= T(t)u_0 + \int_0^t T(t-s)v(s)\,\mathrm{d}s, \quad t \in \mathbb{R}^+.
\end{aligned}$$

Then G_1, G_2 and G_3 belong to $P_{TA}(\mathbb{R};X)$, $C_0(\mathbb{R}^+;X)$, and $AA_T(\mathbb{R}^+;X)$, respectively.

Proof. From (2.1), it is clear that G_1 is well defined and continuous on $\mathbb{R}$. Moreover, we obtain upon changing of variable that for $t \in \mathbb{R}$,

$$\begin{aligned} G_1(t+T) &= \int_{-\infty}^{t+T} T(t+T-s)u_1(s)\,\mathrm{d}s \\ &= \int_{-\infty}^{t} T(t-s)u_1(s+T)\mathrm{d}s = -G_1(t) \end{aligned}$$

in view of the T-anti-periodicity of u_1. Accordingly, G_1 belongs to $P_{TA}(\mathbb{R};X)$.

Given $\epsilon > 0$. Since u_2 vanishes at infinity, one can choose $N > 0$ such that $\|u_2(t)\| < \epsilon$ for all $t > N$. This, together with (2.1), enables us to conclude that

$$\begin{aligned} \left\| \int_0^t T(t-s)u_2(s)\,\mathrm{d}s \right\| &\le \left\| \int_0^N T(t-s)u_2(s)\,\mathrm{d}s \right\| \\ &\quad + \left\| \int_N^t T(t-s)u_2(s)\,\mathrm{d}s \right\| \\ &\le M\delta^{-1}e^{-\delta(t-N)}\|u_2\|_0 + M\delta^{-1}\epsilon, \end{aligned}$$

for $t > N$, from which we see

$$\int_0^t T(t-s)u_2(s)\,\mathrm{d}s \longrightarrow 0 \quad \text{in } X \text{ as } t \to +\infty.$$

Also, a direct calculation gives

$$\left\| T(t)u_0 - \int_{-\infty}^0 T(t-s)u_1(s)\,\mathrm{d}s \right\| \le M(\|u_0\| + \delta^{-1}\|u_1\|'_\infty)e^{-\delta t}.$$

Accordingly, G_2 belongs to $C_0(\mathbb{R}^+;X)$.

Since $v \in AA_T(\mathbb{R}^+;X)$, we have the decomposition as

$$v(t) = v_1(t) + v_2(t), \quad t \in \mathbb{R}^+,$$

where $v_1 \in P_{TA}(\mathbb{R}; X)$ and $v_2 \in C_0(\mathbb{R}^+; X)$. Writing

$$W_1(t) := \int_{-\infty}^{t} T(t-s)v_1(s)\mathrm{d}s, \quad t \in \mathbb{R},$$

$$W_2(t) := T(t)u_0 - \int_{-\infty}^{0} T(t-s)v_1(s)\,\mathrm{d}s + \int_0^t T(t-s)v_2(s)\,\mathrm{d}s, \quad t \in \mathbb{R}^+,$$

it is clear that W_1 and W_2 belong to $P_{TA}(\mathbb{R}; X)$ and $C_0(\mathbb{R}^+; X)$, respectively, as proved in the above arguments. Moreover, noticing $G_3(t) = W_1(t) + W_2(t)$, $t \in \mathbb{R}^+$, one has that G_3 belongs to $AA_T(\mathbb{R}^+; X)$. This completes the proof. □

Set $S_r := \{x \in AA_T(\mathbb{R}^+; X); \|x\|_\infty \leq r\}$ and $\Omega_r := \{x \in C_0(\mathbb{R}^+; X); \|x\|_0 \leq r\}$ for some $r > 0$, which are convex closed subsets of $AA_T(\mathbb{R}^+; X)$ and $C_0(\mathbb{R}^+; X)$, respectively. Let us introduce the following assumptions:

(H_1) $f \backsim (f_1, f_2)$ is semi-Lipschitz continuous with the Lipschitz constant L. Moreover, there exists a function $h \in C_0(\mathbb{R}^+)$ and a nondecreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $t \in \mathbb{R}^+$ and $x \in X$ satisfying $\|x\| \leq r$,

$$\|f_2(t, x)\| \leq h(t)\Phi(r), \quad \text{and} \quad \liminf_{r\to+\infty} \frac{\Phi(r)}{r} = \rho_1. \tag{3.3}$$

(H_2) (i) $H : AA_T(\mathbb{R}^+; X) \to X$ is continuous, there exists a nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $u \in S_r$,

$$\|H(u)\| \leq \Psi(r), \quad \text{and} \quad \liminf_{r\to+\infty} \frac{\Psi(r)}{r} = \rho_2. \tag{3.4}$$

(ii) There exists a $\varsigma > 0$ such that, for $u, v \in AA_T(\mathbb{R}^+; X)$ with $u(t) = v(t)$ for all $t \in [\varsigma, +\infty)$, $H(u) = H(v)$.

Write

$$\sigma_1(t) := \int_0^t e^{-\delta(t-s)}h(s)\,\mathrm{d}s, \quad t \in \mathbb{R}^+.$$

It is not difficult to see that $0 \leq \sigma_1(t) \leq 1/\delta \sup_{s\in\mathbb{R}^+} h(s)$ for every $t \in \mathbb{R}^+$. Furthermore, an analog argument used in Lemma 3.5 gives that $\sigma_1 \in C_0(\mathbb{R}^+)$. Put $\rho_3 := \sup_{t\in\mathbb{R}^+} \sigma_1(t)$.

Remark 3.6. Let us note that assumption (H_2) (ii) is the case when the values of the solution $u(t)$ for t near zero do not affect $H(u)$. A case in point was presented by Deng **[11]**.

