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COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF ρ^*-MIXING RANDOM FIELDS

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ABSTRACT. In this paper we generalize the complete convergence for ρ^*-mixing random fields given by Kuczmaszewska et al. **[7]** to the case of weight sums.

1. Introduction. Let $\mathbb{Z}_+^d$ $(d \geq 2)$ be the set of positive integer lattice points. For $d = 1$, we use the notation $\mathbb{Z}_+$ instead of $\mathbb{Z}_+^1$. For a fixed $d \in \mathbb{Z}_+$, set $\mathbb{Z}_+^d = \{\mathbf{n} = (n_1, n_2, \ldots, n_d) : n_i \in \mathbb{Z}_+, i = 1, 2, \ldots, d\}$ with coordinatewise partial order, $\leq$, i.e., for $\mathbf{m} = (m_1, m_2, \ldots, m_d), \mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}_+^d, \mathbf{m} \leq \mathbf{n}$ if and only if $m_i \leq n_i, i = 1, 2, \ldots, d$. We also use $|\mathbf{n}|$ for $|\mathbf{n}| = \prod_{i=1}^d n_i$, $|\mathbf{n}| \to \infty$ is to be interpreted as $n_i \to \infty$ for $i = 1, 2, \ldots, d$, and $|\mathbf{n}| \to \infty$ is equivalent to $\max\{n_1, n_2, \ldots, n_d\} \to \infty$.

Peligrad and Gut **[10]** investigated a class of dependent random fields based on an interlaced condition which uses the maximal coefficient of correlation, and they defined the condition in the following way:

Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a random field, let $S \subset Z_+^d$, and define

$$\begin{aligned}\mathcal{F}_s &= \sigma(X_{\mathbf{i}}, \mathbf{i} \in S)\\ &= \text{the } \sigma\text{-field generated by the random variables } \{X_{\mathbf{i}}, \mathbf{i} \in S \subset \mathbb{Z}_+^d\}\end{aligned}$$

and

$$\begin{aligned}\rho^*(k) &= \sup \operatorname{corr}(X, Y)\\ &= \sup_{S,T} \left(\sup_{X \in L^2(\mathcal{F}_S), Y \in L^2(\mathcal{F}_T)} \frac{|\operatorname{Cov}(X, Y)|}{(\operatorname{Var} X \operatorname{Var} Y)^{1/2}} \right),\end{aligned}$$

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where the supremum is taken over all $S, T \subset \mathbb{Z}_+^d$ with dist $(S,T) \geq k$, and all $X \in L^2(\mathcal{F}_S, Y \in L^2(\mathcal{F}_T)$ and

$$\text{dist}\,(S,T)^2 = \inf_{x\in S, y\in T} \|x-y\|^2 = \inf_{x\in S, y\in T} \sum_{i=1}^{d} (x_i - y_i)^2,$$

i.e., Euclidean distance. Various limit properties under condition $\rho^*(k) \to 0$ were studied by Bradley **[1, 2]** and Miller **[8]**. Bryc and Smolenski **[3]** and Peligrad **[9]** pointed out the importance of condition

$$\lim_{k\to\infty} \rho^*(k) < 1 \tag{1.1}$$

in estimating the moments of partial sums or of maxima of partial sums. Let us also note that, since $0 \leq \cdots \leq \rho^*(n) \leq \rho^*(n-1) \leq \cdots \leq \rho^*(1) \leq 1$, (1.1) is equivalent to

$$\rho^*(N) < 1 \quad \text{for some } N \geq 1. \tag{1.2}$$

Definition 1.1. A random field $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ is said to be ρ^*-mixing if (1.1) holds.

The ρ^*-mixing random variables were investigated by Bryc and Smolenski **[3]** (moment inequalities of partial sums), Bradley **[1, 2]** (equivalent mixing conditions and various limit properties), Peligrad **[9]** (a moment inequality of maximal partial sums for sequences). Peligrad and Gut **[10]** (a moment inequality of maximal partial sums for fields and almost sure results of Marcinkiewicz-Zygmund type).

