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PLANAR OPEN BOOK DECOMPOSITIONS OF 3-MANIFOLDS

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ABSTRACT. Due to Alexander, every closed oriented 3-manifold has an open book decomposition. In this note, we review two proofs of this theorem, one which constructs a planar open book decomposition. We then present an algorithmic way for determining the monodromy of this planar open book decomposition.

1. Introduction. An *open book decomposition* of a closed, oriented 3-manifold M is a pair (Σ, φ) where Σ is an oriented compact surface with boundary and φ is a diffeomorphism of Σ such that φ is the identity on a neighborhood of the boundary $\partial\Sigma$. The closed 3-manifold M is obtained as:

$$M = \Sigma \times [0,1]/\sim,$$

where $(\varphi(x), 0) \sim (x, 1)$ for $x \in \operatorname{Int}\Sigma$ and $(x,t) \sim (x,1)$ for $x \in \partial\Sigma$ and $t \in [0,1]$. The diffeomorphism φ is called the *monodromy* of the open book decomposition. The fiber surfaces $\Sigma \times t$ are called the *pages* and, identifying Σ with a page $\Sigma \times t'$, for some t', the link $B = \partial\Sigma$ is called the *binding*. The *genus* of the open book decomposition is defined as the genus of the page. In particular, *planar* open book decompositions are genus zero open book decompositions, that is, the pages are disks with punctures.

Existence of open book decompositions for 3-manifolds has been known for a long time. In 1923, Alexander showed that

Theorem 1 ([**1**]). *Every closed orientable* 3*-manifold has an open book decomposition.*

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Open book decompositions are a useful tool in 3-manifold theory. They naturally provide a Heegaard splitting for the 3-manifold where the Heegaard surface is formed by taking the closure of the union of two pages. The fact that the monodromy of an open book decomposition is an element in the mapping class group of the page provides deep connections between open book decompositions and geometric group theory. More recently, the study of open book decompositions has experienced an interesting growth after a theorem of Giroux. In [**7**], Giroux showed that isotopy classes of contact structures on a closed orientable 3-manifold M are in one-to-one correspondence with suitable equivalence classes of open book decompositions of M. Through this correspondence, open book decompositions are used to define invariants for contact structures, [**6, 12**], and to define invariants for knots in contact structures, [**3, 11**].

Among all open book decompositions of a 3-manifold, planar ones are easier to work with. This expository paper aims to provide a method of constructing planar open book decompositions for any given 3-manifold by a combination of surgery techniques. In Section 2, we review some proofs of Theorem 1. In Section 3, we construct explicit planar open book decompositions for 3-manifolds. Namely, we can determine the binding, the page and the monodromy of the constructed planar open book decomposition for M.

2. Two proofs of Theorem 1. The following proof is from Rolfsen's book, [**13,** Theorem 10K1]. It shows the existence of planar open book decompositions for 3-manifolds. However, in this case, the monodromy of the resulting open book is not clear.

First proof of Theorem 1. By Lickorish, [**10,** Theorem 2], and Wallace, [**14,** Theorem 6], every closed oriented 3-manifold M can be obtained from S^3 by a ± 1-surgery along a link L_M of unknots. Given L_M, there is a complex of annuli as shown in Figure 1 onto which L_M may be isotoped so that each unknot component L_i of the link L_M is an S^1 factor of one of the annuli. The extra curve drawn in Figure 1 is an unknot U in S^3 such that L_M is braided about U and each component L_i links U exactly once. In fact, with U as a braid axis, L_M is a pure braid.

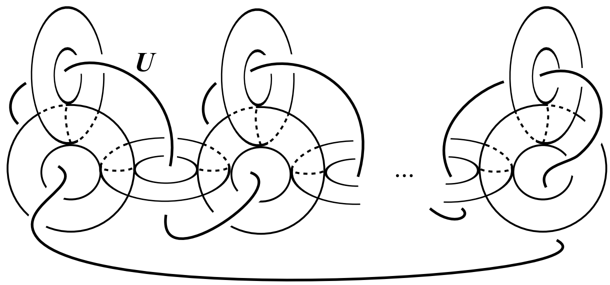

FIGURE 1. Each component of L_M lies on one of these annuli, and L_M is braided about the unknot U.