Let $f \backsim (f_1, f_2)$. Now we consider, for each $m \geq 1$, the following system of integral equations of the form

$$(3.5) \quad \begin{cases} v(t) = \displaystyle\int_{-\infty}^{t} T(t-s) f_1(s, v(s))\,\mathrm{d}s, \quad t \in \mathbb{R}, \\ w(t) = T(t)T\left(\dfrac{1}{m}\right) H(v+w) \\ \qquad + \displaystyle\int_0^t T(t-s)\left[f_1(s, v(s)+w(s)) - f_1(s, v(s))\right]\mathrm{d}s \\ \qquad\qquad - \displaystyle\int_{-\infty}^{0} T(t-s) f_1(s, v(s))\,\mathrm{d}s \\ \qquad\qquad\quad + \displaystyle\int_0^t T(t-s) f_2(s, v(s)+w(s))\,\mathrm{d}s, \quad t \in \mathbb{R}^+. \end{cases}$$

Lemma 3.7. *Let the hypotheses* (H_1) *and* (H_2) (i) *hold. Then the system* (3.5) *has at least one solution* $(v, w_m) \in P_{TA}(\mathbb{R}; X) \times C_0(\mathbb{R}^+; X)$ *for each* $m \geq 1$*, provided that*

$$(3.6) \qquad M(\rho_2 + L\delta^{-1} + \rho_1\rho_3) < 1.$$

Proof. We start by defining a mapping Γ on $P_{TA}(\mathbb{R}; X)$ as follows:

$$(3.7) \qquad (\Gamma v)(t) = \int_{-\infty}^{t} T(t-s) f_1(s, v(s))\,\mathrm{d}s, \quad t \in \mathbb{R}.$$

Set, for $v \in P_{TA}(\mathbb{R}; X)$, $u_1(\cdot) = f_1(\cdot, v(\cdot))$. It easily follows from (H_1) that $u_1 \in P_{TA}(\mathbb{R}; X)$. From this and Lemma 3.5, we obtain that Γ is well defined and maps $P_{TA}(\mathbb{R}; X)$ into itself. Moreover, for any $t \in \mathbb{R}$ and $v_1, v_2 \in P_{TA}(\mathbb{R}; X)$,

$$\begin{aligned} \|(\Gamma v_1)(t) - (\Gamma v_2)(t)\| &\leq ML \int_{-\infty}^{t} e^{-\delta(t-s)} \|v_1(s) - v_2(s)\|\,\mathrm{d}s \\ &\leq ML\delta^{-1} \|v_1 - v_2\|'_\infty \end{aligned}$$

by the semi-Lipschitz continuity of f. This enables us to get

$$\|\Gamma v_1 - \Gamma v_2\|'_\infty \leq ML\delta^{-1} \|v_1 - v_2\|'_\infty,$$

which, together with (3.6), yields that Γ is a strict contraction on $P_{TA}(\mathbb{R};X)$. Thus, in view of the Banach contraction principle we conclude that Γ has a unique fixed point $v \in P_{TA}(\mathbb{R},X)$.

Put $r' := \sup_{t\in\mathbb{R}} \|v(t)\|$ and $M^* := \sup_{t\in\mathbb{R}} \|f_1(t,v(t))\|$.

Next, we introduce the mappings $H_v : C_0(\mathbb{R}^+;X) \to X$ and $J_v, f_{2v} : \mathbb{R}^+ \times X \to X$, defined by

$$\begin{aligned}
H_v(w) &= H(v+w), \quad w \in C_0(\mathbb{R}^+;X),\\
J_v(t,x) &= f_1(t,v(t)+x) - f_1(t,v(t)), \quad t \in \mathbb{R}^+,\ x \in X,\\
f_{2v}(t,x) &= f_2(t,v(t)+x), \quad t \in \mathbb{R}^+,\ x \in X.
\end{aligned}$$

From (H_2) (i), note that H_v is continuous. Also, by the semi-Lipschitz continuity of f, one has

$$\begin{aligned}
&\|J_v(t,x)\| \le L\|x\| \quad \text{for all } t \in \mathbb{R}^+,\ x \in X.\\
&\|J_v(t,x) - J_v(t,x)\| \le L\|x-y\| \quad \text{for all } t \in \mathbb{R}^+,\ x,y \in X.
\end{aligned} \tag{3.8}$$

Let $m \ge 1$ be fixed. Define a mapping Γ_m on $C_0(\mathbb{R}^+;X)$ by

$$\begin{aligned}
(\Gamma_m w)(t) = T(t)T\left(\frac{1}{m}\right)H_v(w) &+ \int_0^t T(t-s)J_v(s,w(s))\,\mathrm{d}s\\
&- \int_{-\infty}^0 T(t-s)f_1(s,v(s))\,\mathrm{d}s\\
&+ \int_0^t T(t-s)f_{2v}(s,w(s))\,\mathrm{d}s, \quad t \in \mathbb{R}^+.
\end{aligned}$$

Noticing, from (H_1) and (H_2) (i), that $f_1(\cdot,v(\cdot)) \in P_{TA}(\mathbb{R};X)$, and, for each $w \in C_0(\mathbb{R}^+;X)$, $J_v(\cdot,w(\cdot)), f_{2v}(\cdot,w(\cdot)) \in C_0(\mathbb{R}^+;X)$ and $T(1/m)H_v(w) \in X$ is independent of t, one has that Γ_m is well defined and maps $C_0(\mathbb{R}^+;X)$ into itself due to Lemma 3.5.

To this end, it suffices to prove that Γ_m possesses at least one fixed point in $C_0(\mathbb{R}^+;X)$. The proof will be divided into four steps.

For the sake of brevity, write

$$(\Gamma_m^1 w)(t) := T(t)T\left(\frac{1}{m}\right)H_v(w),$$

$$(\Gamma^2 w)(t) := \int_0^t T(t-s)J_v(s,w(s))\,\mathrm{d}s,$$

$$(\Gamma^3 w)(t) := -\int_{-\infty}^0 T(t-s)f_1(s,v(s))\,\mathrm{d}s + \int_0^t T(t-s)f_{2v}(s,w(s))\,\mathrm{d}s.$$

Step 1. There exists an $r_0 > 0$ such that Γ_m maps Ω_{r_0} into itself.