Kuczmaszewska and Lagodowski **[7]** proved the convergence rate in the strong law of large numbers for the ρ^*-mixing random field as follows:

Theorem 1.2. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a ρ^*-mixing random field. Let $\alpha p > 1$, $\alpha > 1/2$ and, for some $q \geq 2$,*

(i) $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} \sum_{\mathbf{i}\leq\mathbf{n}} P(|X_{\mathbf{i}}| > |\mathbf{n}|^\alpha) < \infty$,
(ii) $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha(p-q)-2} \sum_{\mathbf{i}\leq\mathbf{n}} E(|X_{\mathbf{i}}|^q I[|X_{\mathbf{i}}| \leq |\mathbf{n}|^\alpha]) < \infty$,
(iii) $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha(p-q)-2} (\log_2 |\mathbf{n}|)^{qd} (\sum_{\mathbf{i}\leq\mathbf{n}} E(X_{\mathbf{i}}^2 I[|X_{\mathbf{i}}| \leq |\mathbf{n}|^\alpha]))^{q/2} < \infty$,
(iv) $\max_{\mathbf{j}\leq\mathbf{n}} |\sum_{\mathbf{i}\leq\mathbf{j}} E(X_{\mathbf{i}} I[|X_{\mathbf{i}}| \leq |\mathbf{n}|^\alpha])| = o(|\mathbf{n}|^\alpha)$.

Then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} P\left\{\max_{\mathbf{j}\le\mathbf{n}} |S_{\mathbf{j}}| > \epsilon|\mathbf{n}|^{\alpha}\right\} < \infty$$

for all $\epsilon > 0$.

Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a field of real random variables, and let $\{a_{\mathbf{n},\mathbf{k}}, \mathbf{n} \in \mathbb{Z}_+^d, \mathbf{k} \in \mathbb{Z}_+^d,\ \mathbf{k} \le \mathbf{n}\}$ be an array of real numbers. The weighted sums $\sum_{\mathbf{k}\le\mathbf{n}} a_{\mathbf{n},\mathbf{k}} X_{\mathbf{k}}$ can play an important role in various applied and theoretical problems, such as those of the least squares estimators (see Kafles and Bhaskara Rao [**5**]) and M-estimates (see Rao and Chao [**12**]) in linear models, the nonparametric regression estimators(see Priestley and Chao [**11**]), the design regression estimators (see Gu, Roussas and Tran [**4**]), etc. So the study of the limiting behavior of weighted sums is very important and significant.

The aim of this paper is to give a result concerning complete convergence of weighted sums $\sum_{\mathbf{i}\le\mathbf{n}} a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}$, where $\{a_{\mathbf{n},\mathbf{i}}, \mathbf{n} \in \mathbb{Z}_+^d, \mathbf{i} \le \mathbf{n}\}$ is an array of real numbers and $\{X_{\mathbf{i}}, \mathbf{i} \in \mathbb{Z}_+^d\}$ is a field of ρ^*-mixing random variables.

2. Main result. We start this section with the following lemma which is useful in the proof of the main result.

Lemma 2.1. [**10**] *Let* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ *be a field of random variables satisfying* (1.1), $EX_{\mathbf{n}} = 0$ *and* $E|X_{\mathbf{n}}|^q < \infty$ *for* $q \ge 2$ *and* $\mathbf{n} \in \mathbb{Z}_+^d$. *Then there exist positive constants* $K_1 = K_1(q, \rho^*(N), d)$ *and* $K_2 = K_2(q, \rho^*(N), d)$ *such that*

$$\begin{aligned}(2.1)\quad & E\max_{\mathbf{k}\le\mathbf{n}} |S_{\mathbf{k}}|^q \\ & \le K_1\left\{\sum_{\mathbf{k}\le\mathbf{n}} E|X_{\mathbf{k}}|^q + (\log_2|\mathbf{n}|)^{qd}\left(\sum_{\mathbf{k}\le\mathbf{n}} EX_{\mathbf{k}}^2\right)^{q/2}\right\} \text{ for all } \mathbf{n} \in \mathbb{Z}_+^d,\end{aligned}$$

and

$$(2.2)\qquad E|S_{\mathbf{k}}|^q \le K_2 E\left(\sum_{\mathbf{k}\le\mathbf{n}} X_{\mathbf{k}}^2\right)^{q/2}\Bigg\} \quad \text{for all } \mathbf{n} \in \mathbb{Z}_+^d.$$

We are going to generalize the result given by Kuczmaszewska and Lagodowski [**7,** Theorem 3.1] to the case of weighted sums.