To get an open book decomposition of M, we use the natural open book decomposition $(D, \varphi = Id)$ of S^3 where the binding is the unknot U, the page is the disk D and the monodromy φ is the identity map. Note that, since U links each component L_i of L_M exactly once, each L_i punctures every disk page, D, transversely once. After performing ±1-surgeries, the boundary of each puncture (which is a meridional curve to each component L_i) becomes a longitude of surgery dual in M. The page of the open book decomposition of M comes from the punctured disk pages after performing ±1-surgeries. Namely, the pages are obtained after gluing an annulus to each puncture on each disk page where the annulus is bounded by L_i and the longitudinal curve on boundary of the neighborhood of L_i. Therefore, the pages are planar and the binding of the open book decomposition of M consists of U and L_M. □

The following proof can be found in [**5,** Theorem 2.11].

Second proof of Theorem 1. By Alexander, [**2**], every closed oriented 3-manifold M is a branched cover of S^3 branching along a link L_M. We may braid the link L_M about an unknot U in S^3, [**1**]. More precisely, we can isotope L_M in the complement of an unknot U in S^3 such that L_M is the closure of a braid in $S^3 \setminus U = D^2 \times S^1$ and L_M meets each disk $D^2 \times \{t\}$ with $t \in S^1$ transversally.

We can get an open book decomposition of M by using the branched covering map $p : M \to S^3$ branching along L_M, and by using the fibering $\varphi_U : S^3 \setminus U \to S^1$ of the complement of U in S^3. By letting $B = p^{-1}(U)$, we get an open book decomposition of M from the fibering $\varphi_U \circ p : M \setminus B \to S^1$ of the complement of B in M. $\square$

Remark 2. The construction using branch coverings usually results in open book decompositions with high genus. For example, take a 3-manifold M which is a 2-fold branched cover of S^3 branching along the closure of an n-braid. According to the second proof of Theorem 1, a page of an open book decomposition of M is obtained by taking the 2-fold branched cover of the disk over n points. For even $n = 2k$, the page is a genus $(k-1)$ surface with two boundary components and, for odd $n = 2k+1$, the page is a genus k surface with only one boundary component. Note that not all 3-manifolds can be obtained as a 2-fold branched cover of S^3; for example, T^3 is not a 2-fold branched cover of S^3, [**9,** Theorem 1].

3. The algorithm for planar open book decompositions. We need the following lemma to construct planar open book decompositions. For the proof of this lemma, see [**5**].

Lemma 3. *Let M be a closed oriented 3-manifold, and let (Σ, φ) be an open book decomposition for M.*

(i) *Let K be a knot in M such that $K = \{x\} \times [0,1]/\sim$ in M for some point x in the interior of Σ of φ. Let D_x denote an open disk which is a neighborhood of x on Σ such that $\varphi \mid_{\{D_x\}} = id$. Then 0-surgery along K gives a new manifold with an open book decomposition having the surgery dual of K as one of the binding components. The surface $\Sigma' = \Sigma - \{D_x\}$ is the page and the map $\varphi' = \varphi \mid_{\Sigma'}$ is the monodromy of the new open book decomposition.*

(ii) *Let K be a knot in M sitting on a page Σ of the open book decomposition (Σ, φ). Then ± 1-surgery along K with respect to the page framing gives a new manifold with an open book decomposition $(\Sigma, \varphi \circ t_K^{\mp 1})$ where t_K^{+1}/ t_K^{-1} denotes right-/ left-handed Dehn twists along the knot, K.*

As we mentioned in the first proof of Theorem 1 above by Lickorish, [**10**], and by Wallace, [**14**], every closed oriented 3-manifold M can be obtained by ±1-surgery on a link L_M of n unknots in S^3. The surgery link L_M may be chosen from a special class of links in S^3 as in Figure 1. We know that the extra curve U in Figure 1 is an unknot in S^3 such that L_M is braided about U with the property that each component of L_M links U exactly once. Therefore, we can present the link L_M as the closure of an n-braid as in Figure 2. Note that, since we have a link of n unknots that is also an n-braid, it is a pure braid. A *pure braid presentation* of the link L_M is obtained by decomposing the pure braid in terms of standard generators of the pure braid group. A generating set of braids A_{ij}, $1 \leq i < j \leq n$, for pure braid group on n-strands is presented in Figure 3.