In fact, from (3.3), (3.4) and (3.6), it follows that there exists an $r_0 > 0$ such that

$$M(\Psi(r_0 + r') + L\delta^{-1}r_0 + M^*\delta^{-1} + \rho_3\Phi(r_0 + r')) \leq r_0,$$

which, together with (2.1) and the first inequality in (3.8), implies that, for any $t \in \mathbb{R}^+$ and $w \in \Omega_{r_0}$,

$$\begin{aligned}\|(\Gamma_m w)(t)\| &\leq \left\|T\left(t + \frac{1}{m}\right)\right\|_{\mathcal{L}(X)} \|H_v(w)\| \\ &\quad + \int_0^t \|T(t-s)\|_{\mathcal{L}(X)}\|J_v(s,w(s))\|\,\mathrm{d}s \\ &\quad + \int_{-\infty}^0 \|T(t-s)\|_{\mathcal{L}(X)}\|f_1(s,v(s))\|\,\mathrm{d}s \\ &\quad + \int_0^t \|T(t-s)\|_{\mathcal{L}(X)}\|f_{2v}(s,w(s))\|\,\mathrm{d}s \\ &\leq M\Psi(r_0 + r') + ML\delta^{-1}r_0 + MM^*\delta^{-1} \\ &\quad + M\Phi(r_0 + r')\sigma_1(t) \\ &\leq r_0.\end{aligned}$$

Accordingly, Γ_m maps Ω_{r_0} into itself.

Step 2. Γ^1 and Γ^3 are completely continuous on Ω_{r_0}.

Taking $w_1, w_2 \in \Omega_r$, we have, by (2.1),

$$\|(\Gamma^1 w_1)(t) - (\Gamma^1 w_2)(t)\|_0 \leq M\|H_v(w_1) - H_v(w_2)\|,$$

which, together with the continuity of H_v, enables us to deduce that Γ^1 is continuous on Ω_{r_0}. Also, since $H_v(\Omega_{r_0})$ is bounded in X in view of (H_2) (i) and $T(1/m)$ is compact in X, we justify by (2.1) that, for each $t \in \mathbb{R}^+$,

$$\left\{T(t)T \leq \left(\frac{1}{m}\right)H_v(w); w \in \Omega_{r_0}\right\} \quad \text{is precompact in } X,$$

and, for $0 \leq s \leq t \leq T$,

$$\begin{aligned} &\left\|T(t)\left(\frac{1}{m}\right)H_v(w) - T(s)T\left(\frac{1}{m}\right)H_v(w)\right\| \\ &\quad = \left\|\Big(T(t) - T(s)\Big)T\left(\frac{1}{m}\right)H_v(w)\right\| \\ &\qquad \longrightarrow 0, \quad \text{as } t \to s \end{aligned}$$

by the strong continuity of $\{T(t)\}_{t\geq 0}$, and the compactness of $T(1/m)$ $H_v(w)$ in X. Moreover, a direct computation gives that, for all $w \in \Omega_{r_0}$,

$$\|T(t)T\left(\frac{1}{m}\right)H_v(w)\| \leq M\Psi(r_0 + r')e^{-\delta t} \longrightarrow 0 \quad \text{as } t \to +\infty,$$

and the limit is independent of $w \in \Omega_{r_0}$. Thus, we verify, with the aid of Lemma 2.1, that Γ^1 is compact on Ω_{r_0}. Consequently, Γ^1 is completely continuous on Ω_{r_0}.

In the sequel, the mapping Γ^3 is treated. Given $\epsilon > 0$, since $h \in C_0(\mathbb{R}^+)$, one can choose t_0 big enough such that

$$h(t) < \frac{\delta\epsilon}{4M\Phi(r_0 + r')} \quad \text{whenever } t \geq t_0,$$

which, together with (2.1) and (3.3), implies that, for all $t \geq t_0$ and $w \in \Omega_{r_0}$,

$$\begin{aligned} \text{(3.9)} \qquad &\left\|\int_{t_0}^{t} T(t-s)f_{2v}(s, w(s))\,\mathrm{d}s\right\| \\ &\quad \leq M\Phi(r_0 + r')\int_{t_0}^{t} e^{-\delta(t-s)}h(s)\,\mathrm{d}s < \frac{\epsilon}{4}. \end{aligned}$$

Taking $\{w_k\}_{k=1}^{+\infty} \subset \Omega_{r_0}$ with $w_k \to w_0$ in $C_0(\mathbb{R}^+; X)$ as $k \to +\infty$, from the continuity of f_{2v} and (3.3), it follows that for each $t \in [0, t_0]$,

$$\lim_{k\to+\infty} \|f_{2v}(t, w_k(t)) - f_{2v}(t, w(t))\| = 0,$$

and

$$\|f_{2v}(t,w_k(t)) - f_{2v}(t,w(t)\| \le 2\Phi(r+r')h(t).$$

This, together with the Lebesgue dominated convergence theorem, yields that there exists a $K>0$ such that

$$M\int_0^{t_0} \|f_{2v}(s,w_k(s)) - f_{2v}(s,w(s))\| \,\mathrm{d}s < \frac{\epsilon}{2}$$

whenever $k \ge K$. Noticing this and (2.1), (3.9), we obtain that for all $t \ge 0$,

$$\begin{aligned}
&\|(\Gamma^3 w_k)(t) - (\Gamma^3 w)(t)\| \\
&\quad \le M\int_0^{t_0} \|f_{2v}(s,w_k(s)) - f_{2v}(s,w(s))\| \,\mathrm{d}s \\
&\qquad + \left\| \int_{t_0}^{\max\{t,t_0\}} T(t-s) f_{2v}(s,w(s)) \,\mathrm{d}s \right\| \\
&\qquad + \left\| \int_{t_0}^{\max\{t,t_0\}} T(t-s) f_{2v}(s,w_k(s)) \,\mathrm{d}s \right\| \\
&\quad < \epsilon,
\end{aligned}$$

whenever $k \ge K$. Accordingly, Γ^3 is continuous on Ω_{r_0}.

Below, we show that Γ^3 is compact on Ω_{r_0}. Since

$$-\int_{-\infty}^0 T(\cdot - s) f_1(s,v(s)) \,\mathrm{d}s$$

belongs to $C_0(\mathbb{R}^+;X)$ due to Lemma 3.5 with (H_1) and is independent of w, we only need to show that the mapping $\Gamma_v^3 : \Omega_{r_0} \to C_0(\mathbb{R}^+;X)$ defined by

$$(\Gamma_v^3 w)(t) := \int_0^t T(t-s) f_{2v}(s,w(s)) \,\mathrm{d}s, \quad t \in \mathbb{R}^+$$

is compact.