Theorem 2.2. *Let* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ *be a field of* ρ^**-mixing random variables with* $EX_{\mathbf{n}} = 0$ *and* $\{a_{\mathbf{n},\mathbf{i}}, \mathbf{n} \in \mathbb{Z}_+^d, \mathbf{i} \leq \mathbf{n}\}$ *an array of real numbers. Let* $\alpha p > 1, \alpha > 1/2$ *and, for some* $q \geq 2$,

(a) $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} \sum_{\mathbf{i}\leq\mathbf{n}} P(|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| > \epsilon|\mathbf{n}|^{\alpha}) < \infty$,

(b) $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha(p-q)-2} \sum_{\mathbf{i}\leq\mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^q E(|X_{\mathbf{i}}|^q I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \leq \epsilon|\mathbf{n}|^{\alpha}]) < \infty$,

(c) $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha(p-q)-2} (\log_2|\mathbf{n}|)^{qd} \left(\sum_{\mathbf{i}\leq\mathbf{n}} a_{\mathbf{n},\mathbf{i}}^2 E(X_{\mathbf{i}}^2 I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \leq \epsilon|\mathbf{n}|^{\alpha}])\right)^{q/2} < \infty$.

Then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} P\left\{\max_{\mathbf{j}\leq\mathbf{n}} \left|\sum_{\mathbf{i}\leq\mathbf{j}} (a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}} - a_{\mathbf{n},\mathbf{i}} EX_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \leq \epsilon|\mathbf{n}|^{\alpha}])\right| > \epsilon|\mathbf{n}|^{\alpha}\right\} < \infty, \tag{2.3}$$

for all $\epsilon > 0$.

Proof. Let, for $\mathbf{i} \leq \mathbf{n}$,

$$X'_{\mathbf{n},\mathbf{i}} = X_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \leq \epsilon|\mathbf{n}|^{\alpha}],$$
$$Y_{\mathbf{n},\mathbf{i}} = a_{\mathbf{n},\mathbf{i}} X'_{\mathbf{n},\mathbf{i}} - a_{\mathbf{n},\mathbf{i}} EX'_{\mathbf{n},\mathbf{i}} \quad \text{and} \quad S'_{\mathbf{n},\mathbf{k}} = \sum_{\mathbf{i}\leq\mathbf{k}} Y_{\mathbf{n},\mathbf{i}}.$$

Let us notice that if the series $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2}$ is convergent, then (2.3) automatically holds. Therefore, we consider only the case such that $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2}$ is divergent.

As in the proof of Theorem 2.1 in Kuczmaszewska [**6**], by (a), for sufficient large $|\mathbf{n}|$, we have

$$P\left\{\max_{\mathbf{j}\le\mathbf{n}}\left|\sum_{\mathbf{i}\le\mathbf{j}}(a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}-a_{\mathbf{n},\mathbf{i}}EX_{\mathbf{i}}I[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}|\le\epsilon|\mathbf{n}|^{\alpha}])\right|>\epsilon|\mathbf{n}|^{\alpha}\right\} \le\sum_{\mathbf{i}\le\mathbf{n}}P(|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}|>\epsilon|\mathbf{n}|^{\alpha})+\epsilon^{-q}|\mathbf{n}|^{-\alpha q}E\left(\max_{\mathbf{i}\le\mathbf{n}}|S'_{\mathbf{n},\mathbf{i}}|\right)^{q}. \tag{2.4}$$

Using the C_r inequality, we can estimate $E|Y_{\mathbf{n},\mathbf{i}}|^r$ in the following way:

$E|Y_{\mathbf{n},\mathbf{i}}|^r\le C(E|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{n},\mathbf{i}}|^rI[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{n},\mathbf{i}}|\le\epsilon|\mathbf{n}|^{\alpha}]+P(|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{n},\mathbf{i}}|\ge\epsilon|\mathbf{n}|^{\alpha})).$

