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SOME K_0-MONOID PROPERTIES PRESERVED BY TRACIAL APPROXIMATION

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ABSTRACT. We show that the following K_0-monoid properties of C^*-algebras in the class Ω are inherited by simple unital C^*-algebras in the class $TA\Omega$:

(1) almost divisible,
(2) directly finite,
(3) strong Riesz interpolation property.

1. Introduction. The Elliott conjecture asserts that all nuclear, separable C^*-algebras are classified up to isomorphism by an invariant, called the Elliott invariant. A first version of the Elliott conjecture might be said to have begun with the K-theoretical classification of AF-algebras in **[2]**. Since then, many classes of C^*-algebras have been classified by the Elliott invariant. Among them, one important class is that of simple unital AH-algebras without dimension growth. A very important axiomatic version of the classification of AH-algebras without dimension growth was given by Lin. Instead of assuming inductive limit structure, he started with a certain abstract approximation property and showed that C^*-algebras with this abstract approximation property and certain additional properties are AH-algebras without dimension growth. More precisely, Lin introduced the class of tracially approximate interval algebras which he also called C^*-algebras of tracial topological rank one.

Following the notion of Lin on the tracial approximation by interval algebras, Elliott and Niu in **[6]** considered tracial approximation by certain C^*-algebras.

The question of the behavior of C^*-algebra properties under passage from a class Ω to the class $TA\Omega$ is interesting and sometimes important. In fact, the property of having tracial states, the property of being of stable rank one, and the property that the strict order on projections

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is determined by traces were used in the proof of the classification theorem in [6, 14] by Elliott and Niu.

In this paper, we show that the following K_0-monoid properties of C^*-algebras in the class Ω are inherited by simple unital C^*-algebras in the class $TA\Omega$:

(1) almost divisible,
(2) directly finite,
(3) strong Riesz interpolation property.

2. Preliminaries and definitions. Let a and b be two positive elements in a C^*-algebra, A. We write $[a] \le [b]$ (cf., [**13,** Definition 3.5.2]), if there exists a partial isometry $v \in A^{**}$ such that, for every $c \in \mathrm{Her}\,(a)$, v^*c, $cv \in A$, $vv^* = P_a$, where P_a is the range projection of a in A^{**}, and $v^*cv \in \mathrm{Her}\,(b)$. We write $[a] = [b]$ if $v^*\mathrm{Her}\,(a)v = \mathrm{Her}\,(b)$. Let n be a positive integer. We write $n[a] \le [b]$, if there are n mutually orthogonal positive elements $b_1, b_2, \ldots, b_n \in \mathrm{Her}\,(b)$ such that $[a] \le [b_i]$, $i = 1, 2, \ldots, n$.

Let $0 < \sigma_1 < \sigma_2 \le 1$ be two positive numbers. Define

$$f_{\sigma_1}^{\sigma_2}(t) = \begin{cases} 1 & \text{if } t \ge \sigma_2 \\ (t-\sigma_1)/(\sigma_2-\sigma_1) & \text{if } \sigma_1 \le t \le \sigma_2 \\ 0 & \text{if } 0 < t \le \sigma_1 \end{cases}$$

Let Ω be a class of unital C^*-algebras. Then the class of C^*-algebras which can be tracially approximated by C^*-algebras in Ω is denoted by $TA\Omega$.

Definition 2.1 ([**6**])**.** A simple unital C^*-algebra A is said to belong to the class $TA\Omega$ if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any nonzero element $a \ge 0$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with $1_B = p$ and $B \in \Omega$, such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,
(2) $pxp \in_\varepsilon B$ for all $x \in F$,
(3) $[1-p] \le [a]$.

Definition 2.2 ([**8**])**.** Let Ω be a class of unital C^*-algebras. A unital C^*-algebra A is said to have property (III) if, for any positive numbers $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, any $\varepsilon > 0$, any finite subset $F \subseteq A$, any nonzero positive element a, and any integer $n > 0$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with $B \in \Omega$ and $1_B = p$, such that:

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,
(2) $pxp \in_\varepsilon B$ for all $x \in F$, $\|pap\| \geq \|a\| - \varepsilon$,
(3) $n[f_{\sigma_1}^{\sigma_2}((1-p)a(1-p))] \leq [f_{\sigma_3}^{\sigma_4}(pap)]$.

Lemma 2.3 ([**6**]). *If the class Ω is closed under tensoring with matrix algebras or closed under taking unital hereditary C^*-subalgebras, then $TA\Omega$ is closed under passing to matrix algebras or unital hereditary C^*-subalgebras.*

Theorem 2.4 ([**8**]). *Let Ω be a class of unital C^*-algebras such that Ω is closed under taking unital hereditary C^*-subalgebras and closed taking finite direct sums. Let A be a simple unital C^*-algebra. Then the following are equivalent:*

(1) *$A \in TA\Omega$,*
(2) *A has property* (III).

