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ON WEAKLY COHERENT RINGS

CHAHRAZADE BAKKARI AND NAJIB MAHDOU

ABSTRACT. In this paper, we define weakly coherent rings and examine the transfer of this property to homomorphic images, trivial ring extensions, localizations and finite direct products. These results provide examples of weakly coherent rings that are not coherent rings. We show that the class of weakly coherent rings is not stable under localization. Also, we show that the class of weakly coherent rings and the class of strongly 2-coherent rings are not comparable.

1. Introduction. Throughout this paper all rings are assumed to be commutative with identity elements, and all modules are unital.

Let R be a commutative ring. We say that an ideal is regular if it contains a regular element, i.e., a non-zerodivisor element.

For a nonnegative integer n, an R-module E is n-presented if there is an exact sequence of R-modules:

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are, respectively, finitely generated and finitely presented R-modules.

A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R [**5**]. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, valuation rings and Prüfer/semihereditary rings. See, for instance, [**5**].

In this paper, we investigate a particular class of coherent rings that we call weakly coherent rings. A ring R is called a weakly coherent

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ring if any finitely generated ideal of R contained in a finitely presented proper ideal of R is itself finitely presented. If R is coherent, then R is naturally weak coherent. Our aim in this paper is to prove that the converse is false in general.

We say that R is strong n-coherent if each n-presented R-module is $(n+1)$-presented. In particular, any coherent ring (i.e., 1-coherent ring) is a strong 2-coherent ring. This led us to consider the relation between the class of weakly coherent rings and the class of strong 2-coherent rings.

Let A be a ring, E an A-module and $R := A \propto E$ the set of pairs (a, e) with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$. R is called the trivial ring extension of A by E. Considerable work has been concerned with trivial ring extensions. Part of it has been summarized in Glaz's book [**5**]) and Huckaba's book (where R is called the idealization of E by A) [**7**]).

In the context of rings containing regular elements, we show that the notion of weak coherent coincides with the definition of coherent ring. The goal of this work is to exhibit a class of non-coherent weakly coherent rings. We show that the class of weakly coherent rings is not stable under localization. Also, we show that the class of weakly coherent rings and the class of strong 2-coherent rings are not comparable. For this purpose, we study the transfer of this property to homomorphic images, trivial ring extensions, and finite direct products.

2. Main results. Recall that, for nonnegative integers n and d, we say that a ring R is an (n, d)-ring if $pd_R(E) \leq d$ for each n-presented R-module E (as usual, $pd_R E$ denotes the projective dimension of E as an R-module). See, for instance, [**2, 8–11**].

We begin this section by giving an example of non-coherent weakly coherent ring.

Example 2.1. Let (A, M) be a local ring with maximal ideal M, let E be an A/M-vector space with infinite rank and let $R := A \propto E$ be the trivial ring extension of A by E. Then:

1) R is a weak coherent ring.

2) R is not a coherent ring.

Proof. 1) It suffices to show that there is no finitely presented proper ideal J of R. Deny. Let J be a finitely presented proper ideal of R. Then J is free since R is a local $(2,0)$-ring by [**11**, Theorem 2.1 (1)], that is, $J = Ra$ for some regular element A of R, a contradiction since $J \subseteq M \propto E$ and $(M \propto E)(0,e) = (0,0)$ for each $e \in E - \{0\}$. Hence, R is a weak coherent ring.

2) We claim that R is not coherent. Deny. Assume that R is coherent. But, R is a $(2,0)$-ring by [**11**, Theorem 2.1 (1)]. Hence, R is Von Neumann regular ring since it is coherent, a contradiction by [**11**, Theorem 2.1 (2)]. Hence, R is not coherent, as desired. $\square$

Now, we give a sufficient condition to have equivalence between coherent and weakly coherent properties.

Proposition 2.2. *Let R be a commutative ring. Then*:

1) *If R is a coherent ring, then R is a weak coherent ring.*

2) *Assume that R contains a regular element* (*that is, R is not a total ring of quotients*). *Then R is a coherent ring if and only if R is a weak coherent ring.*

Proof. 1) Clear.

