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A DIFFERENTIAL INEQUALITY AND STARLIKENESS OF A DOUBLE INTEGRAL

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ABSTRACT. The main objective of this paper is to discuss starlikeness of order β of the solutions of a differential equation and, as a consequence, to obtain conditions on the kernel function g such that the function defined by

$$f(z)=\int_0^1\int_0^1 g(r,s,z)\,dr\,ds$$

is a starlike function of the same order.

1. Introduction. Let $\mathcal{H}$ denote the class of all analytic functions f defined in the open unit disc $E=\{z:|z|<1\}$. For a positive integer n and $a\in\mathcal{C}$, define the classes of functions:

$$\mathcal{H}[a,n]=\{f\in\mathcal{H}:f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\cdots\},$$
$$\mathcal{A}_n=\{f\in\mathcal{H}:f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\cdots\},$$

with $\mathcal{A}_1=\mathcal{A}$. Denote by S the subclass of $\mathcal{A}$ consisting of univalent functions in E. Let $S^*(\beta)$, S^* and K denote the usual classes of starlike functions of order β $(0\le\beta<1)$, starlike functions and convex functions, respectively.

Let $f,g\in\mathcal{H}$, and let g be univalent in E. The function f is said to be subordinate to g (written $f(z)\prec g(z)$ or $f\prec g$) in E if $f(0)=g(0)$ and $f(E)\subset g(E)$.

In 2003, Fournier et al. [**1**] investigated the following differential inequality:

Let $0\le\alpha<2$. If $f\in\mathcal{A}$ satisfies

$$\left|zf''(z)-\alpha\left(\frac{f(z)}{z}-1\right)\right|<1-\frac{\alpha}{2},\quad z\in E,$$

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then $f \in S^*$.

The above result is an extension of results of Obradovic [**3**]. In this paper, we extend the above result to obtain a sufficient condition for starlikeness of order β. As a consequence, we construct new starlike functions of order β which can be expressed in terms of double integrals of some functions in the class $\mathcal{H}$.

2. Preliminary results. We shall need the following lemmas to prove our results.

Lemma 2.1. [**2,** page 71]. *Let h be a convex function with $h(0) = a$, and let* $\mathrm{Re}\,(\gamma) > 0$. *If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1}dt.$$

This result is sharp.

Lemma 2.2. [**2,** page 383]. *Let n be a positive integer and α real, with $0 \le \alpha < n$. Let $q \in \mathcal{H}$, with $q(0) = 0$, $q'(0) \neq 0$ and*

$$Re\frac{zq''(z)}{q'(z)} + 1 > \frac{\alpha}{n}. \tag{1}$$

If $p \in \mathcal{H}[0, n]$ satisfies

$$zp'(z) - \alpha p(z) \prec znq'(z) - \alpha q(z),$$

then $p(z) \prec q(z)$, and this result is sharp.

3. Main result.

Theorem 3.1. *Let $0 \le \alpha < n + 1$, and $0 \le \beta < 1$. If $f \in \mathcal{A}_n$ satisfies*

$$\left| zf''(z) - \alpha\left(\frac{f(z)}{z} - 1\right)\right| < \frac{(1-\beta)[n(n+1)-\alpha]}{(n+1-\beta)}, \quad z \in E, \tag{2}$$

then f is starlike of order β.

Proof. In terms of subordination, the inequality (2) can be written as

$$zf''(z) - \alpha\left(\frac{f(z)}{z} - 1\right) \prec \frac{(1-\beta)[n(n+1)-\alpha]}{(n+1-\beta)} z. \tag{3}$$

If we write

$$P(z) = f'(z) - \gamma\frac{f(z)}{z} = (1-\gamma) + (n+1-\gamma)a_{n+1}z^n + \cdots,$$

then $P \in \mathcal{H}[1-\gamma, n]$ where $\gamma(\gamma-1) = \alpha$, $1 \le \gamma < n+1$. Then, the subordination becomes

$$\gamma P(z) + zP'(z) \prec -\gamma(\gamma-1) + \frac{(1-\beta)[n(n+1)-\gamma(\gamma-1)]}{(n+1-\beta)} z,$$

or

$$P(z) + \frac{zP'(z)}{\gamma} \prec -(\gamma-1) + \frac{(n+\gamma)(n-\gamma+1)(1-\beta)}{(n+1-\beta)\gamma} z = h(z), \quad \text{(say)}.$$