Thus, by the above estimations, (2.4), C_r inequality and Lemma 2.1, we get

$$\begin{aligned}&P\left\{\max_{\mathbf{j}\le\mathbf{n}}\left|\sum_{\mathbf{i}\le\mathbf{j}}(a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}-a_{\mathbf{n},\mathbf{i}}EX_{\mathbf{i}}I[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}|\le\epsilon|\mathbf{n}|^{\alpha}])\right|>\epsilon|\mathbf{n}|^{\alpha}\right\}\\ &\le C\Bigg[\sum_{\mathbf{i}\le\mathbf{n}}P(|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}|>\epsilon|\mathbf{n}|^{\alpha}) \qquad (2.5)\\ &+|\mathbf{n}|^{-\alpha q}\Bigg\{\sum_{\mathbf{i}\le\mathbf{n}}|a_{\mathbf{n},\mathbf{i}}|^qE|X_{\mathbf{i}}|^qI[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}|\le\epsilon|\mathbf{n}|^{\alpha}]\\ &+(\log_2|\mathbf{n}|)^{qd}\left(\sum_{\mathbf{i}\le\mathbf{n}}a_{\mathbf{n},\mathbf{i}}^2EX_{\mathbf{i}}^2I[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}|<\epsilon|\mathbf{n}|^{\alpha}]\right)^{q/2}\Bigg\}\Bigg].\end{aligned}$$

Hence, from (a)–(c) and (2.5), the result (2.3) follows. □

Remark 1. Let us observe that, in the case $q=2$, the above assumptions (b) and (c) reduce to

(b′) $\sum_{\mathbf{n}}|\mathbf{n}|^{\alpha(p-2)-2}(\log_2|\mathbf{n}|)^{2d}\sum_{\mathbf{i}\le\mathbf{n}}a_{\mathbf{n},\mathbf{i}}^2E(X_{\mathbf{i}}^2I[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}|\le\epsilon|\mathbf{n}|^{\alpha}])<\infty.$

Theorem 2.3. *Let* $\{X_{\mathbf{n}},\mathbf{n}\in\mathbb{Z}_+^d\}$ *be a field of random variables satisfying* (1.1) *and* $EX_{\mathbf{n}}=0$. *Let* $\{a_{\mathbf{n},\mathbf{i}},\mathbf{n}\in\mathbb{Z}_+^d,\mathbf{i}\le\mathbf{n}\}$ *be an array of weights. If, for* $\alpha p>1,\alpha>1/2$ *and, for some* $q\ge2$ *Theorem* 2.2 (a)–(c) *and*

(d) $$\max_{\mathbf{j}\le\mathbf{n}}\left|\sum_{\mathbf{i}\le\mathbf{j}}a_{\mathbf{n},\mathbf{i}}E(X_{\mathbf{i}}I[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}|\le\epsilon|\mathbf{n}|^{\alpha}])\right|=o(|\mathbf{n}|^{\alpha})$$

hold. Then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} P\left\{ \max_{\mathbf{j}\le\mathbf{n}} \left| \sum_{\mathbf{i}\le\mathbf{j}} a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}} \right| > \epsilon |\mathbf{n}|^{\alpha} \right\} < \infty \quad \text{for all } \epsilon > 0. \tag{2.6}$$

Proof. Define $X'_{\mathbf{n},\mathbf{i}}$, $Y_{\mathbf{n},\mathbf{i}}$ and $S'_{\mathbf{n},\mathbf{i}}$ as in Theorem 2.2. Noting that $EX_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le \epsilon |\mathbf{n}|^{\alpha}] = -EX_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| > \epsilon |\mathbf{n}|^{\alpha}]$ in view of the fact that $EX_{\mathbf{i}} = 0$, we have