2) It remains to show that, if R is a weak coherent ring and contains a regular element a, then R is a coherent ring. Let I be a finitely generated proper ideal of R. Hence, $aI \subseteq aR$, aI is a finitely generated proper ideal of R and $aR(\cong R)$ is a finitely presented proper ideal of R. Therefore, aI is a finitely presented ideal of R and so $I(\cong aI)$ (since a is regular) is finitely presented, as desired. $\square$

Remark 2.3. By the above result, a non-coherent weak coherent ring is necessarily a total ring of quotients.

Now, we investigate the homomorphic image of weak coherent rings.

Theorem 2.4. *Let R be a weak coherent ring and I a finitely generated ideal of R. Then R/I is a weak coherent ring.*

Proof. Let $L/I \subseteq J/I$ be two finitely generated proper ideals of R/I such that J/I is finitely presented. Our aim is to show that L/I is finitely presented.

We remark that $L \subseteq J$ are two finitely generated proper ideals of R. We claim that J is finitely presented.

Indeed, there exists an exact sequence of (R/I)-modules:

$$(*) \qquad 0 \longrightarrow T \longrightarrow (R/I)^n \longrightarrow J/I \longrightarrow 0$$

where n is a positive integer and T is a finitely generated (R/I)-module. Hence, T is a finitely generated R-module. On the other hand, R/I is a finitely presented R-module (since I is a finitely generated ideal of R). Therefore, J/I is a finitely presented R-module by exact sequence $(*)$ considered as an exact sequence of R-modules. Consequently, J is a finitely presented ideal of R by the exact sequence of R-modules $0 \to I \to J \to J/I \to 0$, as desired.

Now, we have $L \subseteq J$, L is a finitely generated ideal, and J is a finitely presented ideal; so L is a finitely presented ideal of R since R is a weakly coherent ring. Hence, the exact sequence of R-modules $0 \to I \to L \to L/I \to 0$ shows that L/I is a finitely presented R-module. We claim that L/I is a finitely presented ideal of R/I and this completes the proof of Theorem 2.4.

Indeed, since L/I is a finitely generated ideal of R/I, consider the exact sequence of (R/I)-modules:

$$(**) \qquad 0 \longrightarrow S \longrightarrow (R/I)^m \longrightarrow L/I \longrightarrow 0$$

where m is a positive integer and S is an (R/I)-module. The exact sequence $(**)$ is also an exact sequence of R-modules; hence, S is a finitely generated R-module since L/I and R/I are finitely presented R-modules. Therefore, S is also a finitely generated (R/I)-module, and the exact sequence of (R/I)-modules $(**)$ shows that L/I is a finitely presented ideal of R/I, as desired. □

The condition "I is a finitely generated ideal of R" is necessary in Theorem 2.4 as the following example shows.

Example 2.5. Let (A, M) be a non-coherent local domain with maximal ideal M, let E be an A/M-vector space with infinite rank, let

$R := A \propto E$ be the trivial ring extension of A by E and set $I := 0 \propto E$. Then:

1) R is a weak coherent ring by Example 2.1.

2) $R/I (\cong A)$ is not a weak coherent ring (by Proposition 2.2 since A is not a coherent domain).

Now, we investigate a weak coherent property in a particular class of total rings of quotients; namely, those arising as trivial ring extensions of local rings by vector spaces over the residue fields.

Theorem 2.6. *Let (A, M) be a local ring with maximal ideal M, let E be an A-module such that $ME = 0$ and let $R := A \propto E$ be the trivial ring extension of A by E. Then R is a weak coherent ring if and only if one of the following two properties holds*:

1) *E is an A/M-vector space with infinite rank.*

2) *E is an A/M-vector space with finite rank and A is weak coherent.*

Proof. Assume that R is a weak coherent ring and E is an A/M-vector space with finite rank. Our aim is to show that A is weak coherent.

Let $I \subseteq J$ be two proper ideals of A such that I is finitely generated and J is finitely presented. Then $(I \propto 0) \subseteq (J \propto 0)$ are two finitely generated proper ideals of R. We claim that $J \propto 0$ is a finitely presented ideal of R.