Let $t>0$ and $0<\epsilon_0<t$. Since $T(\epsilon_0)$ is compact and

$$\left\| \int_0^{t-\epsilon_0} T(t-\epsilon_0-s) f_{2v}(s,w(s)) \,\mathrm{d}s \right\|$$

is uniformly bounded for $w \in \Omega_{r_0}$ in X in view of (2.1) and (3.3),

$$\left\{ \int_0^{t-\epsilon_0} T(t-s) f_{2v}(s, w(s))\, \mathrm{d}s;\ u \in \Omega_{r_0} \right\}$$
$$= \left\{ T(\epsilon_0) \int_0^{t-\epsilon_0} T(t-\epsilon_0-s) f_{2v}(s, w(s))\, \mathrm{d}s;\ u \in \Omega_{r_0} \right\}$$

is relatively compact in X. Then, for every $w \in \Omega_{r_0}$, as

$$\begin{aligned}
&\left\| (\Gamma_v^3 w)(t) - \int_0^{t-\epsilon_0} T(t-s) f_{2v}(s, w(s))\, \mathrm{d}s \right\| \\
&\leq \int_{t-\epsilon_0}^{t} \| T(t-s) f_{2v}(s, w(s)) \|\, \mathrm{d}s \\
&\leq M\Phi(r_0 + r') \int_{t-\epsilon_0}^{t} e^{-\delta(t-s)} h(s)\, \mathrm{d}s \\
&\longrightarrow 0 \quad \text{as } \epsilon_0 \to 0^+
\end{aligned}$$

in X, we conclude, in view of the total boundedness, that, for each $t \in \mathbb{R}^+$, the set $\{(\Gamma_v^3 w)(t); w \in \Omega_{r_0}\}$ is relatively compact in X.

Now, we consider equicontinuity of the set $\{\Gamma_v^3 w; w \in \Omega_{r_0}\}$. Given $\varepsilon > 0$, we take $t, \tau \in \mathbb{R}^+$ with $t > \tau$, and we can choose an $\eta' > 0$ such that

$$M\delta^{-1}\Phi(r_0 + r') \sup_{s \in \mathbb{R}^+} h(s) \left(1 - e^{-\delta(t-\tau)}\right) \leq \frac{\varepsilon}{4} \quad \text{when } \eta' \geq t - \tau, \tag{3.10}$$

and choose an $\eta'' \in (0, \eta']$ such that

$$M\Phi(r_0 + r') \, \| T(t - \tau + \eta') - T(\eta') \|_{\mathcal{L}(X)} \sup_{s \in \mathbb{R}^+} h(s)\delta^{-1} \leq \frac{\varepsilon}{4} \quad \text{when } \eta'' \geq t - \tau. \tag{3.11}$$

For the case when $\tau \leq \eta'$, we write

$$\begin{aligned}
(\Gamma_v^3 w)(t) - (\Gamma_v^3 w)(\tau) &= \int_0^{\tau} \left(T(t-s) - T(\tau - s) \right) f_{2v}(s, w(s))\, \mathrm{d}s \\
&\quad + \int_{\tau}^{t} T(t-s) f_{2v}(s, w(s))\, \mathrm{d}s.
\end{aligned}$$

From (2.1), (3.3) and (3.10), it follows that, for $\eta'' \geq t-\tau$, $w \in \Omega_{r_0}$,

$$\begin{aligned}
&\left\| \int_0^\tau (T(t-s)-T(\tau-s)) f_{2v}(s,w(s))\,\mathrm{d}s \right\| \\
&\leq \int_0^\tau \|(T(t-s)-T(\tau-s))\|_{\mathcal{L}(X)} \|f_{2v}(s,w(s))\|\,\mathrm{d}s \\
&\leq M\Phi(r_0+r') \int_0^\tau \left(e^{-\delta(t-\tau)}+1\right) e^{-\delta(\tau-s)} h(s)\,\mathrm{d}s \\
&\leq 2M\delta^{-1}\Phi(r_0+r') \sup_{s\in\mathbb{R}^+} h(s)(1-e^{-\delta\tau}) \\
&\leq \frac{\varepsilon}{2},
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \int_\tau^t T(t-s) f_{2v}(s,w(s))\,\mathrm{d}s \right\| \\
&\leq M\Phi(r_0+r') \int_\tau^t e^{-\delta(t-s)} h(s)\,\mathrm{d}s \\
&\leq M\delta^{-1}\Phi(r_0+r') \sup_{s\in\mathbb{R}^+} h(s) \left(1-e^{-\delta(t-\tau)}\right) \\
&\leq \frac{\varepsilon}{4}.
\end{aligned}$$

For the case when $\tau > \eta'$, we write

$$\begin{aligned}
&(\Gamma_v^3 w)(t) - (\Gamma_v^3 w)(\tau) \\
&= \int_0^{\tau-\eta'} (T(t-s)-T(\tau-s)) f_{2v}(s,w(s))\,\mathrm{d}s \\
&\quad + \int_{\tau-\eta'}^{\tau} (T(t-s)-T(\tau-s)) f_{2v}(s,w(s))\,\mathrm{d}s \\
&\quad + \int_\tau^t T(t-s) f_{2v}(s,w(s))\,\mathrm{d}s.
\end{aligned}$$

Noticing (2.1), (3.3) and (3.10), and arguing as above, we can prove that, for $\eta'' \geq t-\tau$, $w \in \Omega_{r_0}$,

$$\left\| \int_{\tau-\eta'}^{\tau} (T(t-s)-T(\tau-s)) f_{2v}(s,w(s))\,\mathrm{d}s \right\|$$

$$\leq 2M\Phi(r_0 + r') \sup_{s\in\mathbb{R}^+} h(s)\delta^{-1}(1 - e^{-\delta\eta'}) \leq \frac{\varepsilon}{2},$$

and

$$\left\| \int_\tau^t T(t-s) f_{2v}(s, w(s))\,\mathrm{d}s \right\| \leq \frac{\varepsilon}{4}.$$

Moreover, in view of (3.11), a direct calculation yields

$$\begin{aligned}
&\left\| \int_0^{\tau-\eta'} (T(t-s) - T(\tau - s)) f_{2v}(s, w(s))\,\mathrm{d}s \right\| \\
&\leq \|T(t-\tau+\eta') - T(\eta')\|_{\mathcal{L}(X)} \\
&\quad \times \int_0^{\tau-\eta'} \|T(\tau-\eta-s) f_{2v}(s, w(s))\|\,\mathrm{d}s \\
&\leq M\Phi(r_0+r')\,\|T(t-\tau+\eta') - T(\eta')\|_{\mathcal{L}(X)} \\
&\quad \times \int_0^{\tau-\eta'} e^{-(t-\eta'-s)} h(s)\,\mathrm{d}s \\
&\leq M\delta^{-1}\Phi(r_0+r')\,\|T(t-\tau+\eta') - T(\eta')\|_{\mathcal{L}(X)} \sup_{s\in\mathbb{R}^+} h(s) \\
&\leq \frac{\varepsilon}{4},
\end{aligned}$$

when $\eta'' \geq t - \tau$ and $w \in \Omega_{r_0}$.