Clearly, h is convex and $h(0) = P(0)$. Applying Lemma 2.1, we obtain

$$P(z) \prec \frac{\gamma}{nz^{\gamma/n}} \int_0^z \left\{ -(\gamma-1) + \frac{(n+\gamma)(n-\gamma+1)(1-\beta)}{(n+1-\beta)\gamma}\, t \right\} t^{\gamma/n-1} dt,$$

or equivalently,

$$f'(z) - \gamma\frac{f(z)}{z} \prec -(\gamma-1) + \frac{(1-\beta)(n-\gamma+1)}{n+1-\beta} z. \tag{4}$$

If we write

$$p(z) = \frac{f(z)}{z} - 1 = a_{n+1}z^n + a_{n+2}z^{n+1} + \cdots,$$

then $p \in \mathcal{H}[0, n]$. Writing

$$Q(z) = \frac{(1-\beta)}{(n+1-\beta)} z,$$

we see that Q is analytic in E and $Q(0) = 0$, $Q'(0) = (1-\beta)/(n+1-\beta) \neq 0$. Now, the subordination (4) can be written as

$$\begin{aligned} zp'(z) - (\gamma - 1)p(z) &\prec \frac{(1-\beta)(n-\gamma+1)}{n+1-\beta} z \\ &= znQ'(z) - (\gamma - 1)Q(z). \end{aligned} \tag{5}$$

Since $0 \leq \gamma - 1 < n$ and the function Q satisfies the criteria of Lemma 2.2, we obtain the subordination $p \prec Q$, i.e.,

$$\frac{f(z)}{z} - 1 \prec \frac{(1-\beta)}{(n+1-\beta)} z. \tag{6}$$

It follows from the subordination (4) that

$$\begin{aligned} \left| f'(z) - \gamma \frac{f(z)}{z} \right| &< (\gamma - 1) + \frac{(1-\beta)(n-\gamma+1)}{n+1-\beta} \\ &= \frac{n(\gamma - \beta)}{n - \beta + 1}, \end{aligned} \tag{7}$$

while subordination (6) implies that

$$\left| \frac{f(z)}{z} \right| > 1 - \frac{1-\beta}{(n-\beta+1)} = \frac{n}{n-\beta+1}. \tag{8}$$

Combining the above two inequalities, we get

$$\begin{aligned} \frac{n}{n-\beta+1} \left| \frac{zf'(z)}{f(z)} - \gamma \right| &< \left| \frac{f(z)}{z} \right| \left| \frac{zf'(z)}{f(z)} - \gamma \right| \\ &= \left| f'(z) - \gamma \frac{f(z)}{z} \right| < \frac{n(\gamma - \beta)}{n+1}, \end{aligned}$$

which implies

$$\left| \frac{zf'(z)}{f(z)} - \gamma \right| < \gamma - \beta, \quad z \in E.$$

It proves that f is starlike of order β. $\square$

Letting $n = 1$ and $\beta = 0$ in Theorem 3.1, we obtain the following result of Fournier et al. [**1**].

Corollary 3.2. *Let $0 \le \alpha < 2$. If $f \in \mathcal{A}$ satisfies*

$$\left|zf''(z) - \alpha\left(\frac{f(z)}{z} - 1\right)\right| < 1 - \frac{\alpha}{2}, \quad z \in E, \tag{9}$$

then $f \in S^*$.

4. Applications. As an application of Theorem 3.1 in the following result, we construct a function f which is starlike of order β in E.

Theorem 4.1. *If $g \in \mathcal{H}$ is such that*

$$|g(z)| \le \frac{(1-\beta)[n(n+1) - \gamma(\gamma-1)]}{(n+1-\beta)}, \quad z \in E,$$

for some $1 \le \gamma < n+1$ and $0 \le \beta < 1$. Then the function f given by

$$f(z) = z + z^{n+1} \int_0^1 \int_0^1 g(rsz) r^{n+\gamma-1} s^{n-\gamma} dr\, ds \tag{10}$$

is starlike of order β in E.