Call projections $p, q \in M_\infty(A)$ equivalent, denoted $p \sim q$, when there is a partial isometry $v \in M_\infty(A)$ such that $p = v^*v$, $q = vv^*$. The equivalent classes are denoted by [.], and the set of all these is:

$$V(A) := \{[p] | p = p^* = p^2 \in M_\infty(A)\}.$$

Addition in $V(A)$ is defined by

$$[p] + [q] := [\mathrm{diag}\,(p, q)],$$

$V(A)$ becomes an abelian monoid, and we call $V(A)$ the K_0-monoid of A.

All abelian monoids have a natural pre-order, the *algebraic ordering*, defined as follows: if $x, y \in M$, we write $x \leq y$ if there is a z in M such that $x + z = y$. In the case of $V(A)$, the algebraic ordering is given by Murray-von Neumann subequivalence, that is, $[p] \leq [q]$ if and only if there is a projection $p' \leq q$ such that $p \sim p'$. We also write, as is customary, $p \preceq q$ to mean that p is subequivalent to q.

Let M be an abelian monoid. If $x, y \in M$, we will write $x \leq^* y$ if there is a nonzero element z in M, such that $x + z = y$.

We say that a monoid M is conical if $x+y=0$ only when $x=y=0$. Note that, for any C^*-algebra A, the monoid $V(A)$ is conical.

We say that an order abelian monoid M is almost divisible when it satisfies the statement that, for any $x \in M$ and any $n \in \mathbf{N}$, there is $y \in M$ such that $ny \leq x \leq (n+1)y$.

Let M be an abelian monoid. An element x of M is directly finite if $x+y=x$ implies that $y=0$, for all $y \in M$. M is directly finite if for any $x \in M$ is directly finite. M is said to satisfy the strong Riesz interpolation property provided that M satisfies the conditions that, given any x_1, x_2, y_1, y_2 in M such that $x_i \leq^* y_j$ for all i, j, there exists z such that $x_i \leq^* z \leq^* y_j$ for all i, j.

3. The main results.

Theorem 3.1. *Let Ω be a class of unital C^*-algebras such that, for any $B \in \Omega$, the K_0-monoid $V(B)$ is almost divisible. Then the K_0-monoid $V(A)$ is almost divisible for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof. We need to show that there exist $y \in V(A)$ such that $ny \leq x \leq (n+1)y$ for any $x \in V(A)$ and any $n \in \mathbf{N}$. By Lemma 2.3, we may assume that $x=[p]$ for some projection $p \in \mathrm{proj}\,(A)$. For $F=\{p\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $r \in A$ and a C^*-subalgebra $B \subseteq A$ with $B \in \Omega$, $1_B = r$ such that

(1) $\|pr - rp\| < \varepsilon$,
(2) $rpr \in_\varepsilon B$.

By (1) and (2) there exist projections $p_1 \in B$ and $p_2 \in (1-r)A(1-r)$ such that
$$\|p - p_1 - p_2\| < \varepsilon.$$
Therefore, we have
$$[p] = [p_1] + [p_2].$$

Since $B \in \Omega$ and $V(B)$ is almost divisible, we may assume that there exists a projection $e_1 \in B$ such that $n[e_1] \leq [p] \leq (n+1)[e_1]$ in $V(B)$. We prove this theorem by two steps.

Firstly, we may assume that $n[e_1] = [p_1]$.

For $G = \{p_2,\ e_1\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $s \in A$ and a C^*-subalgebra $C \subseteq A$ with $C \in \Omega,\ 1_C = s$ such that

(1′) $\|xs - sx\| < \varepsilon$ for all $x \in G$,
(2′) $sxs \in_\varepsilon C$ for all $x \in G$,
(3′) $[1 - s] \leq [e_1]$.

By (1′) and (2′) there exist projections $p_3 \in C$ and $p_4 \in (1 - s)A(1 - s)$ such that

$$\|p_2 - p_3 - p_4\| < \varepsilon.$$

Therefore, we have

$$[p_2] = [p_3] + [p_4].$$

Since $C \in \Omega$ and $V(C)$ is almost divisible, we may assume that there exists a projection $e_3 \in C$ such that $n[e_3] \leq [p_3] \leq (n+1)[e_3]$ in $V(C)$.

By (3′) we have $[p_4] \leq [e_1]$; therefore, we have

$$\begin{aligned} n([e_1] + [e_3]) &= n[e_1] + n[e_3] \\ &\leq [p_1] + [p_3] + [p_4] \leq n[e_1] + (n+1)[e_3] + [p_4] \\ &\leq (n+1)[e_1] + (n+1)[e_3] \\ &= (n+1)([e_1] + [e_3]). \end{aligned}$$

Secondly, we assume that $n[e_1] \leq^* [p_1] \leq (n+1)[e_1]$. Since $n[e_1] \leq^* [p_1]$, there exist a nonzero projection $g \in A$ such that $n[e_1] + [g] = [p_1]$.