Summarizing the above, one can deduce that

$$\|(\Gamma_v^3 w)(t) - (\Gamma_v^3 w)(\tau)\| \leq \varepsilon$$

when $\eta'' \geq t-\tau$ and $w \in \Omega_{r_0}$, which proves that the set $\{\Gamma_v^3 w; w \in \Omega_{r_0}\}$ is equicontinuous.

At the end of this step, it remains to show, in view of Lemma 2.1, that $(\Gamma_v^3 w)(\cdot)$ vanishes at infinity uniformly for $w \in \Omega_{r_0}$. Since $\sigma_1 \in C_0(\mathbb{R}^+)$, as

$$\|(\Gamma_v^3 w)(t)\| \leq M\Phi(r_0 + r')\sigma_1(t) \quad \text{for } t \geq 0,\ w \in \Omega_{r_0},$$

we deduce that the conclusion follows.

Thus, applying Lemma 2.1, we obtain that Γ_v^3 is compact on Ω_{r_0}, so is Γ^3.

Step 3. Γ^2 is a strict contraction on Ω_{r_0}.

For any $t \in \mathbb{R}^+$ and $w_1, w_2 \in \Omega_{r_0}$, we obtain by (H_1) that

$$\begin{aligned}
&\|(\Gamma^2 w_1)(t) - (\Gamma^2 w_2)(t)\| \\
&\leq \left\| \int_0^t T(t-s)\left(J_v(s, w_1(s)) - J_v(s, w_2(s))\right) \mathrm{d}s \right\| \\
&\leq ML \int_0^t e^{-\delta(t-s)} \|w_1(s) - w_2(s)\| \,\mathrm{d}s \\
&\leq ML\delta^{-1} \|w_1 - w_2\|_0,
\end{aligned}$$

which means that Γ^2 is a strict contraction due to (3.6).

Step 4. Combining the considerations in the above three steps with (a), (b) and (d), we obtain that

$$\begin{aligned}
\mu(\Gamma_m(\Omega_{r_0})) &\leq \mu(\Gamma_m^1(\Omega_{r_0})) + \mu(\Gamma^2(\Omega_{r_0})) + \mu(\Gamma^3(\Omega_{r_0})) \\
&\leq ML\delta^{-1}\mu(\Omega_{r_0}),
\end{aligned}$$

which together with (3.6) implies that Γ_m is a μ-contraction. Applying Darbo-Sadovskii's fixed point theorem, one finds that Γ_m possesses at least one fixed point $w_m \in \Omega_{r_0}$. This completes the proof. □

Below, let $(v, w_m) \in P_{TA}(\mathbb{R}; X) \times C_0(\mathbb{R}^+; X)$, coming from Lemma 3.7, be a solution of the system (3.5) corresponding to $m \geq 1$.

Lemma 3.8. *Under the hypotheses* (H_1)*,* (H_2) *and* (3.6)*, the set* $\{w_m; m \geq 1\}|_{[\varsigma,+\infty)}$ *is, with* ς *being the constant in* (H_2) (ii)*, relatively compact in* $C_0([\varsigma, +\infty); X)$*.*

Proof. As proved in Lemma 3.7, there exists an $r_0 > 0$ such that $w_m \in \Omega_{r_0}$ for all $m \geq 1$ and w_m satisfies the integral equation

$$\begin{gathered}
w_m(t) = (\Gamma_m^1 w_m)(t) + (\Gamma^2 w_m)(t) + (\Gamma^3 w_m)(t), \\
t \in \mathbb{R}^+, \quad m \geq 1,
\end{gathered}$$

where the mappings Γ_m^1, Γ^2, and Γ^3 are defined the same as in Lemma 3.7. To this end, it suffices to show that the sets $\{\Gamma_m^1 w_m; m \geq 1\}|_{[\varsigma,+\infty)}$, $\{\Gamma^2 w_m; m \geq 1\}|_{[\varsigma,+\infty)}$, and $\{\Gamma^3 w_m; m \geq 1\}|_{[\varsigma,+\infty)}$ are relatively compact in $C_0([\varsigma, +\infty); X)$.

From the compactness of $T(t)$ $(t \geq \varsigma)$ in X, the boundedness of $T(1/m)$ and (H_2) (i), it follows that the set $\{(\Gamma_m^1 w_m)(t); m \geq 1\}$ for each $t \in [\varsigma, +\infty)$ is precompact in X, and for $s_1, s_2 \in [\varsigma, +\infty)$ with $s_1 \le s_2$,

$$\begin{aligned}\big\|(\Gamma_m^1 w_m)(s_2) - (\Gamma_m^1 w_m)(s_1)\big\| &\\ = \left\|\Big(T(s_2) - T(s_1)\Big)T\left(\frac{1}{m}\right)H_v(w_m)\right\| &\\ \longrightarrow 0, \quad \text{as } s_2 \to s_1&\end{aligned}$$

uniformly for $m \ge 1$ (since the compactness of $T(t)$ for $t > 0$ implies continuity in the uniform operator topology). Moreover, by (2.1) and (H_2) (i), one has

$$\|\Gamma_m^1 w_m)(t)\| \le Me^{-\delta t}\Psi(r + r') \longrightarrow 0 \quad \text{as } t \to +\infty$$

uniformly for $m \ge 1$. Hence, an application of Lemma 2.1 justifies that the set $\{\Gamma_m^1 w_m; m \ge 1\}|_{[\varsigma,+\infty)}$ is relatively compact in $C_0([\varsigma, +\infty); X)$.