Proof. Let $f \in \mathcal{A}_n$ satisfy the differential equation

$$zf''(z) - \gamma(\gamma-1)\left(\frac{f(z)}{z} - 1\right) = z^n g(z). \tag{11}$$

Clearly,

$$\left|zf''(z) - \gamma(\gamma-1)\left(\frac{f(z)}{z} - 1\right)\right| < \frac{(1-\beta)[n(n+1) - \gamma(\gamma-1)]}{(n+1-\beta)}, \quad z \in E.$$

Equation (11) simplifies to

$$\begin{aligned} z^n g(z) &= z^{1-\gamma}\left(z^\gamma f''(z) - \gamma(\gamma-1) z^{\gamma-1}\left(\tfrac{f(z)}{z} - 1\right)\right) \\ &= z^{1-\gamma}\left(z^\gamma\left(f'(z) - \gamma\tfrac{f(z)}{z}\right)' \right.\\ &\quad \left. + \gamma z^{\gamma-1}\left(f'(z) - \gamma\tfrac{f(z)}{z}\right)\right) + \gamma(\gamma-1) \\ &= z^{1-\gamma}\left(z^\gamma\left(f'(z) - \gamma\tfrac{f(z)}{z}\right)\right)' + \gamma(\gamma-1). \end{aligned}$$

Thus,

$$z^{\gamma}\left(f'(z)-\gamma\frac{f(z)}{z}\right)=\int_0^z\left(\zeta^{n+\gamma-1}g(\zeta)-\gamma(\gamma-1)\zeta^{\gamma-1}\right)d\zeta.$$

Substituting $\zeta = rz$ in the above integral, we get

$$\left(f'(z)-\gamma\frac{f(z)}{z}\right)=\int_0^1 r^{n+\gamma-1}z^n g(rz)\,dr-(\gamma-1),$$

which further simplifies to

$$\begin{aligned}\int_0^1 r^{n+\gamma-1} \quad & z^n g(rz)\,dr\\ &=\left(f'(z)-\gamma\tfrac{f(z)}{z}\right)+(\gamma-1)\\ &=z^{\gamma}\left(z^{-\gamma}f'(z)-\gamma z^{-1-\gamma}f(z)\right)+(\gamma-1)\\ &=z^{\gamma}\left(z^{1-\gamma}\left(\tfrac{f(z)}{z}-1\right)'+(1-\gamma)z^{-\gamma}\left(\tfrac{f(z)}{z}-1\right)\right)\\ &=z^{\gamma}\left(z^{1-\gamma}\left(\tfrac{f(z)}{z}-1\right)\right)'.\end{aligned}$$

Thus,

$$z^{1-\gamma}\left(\frac{f(z)}{z}-1\right)=\int_0^z\zeta^{-\gamma}\left(\int_0^1 r^{n+\gamma-1}\zeta^n g(r\zeta)\,dr\right)d\zeta.$$

Again, substituting $\zeta = sz$ in the integral, we get

$$\left(\frac{f(z)}{z}-1\right)=z^n\int_0^1\int_0^1 g(rsz)r^{n+\gamma-1}s^{n-\gamma}dr\,ds,$$

or

$$f(z)=z+z^{n+1}\int_0^1\int_0^1 g(rsz)r^{n+\gamma-1}s^{n-\gamma}dr\,ds.$$

This completes the proof. □

Taking various permissible values of γ and n, we obtain several special cases of above result. However, we mention only one such result by taking $\gamma = 1$ and $n = 1$.

Corollary 4.2. *If $g \in \mathcal{H}$ and $|g(z)| < 2(1-\beta)/(2-\beta)$ for $z \in E$, then for some β, $0 \le \beta < 1$,*

$$f(z)=z+z^2\int_0^1\int_0^1 g(rsz)r\,dr\,ds\in S^*(\beta).$$

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