For $H = \{p_2,\ g\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $t \in A$ and a C^*-subalgebra $D \subseteq A$ with $D \in \Omega$, $1_D = t$ such that

(1″) $\|xt - tx\| < \varepsilon$ for all $x \in G$,
(2″) $txt \in_\varepsilon C$ for all $x \in G$,
(3″) $n[1 - t] \leq [g]$.

By (1″) and (2″) there exist projections $p_3 \in D$ and $p_4 \in (1 - t)A(1 - t)$ such that

$$\|p_2 - p_3 - p_4\| < \varepsilon.$$

Therefore, we have

$$[p_2] = [p_3] + [p_4].$$

Since $D \in \Omega$ and $V(D)$ is almost divisible, we may assume that there exists a projection $e_3 \in C$ such that $n[e_3] \le [p_3] \le (n+1)[e_3]$ in $V(C)$. By ($3''$), $n[p_4] \le n[1-t] \le [g]$; therefore, we have

$$\begin{aligned} n[e_1] + n[e_3] + n[p_4] &\le [p_1] + [p_3] + [p_4] \\ &\le (n+1)[e_1] + (n+1)[e_3] + (n+1)[p_4] \\ &= (n+1)([e_1] + [e_3] + [p_4]). \quad \square \end{aligned}$$

Let Ω denote the class of all finite dimensional C^*-algebras. Then the K_0-monoid $V(A)$ is almost divisible for any simple unital C^*-algebra $A \in TA\Omega$. That is to say, any K_0-monoid $V(A)$ is almost divisible for any simple unital C^*-algebra A with tracial topological rank zero.

Theorem 3.2. *Let Ω be a class of unital C^*-algebras such that, for any $B \in \Omega$, the K_0-monoid $V(B)$ is directly finite. Then the K_0-monoid $V(A)$ is directly finite for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof. We need to show that $y = 0$ for any $x, y \in V(A)$ with $x + y = x$. By Lemma 2.3, we may assume that $x = [p]$, $y = [q]$ for some projections $p, q \in \operatorname{proj}(A)$. For $F = \{p, q\}$, any $\varepsilon > 0$ and any positive numbers $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, since $A \in TA\Omega$, there exist a projection $r \in A$ and a C^*-subalgebra $B \subseteq A$ with $B \in \Omega$, $1_B = r$ such that

(1) $\|rx - xr\| < \varepsilon$ for all $x \in F$,
(2) $rxr \in_{\varepsilon} B$ for all $x \in F$.
(3) $[f_{\sigma_1}^{\sigma_2}((1-r)q(1-r))] \le [f_{\sigma_3}^{\sigma_4}(rqr)]$

By (1) and (2), there exist projections $p_1, q_1 \in B$ and $p_2, q_2 \in (1-r)A(1-r)$ such that

$$\|p - p_1 - p_2\| < \varepsilon, \qquad \|q - q_1 - q_2\| < \varepsilon.$$

Therefore, we have

$$[p] = [p_1] + [p_2], \qquad [q] = [q_1] + [q_2],$$

and

$$[p_1] + [q_1] = [p_1], \qquad [p_2] + [q_2] = [p_2].$$

Since $B \in \Omega$ and $V(B)$ is directly finite, we have $[q_1] = 0$. By (3), $[q_2] \le [q_1]$; therefore, $[q_2] = 0$. We have $[q] = [q_1] + [q_2] = 0$. $\square$

Let Ω denote the class of all interval C^*-algebras. Then the K_0-monoid $V(A)$ is directly finite for any simple unital C^*-algebra $A \in TA\Omega$. That is to say, K_0-monoid $V(A)$ is directly finite for any simple unital C^*-algebra A with tracial topological rank no more than one.

Theorem 3.3. *Let Ω be a class of unital C^*-algebras such that, for any $B \in \Omega$, the K_0-monoid $V(B)$ has the strong Riesz interpolation property. Then the K_0-monoid $V(A)$ has the strong Riesz interpolation property for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof. We need to show that there exist $z \in V(A)$ such that $x_i \leq^* z \leq^* y_j$ for any x_1, x_2, y_1, y_2 in $V(A)$ with $x_i \leq^* y_j$ for all i, j. By Lemma 2.3, we may assume that $x_1 = [p_1]$, $x_2 = [p_2]$, $y_1 = [q_1]$, $y_2 = [q_2]$ for some projections $p_1, p_2, q_1, q_2 \in \operatorname{proj}(A)$. For $F = \{p_1, p_2, q_1, q_2\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $r \in A$ and a C^*-subalgebra $B \subseteq A$ with $B \in \Omega$, $1_B = r$ such that

(1) $\|xr - rx\| < \varepsilon$ for all $x \in F$,
(2) $rxr \in_\varepsilon B$ for all $x \in F$.