The same idea as in Lemma 3.7 step 2 can be used to prove that the set $\{\Gamma^3 w_m; m \ge 1\}|_{[\varsigma,+\infty)}$ is relatively compact in $C_0([\varsigma, +\infty); X)$.

Next, we consider the set $\{\Gamma^2 w_m; m \ge 1\}_{[\varsigma,+\infty)}$. Let us decompose the mapping $\Gamma^2 = \Gamma' + \Gamma''$ as

$$(\Gamma' w_m)(t) = \int_\varsigma^t T(t-s)J_v(s, w_m(s))\,\mathrm{d}s, \quad t \in [\varsigma, +\infty),$$
$$(\Gamma'' w_m)(t) = \int_0^\varsigma T(t-s)J_v(s, w_m(s))\,\mathrm{d}s, \quad t \in [\varsigma, +\infty).$$

It follows from (H_1) that, for any $t \in [\varsigma, +\infty)$ and $w_1, w_2 \in C_0([\varsigma, +\infty); X)$,

$$\begin{aligned}\|(\Gamma' w_1)(t) - (\Gamma' w_2)(t)\| &\le \left\|\int_\varsigma^t T(t-s)\left(J_v(s, w_1(s)) - J_v(s, w_2(s))\right)\mathrm{d}s\right\| \\ &\le ML\int_\varsigma^t e^{-\delta(t-s)}\|w_1(s) - w_2(s)\|\,\mathrm{d}s \\ &\le ML\delta^{-1}\|w_1 - w_2\|_\varsigma,\end{aligned}$$

that is,

$$\|(\Gamma' w_1)(t) - (\Gamma' w_2)(t)\|_\varsigma \le ML\delta^{-1}\|w_1 - w_2\|_\varsigma.$$

Also, noticing $w_m \in \Omega_{r_0}$ for all $m \geq 1$, the first inequality in (3.8), and the compactness of $T(t)$ for $t > 0$, one can show by a similar argument as that in Lemma 3.7 step 2, that the set $\{\Gamma'' w_m; m \geq 1\}|_{[\varsigma,+\infty)}$ is relatively compact in $C_0([\varsigma,+\infty);X)$.

Thus, we obtain, thanks to (a), (b) and (d), that

$$\begin{aligned}\mu(\{w_m; m \geq 1\}|_{[\varsigma,+\infty)}) &\leq \mu(\{(\Gamma_m^1 w_m)(t); m \geq 1\}_{[\varsigma,+\infty)}) \\ &\quad + \mu(\{\Gamma' w_m; m \geq 1\}_{[\varsigma,+\infty)}) \\ &\quad + \mu(\{\Gamma'' w_m; m \geq 1\}_{[\varsigma,+\infty)}) \\ &\quad + \mu(\{\Gamma^3 w_m; m \geq 1\}_{[\varsigma,+\infty)}) \\ &\leq ML\delta^{-1}\mu(\{w_m; m \geq 1\}_{[\varsigma,+\infty)}),\end{aligned}$$

from which, together with (3.6), we see that $\mu(\{w_m; m \geq 1\}|_{[\varsigma,+\infty)}) = 0$. The proof is then completed. $\square$

Having at hand the above auxiliary results, we can state our main results of this section.

Theorem 3.9. *Let the hypotheses* (H_1), (H_2) *and* (3.6) *hold. Then the Cauchy problem* (1.1) *admits at least an asymptotically T-anti-periodic mild solution.*

Proof. Assume that the mappings Γ_m^1, Γ^2, and Γ^3 are defined the same as in Lemma 3.7 and $(v, w_m) \in P_{TA}(\mathbb{R};X) \times C_0(\mathbb{R}^+;X)$, coming from Lemma 3.7, is a solution of the system (3.5) corresponding to $m \geq 1$. It has been shown, thanks to Lemma 3.8, that the set $\{w_m; m \geq 1\}|_{[\varsigma,+\infty)}$ is relatively compact in $C_0([\varsigma,+\infty);X)$. Writing

$$\overline{w}_m(t) = \begin{cases} u_m(t) & \text{if } t \in [\varsigma,+\infty), \\ w_m(\varsigma) & \text{if } t \in [0,\varsigma], \end{cases}$$

one can assume, without loss of generality, that

$$w_m \longrightarrow w' \quad \text{in } C_0(\mathbb{R}^+;X)$$

as $m \to +\infty$, which, together with the continuity of H and the strong continuity of $T(t)$, gives that for any $t \in [0, \varsigma]$ and fixed $v \in P_{TA}(\mathbb{R}; X)$,

$$\begin{aligned}
&\left\|(\Gamma_m^1 w_m)(t) - T(t)H_v(w')\right\| \\
&\le M\left\|T\left(\frac{1}{m}\right)H(\overline{w}_m + v) - H(w' + v)\right\| \\
&\le M\left\|\left(T\left(\frac{1}{m}\right) - I\right)H(w' + v)\right\| \\
&\quad + M\left\|T\left(\frac{1}{m}\right)H(\overline{w}_m + v) - T\left(\frac{1}{m}\right)H(w' + v)\right\| \\
&\le \left\|\left(T\left(\frac{1}{m}\right) - I\right)H(w' + v)\right\| \\
&\quad + M^2\left\|H(\overline{w}_m + v) - H(w' + v)\right\| \\
&\longrightarrow 0 \quad \text{as } m \to +\infty.
\end{aligned}$$

This proves that the set $\{(\Gamma_m^1 w_m)(t); m \ge 1\}$ for any $t \in [0, \varsigma]$ is relatively compact in X, which means, in particular, that $\{T(1/m)H_v(w_m); m \ge 1\}$ is relatively compact in X. From this and the strong continuity of $T(t)$, it follows readily that, for $t, \tau \in [0, \varsigma]$, $t \ge \tau$,

$$\begin{aligned}
&\left\|(\Gamma_m^1 w_m)(t) - (\Gamma_m^1 w_m)(\tau)\right\| \\
&\qquad \le \left\|(T(t) - T(\tau))\,T\left(\frac{1}{m}\right)H_v(w_m)\right\| \\
&\qquad\qquad \longrightarrow 0 \quad \text{as } t - \tau \to 0
\end{aligned}$$

uniformly for $m \ge 1$. Therefore, we conclude that the set $\{\Gamma_m^1 w_m; m \ge 1\}|_{[0,\varsigma]}$ is relatively compact in $C([0, \varsigma]; X)$ due to Arzela-Ascoli's theorem. This proves, with the aid of Lemma 2.2, that the set $\{\Gamma_m^1 w_m; m \ge 1\}$ is relatively compact in $C_0([0, +\infty); X)$, since the set $\{\Gamma_m^1 w_m; m \ge 1\}|_{[\varsigma,+\infty)}$ is relatively compact in $C_0([\varsigma, +\infty); X)$. On the other hand, as proved in Lemma 3.7, Γ^2 is a strict contraction on Ω_{r_0} with Lipschitz constant $ML\delta^{-1}$, and Γ^3 is completely continuous on Ω_{r_0}.