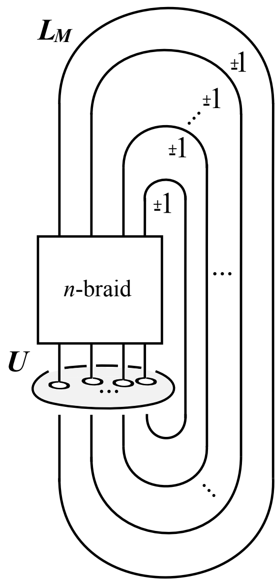

FIGURE 2. The surgery link L_M.

In our construction, we mainly use the blowing up operation. The simplest form of the blowing up operation consists of adding a ±1-framed unknot to a diagram as in Figure 4. In the first case, in Figure 4 (a), the framing of the strand will increase by ±1. In the second case, in Figure 4 (b), if the strands belong to different components of the link diagram, their framings increase by ±1 each;

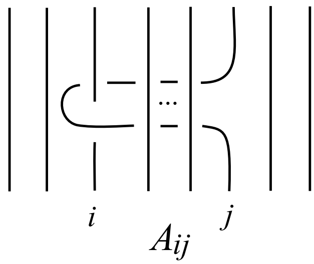

FIGURE 3. A generating set of braids A_{ij}, $1 \leq i < j \leq n$, for pure braid group on n-strands.

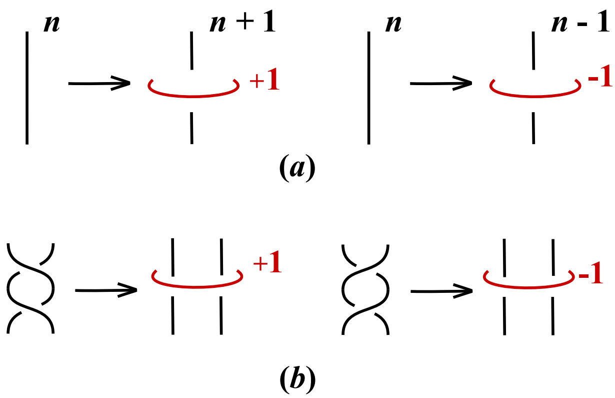

FIGURE 4. Blowing up operation to remove twists.

if the strands belong to the same component, the framing changes by either 0 or ± 4. For the general case of blowing up operation and how blowing up changes the framings, see [**8**].

Theorem 4. *Let M be a closed, oriented 3-manifold given by ± 1-surgery on a link L_M of n unknots in S^3 that is the closure of a pure braid as in Figure 2. Then, M has a planar open book decomposition whose page is a disk with n punctures and whose monodromy presentation can be obtained from a pure braid presentation of the link L_M and its surgery coefficients.*

Proof. We start with a pure braid presentation of the surgery link L_M as in Figure 2. Note that the unknot U in Figure 2 bounds a disk in S^3, and each component of the link L_M punctures this disk transversely once. We will construct a planar open book for M by using Lemma 3, and by using the open book decomposition $(D, \varphi = Id)$ of S^3 where the binding is the unknot $U = \partial D$.

First, we remove each linking between the components of L_M by blowing up. This is possible since the generating set A_{ij} differs from the trivial braid by a single clasp, and these blow ups will remove the clasp. In this case, adding a ± 1-framed unknot to a diagram will add ± 1 to the framing of each corresponding component of L_M. Note that each resulting ± 1-framed unknots can be isotoped to sit on the punctured disk pages that U bounds.

Next, we continue blowing up to arrange the framing coefficient of each component of L_M to be zero. Note that we have obtained a surgery description of M as a link of a 0-framed trivial n-braid puncturing the pages transversely once and a collection of ± 1-framed unknots lying on different pages. After performing surgeries by Lemma 3, we obtain a planar open book decomposition for M where the pages are disk with n-punctures and the monodromy is a product of negative/positive Dehn twists along the ± 1-framed surgery curves on the punctured disk that U bounds. □

Let us now give an illustrative example.

Example 5. The Poincaré homology sphere $\Sigma(2,3,5)$ can be given by a surgery on the Borromean link as in Figure 5.