$$\begin{aligned}
&\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} P\left\{ \max_{\mathbf{j}\le\mathbf{n}} \left| \sum_{\mathbf{i}\le\mathbf{j}} a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}} \right| > \epsilon |\mathbf{n}|^{\alpha} \right\} \\
\le\ & \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} P\left\{ \max_{\mathbf{i}\le\mathbf{n}} |a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha} \right\} \\
&\quad + \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} P\left\{ \max_{\mathbf{j}\le\mathbf{n}} \left| \sum_{\mathbf{i}\le\mathbf{j}} a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le \epsilon |\mathbf{n}|^{\alpha}] \right| > \epsilon |\mathbf{n}|^{\alpha} \right\} \\
\le\ & \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} \sum_{\mathbf{i}\le\mathbf{n}} P\{|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha}\} \\
&\quad + \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} P\left\{ \max_{\mathbf{j}\le\mathbf{n}} |S'_{\mathbf{n},\mathbf{j}}| > \epsilon |\mathbf{n}|^{\alpha} \right. \\
&\qquad\qquad \left. - \max_{\mathbf{j}\le\mathbf{n}} \left| \sum_{\mathbf{i}\le\mathbf{j}} a_{\mathbf{n},\mathbf{i}} E(X_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le \epsilon |\mathbf{n}|^{\alpha}]) \right| \right\}
\end{aligned}$$

(see the proof of Theorem 5 in **[10]**). □

In this case the first sum of the right-hand side is finite by Theorem 2.2 (a). Because of (d), we conclude that it remains to show that

$$I = \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} P\{\max_{\mathbf{j}\le\mathbf{n}} |S'_{\mathbf{n},\mathbf{j}}| > \epsilon |\mathbf{n}|^{\alpha}\} < \infty \quad \text{for all } \epsilon > 0.$$

By (2.1), C_r inequality and Chebyshev's inequality, we can estimate

$$\begin{aligned} I &\leq C \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2 - \alpha q} \Bigg\{ \sum_{\mathbf{i} \leq \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^q E(|X_{\mathbf{i}}|^q I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \leq \epsilon |\mathbf{n}|^\alpha]) \\ &\qquad + (\log_2 |\mathbf{n}|)^{qd} \bigg(\sum_{\mathbf{i} \leq \mathbf{n}} a_{\mathbf{n},\mathbf{i}}^2 E(X_{\mathbf{i}}^2 I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \leq \epsilon |\mathbf{n}|^\alpha]) \bigg)^{q/2} \Bigg\} \\ &= I_1 + I_2. \end{aligned}$$

It is clear that $I_1 < \infty$ by (b) and $I_2 < \infty$ by (c). Hence, $I < \infty$ and the proof is complete.

Corollary 2.4. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a field of ρ^*-mixing random variables with $EX_{\mathbf{n}} = 0$ and $E|X_{\mathbf{n}}|^q < \infty$ for $q \geq 2$ and for all $\mathbf{n} \in \mathbb{Z}_+^d$. Let $\{a_{\mathbf{n},\mathbf{i}}, \mathbf{n} \in \mathbb{Z}_+^d, \mathbf{i} \leq \mathbf{n}\}$ be a field of real numbers. Let $\alpha p > 1, \alpha > 1/2$. Assume that, for some field $\{\lambda_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ with $0 < \lambda_{\mathbf{n}} \leq 1$. If*

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} |\mathbf{n}|^{-\alpha(1+\lambda_{\mathbf{n}})} \sum_{\mathbf{i} \leq \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} E|X_{\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} < \infty \tag{2.7}$$

then, for all $\epsilon > 0$,

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} P\bigg(\max_{\mathbf{j} \leq \mathbf{n}} \bigg| \sum_{\mathbf{i} \leq \mathbf{j}} a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}} \bigg| > \epsilon |\mathbf{n}|^\alpha \bigg) < \infty. \tag{2.8}$$

Proof. First, note that $E|X_{\mathbf{n}}|^{1+\lambda_{\mathbf{n}}} < \infty$ since $q \geq 1 + \lambda_{\mathbf{n}} > 1$. If $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} < \infty$, then (2.8) automatically holds. Hence, we consider only the case $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} = \infty$. It follows from (2.7) that

$$|\mathbf{n}|^{-\alpha(1+\lambda_{\mathbf{n}})} \sum_{\mathbf{i} \leq \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} E|X_{\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} < 1. \tag{2.9}$$