Thus, again by (a), (b) and (d), one has

$$\begin{aligned}
\mu(\{w_m; m \ge 1\}) &\le \mu(\big\{(\Gamma_m^1 w_m)(t); m \ge 1\big\}) \\
&\qquad + \mu(\{\Gamma^2 w_m; m \ge 1\}) + \mu(\{\Gamma^3 w_m; m \ge 1\})
\end{aligned}$$

$$\leq ML\delta^{-1}\mu(\{w_m; m \geq 1\}).$$

This proves that $\mu(\{w_m; m \geq 1\}) = 0$ due to (3.6), that is, $\{w_m; m \geq 1\}$ is relatively compact in $C_0([0, +\infty); X)$. Hence, there is a subsequence of $\{w_m; m \geq 1\}$, again denoted by $\{w_m\}$, and a $w \in C_0([0, +\infty); X)$ such that $w_m \to w$ in $C_0([0, +\infty); X)$ as $m \to \infty$.

Recall that $(v, w_m) \in P_{TA}(\mathbb{R}; X) \times C_0(\mathbb{R}^+; X)$ satisfies the integral equation

$$\begin{cases} v(t) = \displaystyle\int_{-\infty}^{t} T(t-s) f_1(s, v(s))\,\mathrm{d}s, \quad t \in \mathbb{R}, \\ w_m(t) = T(t)T\left(\dfrac{1}{m}\right) H(v + w_m) \\ \qquad\qquad + \int_0^t T(t-s)\left[f_1(s, v(s) + w_m(s)) - f_1(s, v(s))\right]\,\mathrm{d}s \\ \qquad\qquad - \int_{-\infty}^{0} T(t-s) f_1(s, v(s))\,\mathrm{d}s \\ \qquad\qquad + \int_0^t T(t-s) f_2(s, v(s) + w_m(s))\,\mathrm{d}s, \quad t \in \mathbb{R}^+. \end{cases}$$

Letting $m \to \infty$ on both sides, one finds, noticing the continuity of H, f_1 and f_2 with respect to the second argument, that (v, w) satisfies system (3.5) which, in particular, implies that $v+w$ is an asymptotically T-anti-periodic mild solution. This completes the proof. $\square$

Remark 3.10. Let us note that Lemma 3.4 and Theorem 3.9 can be easily extended to the case of the nonlinear item f being locally semi-Lipschitz continuous.

The following corollaries are generalizations of Theorem 3.9.

Corollary 3.11. *Under the hypotheses* (H_1), *for every* $u_0 \in X$, *the Cauchy problem*

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \quad t > 0, \\ u(0) = u_0, \end{cases}$$

has at least one asymptotically T*-anti-periodic mild solution provided that* $M(L\delta^{-1} + \rho_1\rho_3) < 1$.

Corollary 3.12. *Assuming that the hypothesis* (H_1) *is satisfied, the*

Cauchy problem with nonlocal initial condition in the form

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \quad t > 0, \\ u(0) = \displaystyle\sum_{i=1}^{p} C_i u(s_i), \quad 0 < s_1 < \cdots < s_{p-1} < s_p < +\infty, \end{cases}$$

where C_i $(i = 1, \ldots, p)$ *are given constants, has at least one asymptotically* T*-anti-periodic mild solution provided that* $M(\sum_{i=1}^{p} |C_i| + L\delta^{-1} + \rho_1\rho_3) < 1$.

Proof. Define

$$H(u) = \sum_{i=1}^{p} C_i u(s_i), \quad u \in AA_T(\mathbb{R}^+; X).$$

It is clear that H verifies the hypothesis (H_2) with

$$\Psi(r) = r \sum_{i=1}^{p} |C_i|, \qquad \rho_2 = \sum_{i=1}^{p} |C_i|$$

Hence, the conclusion holds due to Theorem 3.9. The proof is completed. □

In the following, we establish the existence and uniqueness result of the asymptotically T-anti-periodic mild solution to the Cauchy problem (1.1) under the hypotheses of f and H being Lipschitzian.

Theorem 3.13. *Assume that*

- (H_1') $f : \mathbb{R}^+ \times X \to X$ *is asymptotically* T*-anti-periodic, and there exists a constant* $L_f > 0$ *such that*

$$\|f(t, x) - f(t, y)\| \le L_f \|x - y\|$$

for all $t \in \mathbb{R}^+$ *and* $x, y \in X$,
- (H_2') $H : AA_T(\mathbb{R}^+; X) \to X$ *is Lipschitz continuous with the Lipschitz constant* L_H, *and*
- (H_3) $ML_{H} + ML_f\delta^{-1} < 1$.