We will construct a planar open book for $\Sigma(2,3,5)$. First, we present the Borromean link as a pure 3-braid as in Figure 6. We decompose this pure braid in terms of standard generators of the pure braid group on 3-strands. Next, we remove each linking between the components of the Borromean link by blowing up, and we continue blowing up to arrange the framing of each component of the Borromean link to be 0.

Now, the unknot U in Figure 6 bounds a disk in S^3 and each 0-framed unknot punctures this disk transversely once. Moreover, the ± 1-framed unknots in Figure 6 may be isotoped onto the punctured disk pages. By Lemma 3, after performing surgeries, we obtain a planar open book decomposition (Σ, φ) for $\Sigma(2,3,5)$ where the page Σ is a disk

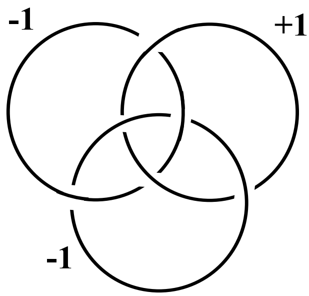

FIGURE 5. Surgery on a Borromean link giving the Poincaré homology sphere.

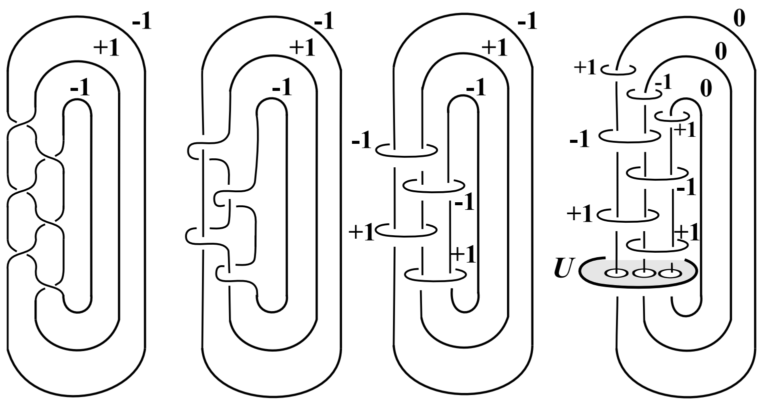

FIGURE 6. Pure braid representation of Borromean link and a way of resolving the linking.

with 3 punctures. Each 0-framed unknot becomes a binding component δ_1, δ_2 and δ_3, respectively, where we set the notation for binding components from inner component to outer component. By Lemma 3 again, each ±1-framed surgery curve contributes a negative/positive Dehn twist to the monodromy of the starting open book. In this case, the monodromy of the starting open book is the identity. Hence, the monodromy of the open book for $\Sigma(2,3,5)$ is $\varphi = t_\alpha^{-1} t_\beta^{-1} t_\alpha t_\beta t_{\delta_1}^{-1} t_{\delta_2} t_{\delta_3}^{-1}$ where α is the simple closed curve enclosing the holes δ_1, δ_2 and β is the simple closed curve enclosing the holes δ_2, δ_3 as in Figure 7. Note

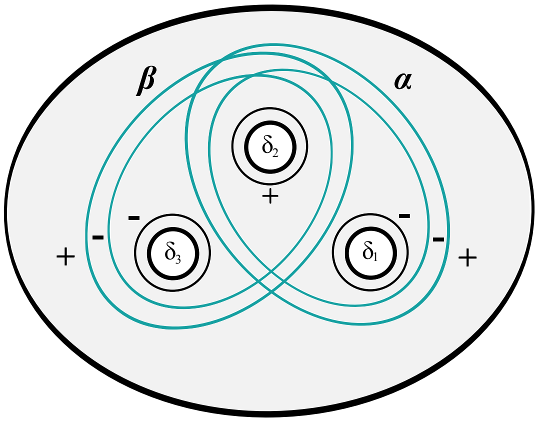

FIGURE 7. A planar open book for the Poincaré homology sphere, where the page Σ is a disk with 3 punctures and the monodromy is $\varphi = t_\alpha^{-1} t_\beta^{-1} t_\alpha t_\beta t_{\delta_1}^{-1} t_{\delta_2} t_{\delta_3}^{-1}$.

that there are Dehn twists with intersecting curves in this monodromy factorization, and hence these Dehn twists do not commute. Therefore, an order of the Dehn twists should be carefully specified.

For more information on this surgery technique, see [**4**].

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