By assumption (2.7),

$$\begin{aligned} &\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} \sum_{\mathbf{i} \leq \mathbf{n}} P(|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \geq \epsilon |\mathbf{n}|^\alpha) \\ &\quad \leq \epsilon^{-1-\lambda_n} \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} |\mathbf{n}|^{-\alpha(1+\lambda_{\mathbf{n}})} \sum_{\mathbf{i} \leq \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} E|X_{\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} < \infty, \end{aligned} \tag{2.10}$$

and, for $q \geq 2$,

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2}|\mathbf{n}|^{-\alpha q}\sum_{\mathbf{i}\leq\mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^q E|X_{\mathbf{i}}|^q I[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}| \leq \epsilon|\mathbf{n}|^{\alpha}] \leq \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2-\alpha(1+\lambda_{\mathbf{n}})}\sum_{\mathbf{i}\leq\mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} E|X_{\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} < \infty. \tag{2.11}$$

By (2.7) and (2.9), we estimate

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2}|\mathbf{n}|^{-\alpha q}\left(\sum_{\mathbf{i}\leq\mathbf{n}} a^2_{\mathbf{n},\mathbf{i}} EX^2_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}| < \epsilon|\mathbf{n}|^{\alpha}]\right)^{q/2} \leq \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2-\alpha(1+\lambda_{\mathbf{n}})q/2}\left(\sum_{\mathbf{i}\leq\mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} E|X_{\mathbf{i}}|^{1+\lambda_{\mathbf{n}}}\right)^{q/2} < \infty. \tag{2.12}$$

Hence (2.10)–(2.12) satisfy (a), (b) and (c), respectively.

Finally, we have

$$\begin{aligned}
&|\mathbf{n}|^{-\alpha}\max_{\mathbf{j}\leq\mathbf{n}}\left|\sum_{\mathbf{i}\leq\mathbf{j}} a_{\mathbf{n},\mathbf{i}} EX_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}| \leq \epsilon|\mathbf{n}|^{\alpha}]\right| \\
&\quad = |\mathbf{n}|^{-\alpha}\max_{\mathbf{j}\leq\mathbf{n}}\left|\sum_{\mathbf{i}\leq\mathbf{j}} a_{\mathbf{n},\mathbf{i}} EX_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}| > \epsilon|\mathbf{n}|^{\alpha}]\right| \\
&\quad \leq |\mathbf{n}|^{-\alpha}\max_{\mathbf{j}\leq\mathbf{n}}\sum_{\mathbf{i}\leq\mathbf{j}} |a_{\mathbf{n},\mathbf{i}}| E|X_{\mathbf{i}}| I[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}| > \epsilon|\mathbf{n}|^{\alpha}] \\
&\quad = |\mathbf{n}|^{-\alpha(1+\lambda_{\mathbf{n}})}\sum_{\mathbf{i}\leq\mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} E|X_{\mathbf{i}}|^{1+\lambda_n} \to 0 \quad \text{as } |\mathbf{n}| \to \infty,
\end{aligned}$$

which satisfies condition (d). Hence, the proof is completed. □

Corollary 2.5. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^d_+\}$ be a field of ρ^*-mixing random variables satisfying* (1.1), *$EX_{\mathbf{n}} = 0$ and $E|X_{\mathbf{n}}|^p < \infty$ for $1 < p \leq 2$. Let $\{a_{\mathbf{n},\mathbf{i}}, \mathbf{n} \in \mathbb{Z}^d_+, \mathbf{i} \leq \mathbf{n}$ be a field of real numbers satisfying*

$$\sum_{\mathbf{i}\leq\mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^p E|X_{\mathbf{i}}|^p = O(|\mathbf{n}|^{\delta}) \tag{2.13}$$

for some $0 < \delta < 1$. *Then, for any* $\epsilon > 0$, $\alpha > 1/2$ *and* $\alpha p > 1$

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} P\Big(\max_{\mathbf{j}\le\mathbf{n}} |\sum_{\mathbf{i}\le\mathbf{j}} a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| > \epsilon|\mathbf{n}|^{\alpha}\Big) < \infty. \tag{2.14}$$