Then the Cauchy problem (1.1) *has a unique asymptotically* T*-anti-periodic mild solution.*

Proof. Let us define a mapping Λ on $AA_T(\mathbb{R}^+;X)$ as

$$(\Lambda u)(t) := T(t)H(u) + \int_0^t T(t-s)f(s,u(s))\,\mathrm{d}s, \quad t \in \mathbb{R}^+.$$

Set, for $u \in AA_T(\mathbb{R}^+;X)$, $F(\cdot) := f(\cdot, u(\cdot))$. From Remark 3.3 and Lemma 3.4, it follows that $F \in AA_T(\mathbb{R}^+;X)$, which together with Lemma 3.5 implies that Λ is well defined, and it maps $AA_T(\mathbb{R}^+;X)$ into itself. Moreover, for any $t \in \mathbb{R}^+$ and $u_1, u_2 \in AA_T(\mathbb{R}^+;X)$, by (H_1') and (H_2'), we have

$$\begin{aligned}
&\|(\Lambda u_1)(t) - (\Lambda u_2)(t)\| \\
&\quad\le \|T(t)\left(H(u_1) - H(u_2)\right)\| \\
&\qquad\qquad + \int_0^t \|T(t-s)\left(f(s,u_1(s)) - f(s,u_2(s))\right)\|\,\mathrm{d}s \\
&\quad\le ML_H\|u_1 - u_2\|_\infty \\
&\qquad\qquad + ML_f\int_0^t e^{-\delta(t-s)}\|u_1(s) - u_2(s)\|\,\mathrm{d}s \\
&\quad\le (ML_H + ML_f\delta^{-1})\|u_1 - u_2\|_\infty.
\end{aligned}$$

Consequently,

$$\|(\Lambda u_1)(t) - (\Lambda u_2)(t)\|_\infty \le (ML_H + ML_f\delta^{-1})\|u_1 - u_2\|_\infty,$$

which, together with (H_3), implies that Λ is a strict contraction on $AA_T(\mathbb{R}^+;X)$. Thus, we conclude, using the Banach contraction principle, that Λ has a unique fixed point in $AA_T(\mathbb{R}^+;X)$, which is an asymptotically T-anti-periodic mild solution to the Cauchy problem (1.1). This completes the proof. □

4. An example. In this section, we present an example as an application of our abstract results.

Consider the partial differential equation with homogeneous Dirichlet boundary condition and nonlocal initial condition

$$
(4.1) \qquad \begin{cases} \dfrac{\partial u(t,\xi)}{\partial t} - a\dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} = \cos\dfrac{\pi t}{T}\sin u(t,\xi) \\ \qquad\qquad\qquad\qquad\qquad +e^{-t}u(t,\xi)\sin u^2(t,\xi), \\ \qquad\qquad\qquad\qquad\qquad (t,\xi)\in(0,+\infty)\times[0,\pi], \\ u(t,0)=u(t,\pi)=0, \quad t\in\mathbb{R}^+, \\ u(0,\xi)=u_0(\xi)+\sum_{i=1}^p C_i u^{\frac{1}{3}}(t_i,\xi), \quad \xi\in[0,\pi], \end{cases}
$$

where $0<t_1<\cdots<t_p<+\infty$, C_i $(i=1,\dots,p)$ and $a>0$ are given constants, and $u_0\in L^2[0,\pi]$. Here, our objective is to show the existence of asymptotically T-anti-periodic mild solution for the partial differential equation (4.1).

Take $X=L^2[0,\pi]$ with the norm $\|\cdot\|_{L^2[0,\pi]}$ and inner product $(\cdot,\cdot)_2$. Define an operator $A: D(A)\subset X\to X$ by

$$
\begin{aligned}
Ax &= a\frac{\partial^2 x}{\partial \xi^2}, \quad x\in D(A),\\
D(A) &= \{x\in X; x, x' \text{ are absolutely continuous},\\
&\qquad\qquad x''\in X, \text{and } x(0)=x(\pi)=0\}.
\end{aligned}
$$

It is well known that A has a discrete spectrum, and its eigenvalues are $-an^2, n\in\mathbb{N}^+$ with the corresponding normalized eigenvectors $y_n(\xi)=\sqrt{2/\pi}\sin(n\xi)$. Moreover, A generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X as

$$
T(t)x=\sum_{n=1}^{+\infty} e^{-an^2t}(x,y_n)_2 y_n, \quad \text{for all } t\geq 0,\ x\in X.
$$

More details about these facts can be seen from the monograph **[26]** of Pazy.

A direct computation yields

$$
\|T(t)\|_{\mathcal{L}(X)}\leq e^{-at} \quad \text{for all } t\geq 0,
$$

which means that $\{T(t)\}_{t\geq 0}$ is uniformly exponentially stable with $M=1$ and $\delta=a$. Note also that, for each $t>0$, $T(t)$ is a nuclear operator, which gives the compactness of $T(t)$ for $t>0$.

Define

$$f(t,x(\xi)) := \cos\frac{\pi t}{T}\sin x(\xi) + e^{-t}x(\xi)\sin x^2(\xi), \quad t\in\mathbb{R}^+,\ x\in X,$$
$$f_1(t,x(\xi)) := \cos\frac{\pi t}{T}\sin x(\xi), \quad t\in\mathbb{R},\ x\in X,$$
$$f_2(t,x(\xi)) := e^{-t}x(\xi)\sin x^2(\xi), \quad t\in\mathbb{R}^+,\ x\in X,$$
$$H(u(t,\xi)) := u_0(\xi) + \sum_{i=1}^{p} C_i u^{1/3}(t_i,\xi), \quad u\in AA_T(\mathbb{R}^+;X).$$

Then it is not difficult to verify that $H : AA_T(\mathbb{R}^+;X)\to X$, $f_1 : \mathbb{R}\times X\to X$ and $f_2 : \mathbb{R}^+\times X\to X$ are continuous, $f_1(t+T,-x) = -f_1(t,x)$ for all $t\in\mathbb{R}$ and $x\in X$,

$$\|f_1(t,x)-f_1(t,y)\| \le \|x-y\|$$
$$\text{for all } t\in\mathbb{R}^+,\ x,y\in X,$$

and

$$\|f_2(t,x)\| \le e^{-t}\|x\| \quad \text{for } t\in\mathbb{R}^+ \text{ and all } x\in X,$$

which implies that $f \backsim (f_1,f_2)$ is semi-Lipschitz continuous with the Lipschitz constant $L=1$. Therefore, (4.1) can be reformulated as the abstract Cauchy problem (1.1), and the assumptions (H_1) and (H_2) hold with

$$L=1, \quad \Phi(r)=r, \quad \Psi(r)=\|u_0\|+\pi^{1/3}r^{1/3}\sum_{i=1}^{p}|C_i|,$$
$$h(t)=e^{-t}, \quad \rho_1=1, \quad \rho_2=0, \quad \rho_3\le\frac{1}{a}.$$

Hence, we deduce by Theorem 3.9 that, when $a>2$, the partial differential equation (4.1) has at least one asymptotically T-anti-periodic mild solution.

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