By (1) and (2) there exist projections $p_1', q_1', p_2', q_2' \in B$ and p_2'', q_2'', p_1'', $q_1'' \in (1-r)A(1-r)$ such that

$$\|p_1 - p_1' - p_1''\| < \varepsilon, \qquad \|q_1 - q_1' - q_1''\| < \varepsilon,$$
$$\|p_2 - p_2' - p_2''\| < \varepsilon, \qquad \|q_2 - q_2' - q_2''\| < \varepsilon.$$

Therefore, we have

$$[p_1] = [p_1'] + [p_1''], \qquad [q_1] = [q_1'] + [q_1''],$$
$$[p_2] = [p_2'] + [p_2''], \qquad [q_2] = [q_2'] + [q_2''],$$

and $[p_i'] \leq^* [q_j']$, $[p_i''] \leq^* [q_j'']$ for all i, j.

Since $B \in \Omega$ and $V(B)$ has the strong Riesz interpolation property, we may assume that there exists a projection $e' \in B$ such that $[p_i'] \leq^* [e'] \leq^* [q_j']$ for all i, j.

Since $[e'] \leq^* [q_1']$ and $[e'] \leq^* [q_2']$, there exist nonzero projections $g, h \in A$ such that $[e'] + [g] = [q_1']$, $[e'] + [h] = [q_2']$.

For $G = \{p_1'', p_2'', q_1'', q_2'', g, h\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $s \in A$ and a C^*-subalgebra $D \subseteq A$ with $D \in \Omega$, $1_D = s$, such that

(1′) $\|xs - sx\| < \varepsilon$ for all $x \in G$,
(2′) $sxs \in {}_{\varepsilon}D$ for all $x \in G$,
(3′) $2[1-s] \le [g],\ 2[1-s] \le [h]$.

By (1′) and (2′), there exist projections $p_1''', q_1''', p_2''', q_2''' \in D$ and $p_2'''', q_2'''', p_1'''', q_1'''' \in (1-s)A(1-s)$ such that

$$\|p_1'' - p_1''' - p_1''''\| < \varepsilon,\ \|q_1'' - q_1''' - q_1''''\| < \varepsilon,$$
$$\|p_2'' - p_2''' - p_2''''\| < \varepsilon,\ \|q_2'' - q_2''' - q_2''''\| < \varepsilon.$$

Therefore, we have

$$[p_1''] = [p_1'''] + [p_1''''], \qquad [q_1''] = [q_1'''] + [q_1''''],$$
$$[p_2''] = [p_2'''] + [p_2''''], \qquad [q_2''] = [q_2'''] + [q_2''''],$$

and $[p_i'''] \le^* [q_j''']$, $[p_i''''] \le^* [q_j''']$ for all i, j.

Since $D \in \Omega$ and $V(D)$ has the strong Riesz interpolation property, we may assume that there exists a projection $e''' \in D$ such that $[p_i'''] \le^* [e'''] \le^* [q_j''']$ for all i, j.

By (3′), we have $[p_1''''] + [p_2''''] \le [g]$, $[p_1''''] + [p_2''''] \le [h]$. We have

$$[p_1'] + [p_1'''] + [p_1''''] \le^* [e'] + [e'''] + [p_1''''] + [p_2''''] \le^* [q_1'] + [q_1'''] + [q_1''''],$$
$$[p_2'] + [p_2'''] + [p_2''''] \le^* [e'] + [e'''] + [p_1''''] + [p_2''''] \le^* [q_1'] + [q_1'''] + [q_1''''],$$
$$[p_1'] + [p_1'''] + [p_1''''] \le^* [e'] + [e'''] + [p_1''''] + [p_2''''] \le^* [q_2'] + [q_2'''] + [q_2''''],$$
$$[p_2'] + [p_2'''] + [p_2''''] \le^* [e'] + [e'''] + [p_1''''] + [p_2''''] \le^* [q_1'] + [q_1'''] + [q_1''''].$$

Therefore, we have $[p_i] \le^* [e'] + [e'''] + [p_1''''] + [p_2''''] \le^* [q_j]$ for all i, j. $\square$

Let Ω denote the class of all finite-dimensional C^*-algebras. Then the K_0-monoid $V(A)$ has the strong Riesz interpolation property for any simple unital C^*-algebra $A \in TA\Omega$. That is to say, K_0-monoid $V(A)$ has the strong Riesz interpolation property for any simple unital C^*-algebra A with tracial topological rank zero.

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