Proof. Let $q = 2/\delta$. By (2.13), and the Chebyshev inequality, we have

$$\begin{aligned} (2.15)\quad & \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} \sum_{\mathbf{i}\le\mathbf{n}} P(|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| > \epsilon|\mathbf{n}|^{\alpha}) \\ & \le \epsilon^{-p} \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-2} \sum_{\mathbf{i}\le\mathbf{n}} \frac{|a_{\mathbf{n},\mathbf{i}}|^p E|X_{\mathbf{i}}|^p}{|\mathbf{n}|^{\alpha p}} \le C \sum_{\mathbf{n}} |\mathbf{n}|^{-2+\delta} < \infty, \end{aligned}$$

$$\begin{aligned} (2.16)\quad & \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha(p-q)-2} \sum_{\mathbf{i}\le\mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^q E|X_{\mathbf{i}}|^q I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le \epsilon|\mathbf{n}|^{\alpha}] \\ & \le C \sum_{\mathbf{n}} |\mathbf{n}|^{-2} \sum_{\mathbf{i}\le\mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^p E|X_{\mathbf{i}}|^p \le C \sum_{\mathbf{n}} |\mathbf{n}|^{-2+\delta} < \infty, \end{aligned}$$

and, for some $q \ge 2$,

(2.17)

$$\begin{aligned} & \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha(p-q)-2} (\log_2 |\mathbf{n}|)^{qd} \Big(\sum_{\mathbf{i}\le\mathbf{n}} a_{\mathbf{n},\mathbf{i}}^2 E(X_{\mathbf{i}}^2 I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le \epsilon|\mathbf{n}|^{\alpha}]) \Big)^{q/2} \\ & \le C \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha(p-q)-2} (\log_2 |\mathbf{n}|)^{qd} (|\mathbf{n}|^{\alpha(2-p)})^{q/2} \Big(\sum_{\mathbf{i}\le\mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^p E|X_{\mathbf{i}}|^p \Big)^{q/2} \\ & \le C \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p-\alpha q-2+\alpha q-\alpha pq/2+q\delta/2} (\log_2 |\mathbf{n}|)^{qd} \\ & \le C \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p(1-q/2)-1} (\log_2 |\mathbf{n}|)^{qd} < \infty \quad \text{since } \frac{\delta q}{2} \le 1. \end{aligned}$$

Hence, by (2.15)–(2.17), conditions (a)–(c) in Theorem 2.2 are satisfied, respectively.

To complete the proof, it is enough to note that, by the assumption $EX_{\mathbf{n}} = 0$ for $\mathbf{n} \in \mathbb{Z}_+^d$ and by (2.13), we get for $\mathbf{j} \le \mathbf{n}$

$$|\mathbf{n}|^{-\alpha} \sum_{\mathbf{i}\le\mathbf{j}} |a_{\mathbf{n},\mathbf{i}}| E|X_{\mathbf{i}}| I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le \epsilon|\mathbf{n}|^{\alpha}] \to 0 \quad \text{as } |\mathbf{n}| \to \infty,$$

which satisfies (d). Hence, by Theorem 2.3, the proof is complete. $\square$

Definition 2.6. A field $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a constant D such that

$$P(|X_{\mathbf{n}}| > x) \leq DP(|X| > x)$$

for all $x \geq 0$ and $\mathbf{n} \in \mathbb{Z}_+^d$.

Corollary 2.7. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a field of ρ^*-mixing random variables with $EX_{\mathbf{n}} = 0$ and $E|X_{\mathbf{n}}|^p < \infty$ for $\mathbf{n} \in \mathbb{Z}_+^d$ and $1 < p \leq 2$. Let the random variables $\{X_{\mathbf{n}}\}$ be stochastically dominated by a random variable X, such that $E|X|^p < \infty$ and $\{a_{\mathbf{n},\mathbf{i}}, \mathbf{n} \in \mathbb{Z}_+^d, \mathbf{i} \leq \mathbf{n}\}$ are a field of real numbers satisfying the condition*

$$\sum_{\mathbf{i} \leq \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^p = O(|\mathbf{n}|^\delta)$$

for some $0 < \delta < 2/q$ and $q \geq 2$. Then, for any $\epsilon > 0$, $\alpha > 1/2$ and $\alpha p > 1$ (2.14) holds.

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