Indeed, let $J := \sum_{i=1}^{n} Aa_i$ for some positive integer n and some $a_i \in J$, and consider the exact sequence of A-modules:

$$0 \longrightarrow \mathrm{Ker}\,(u) \longrightarrow A^n \xrightarrow{u} J \longrightarrow 0,$$

where u is defined by $u((b_i)_{1\le i\le n}) = \sum_{i=1}^{n} b_i a_i$. Hence,

$$\mathrm{Ker}\,(u)\left(\left\{(b_i)_{1\le i\le n} \in A^n / \sum_{i=1}^{n} a_i b_i = 0\right\}\right)$$

is a finitely generated A-module (since J is a finitely presented ideal of A). Now, consider the exact sequence of R-modules:

$$0 \longrightarrow \mathrm{Ker}\,(v) \longrightarrow R^n \xrightarrow{v} J \propto 0 \longrightarrow 0,$$

where $v((b_i,e_i)_{1\le i\le n}) = \sum_{i=1}^n (b_i,e_i)(a_i,0)$. But,

$$\begin{aligned}\operatorname{Ker}(v) &= \left\{(b_i,e_i)_{1\le i\le n} \in R^n / \sum_{i=1}^n (a_i,0)(b_i,e_i) = 0\right\}\\ &= \left\{(b_i,e_i)_{1\le i\le n} \in R^n / \sum_{i=1}^n a_i b_i = 0\right\}\\ &= \operatorname{Ker}(u) \propto E^n\end{aligned}$$

(since $a_i \in J \subseteq M$) which is finitely generated R-module (since $\operatorname{Ker}(u)$ and E are finitely generated A-modules). Hence, $J \propto 0$ is a finitely presented (proper) ideal of R and so $(I \propto 0)$ $(\subseteq (J \propto 0))$ is a finitely presented ideal of R since R is a weak coherent ring. Therefore, by the same reasoning as for J above, we can show that I is finitely presented and this shows that A is a weak coherent ring.

Conversely, if $\dim_{A/M} E = \infty$, then R is a weak coherent ring by Example 2.1. Now, assume that $\dim_{A/M} E < \infty$ and A is a weak coherent ring and our aim is to show that R is a weak coherent ring.

Let $I(:= \sum_{i=1}^n R(a_i,e_i)) \subseteq J(:= \sum_{i=1}^m R(b_i,f_i))$ be two proper ideals of R such that n, m are positive integers, $a_i, b_j \in A$ and $e_i, f_j \in E$ for each i, j, and J is finitely presented. We which to show that I is finitely presented. Two cases are then possible.

Case 1. $b_i = 0$ for all $i = 1, \dots, m$. In this case, $a_i = 0$ for all $i = 1, \dots, n$ and $I := 0 \propto E_1$ and $J := 0 \propto E_2$ for some (A/M)-vector subspace E_1 and E_2 of E. Assume that $(e_i)_{i=1,\dots,n}$ and $(f_i)_{i=1,\dots,m}$ are, respectively, the basis of the (A/M)-vector space E_1 and E_2. Consider the exact sequence of R-modules:

$$0 \longrightarrow \operatorname{Ker}(u) \longrightarrow R^m \xrightarrow{u} J(:= 0 \propto E_2) \longrightarrow 0,$$

where $u((c_i,g_i)_{1\le i\le m}) = \sum_{i=1}^m (c_i,g_i)(0,f_i) = (0, \sum_{i=1}^m c_i f_i)$. Hence, $\operatorname{Ker}(u) = M^m \propto E^m (= (M \propto E)^m)$ since $(f_i)_{i=1,\dots,m}$ is a basis of the (A/M)-vector space E_2 and so M is a finitely generated ideal of A (since J is a finitely presented ideal of R). Therefore, the exact sequence of R-modules

$$0 \longrightarrow M^n \propto E^n \longrightarrow R^n \xrightarrow{v} I(:= 0 \propto E_1) \longrightarrow 0,$$

where $v((c_i, g_i)_{1\leq i\leq n}) = \sum_{i=1}^{n}(c_i, g_i)(0, e_i)$ shows that I is a finitely presented ideal of R (since $M^n \propto E^n$ is a finitely generated R-module), as desired.

Case 2. $b_i \neq 0$ for some $i = 1, \dots, m$. We may assume that $((a_i, e_i)_{i=1,\dots,n})$ and $((b_i, f_i)_{i=1,\dots,n})$ are minimal generating sets, respectively, of I and J. Set $I_0 := \sum_{i=1}^{n} Aa_i$ and $J_0 := \sum_{i=1}^{m} Ab_i$. Consider the exact sequence of R-modules

$$0 \longrightarrow \mathrm{Ker}\,(u) \longrightarrow R^m \xrightarrow{u} J \longrightarrow 0,$$

where $u((c_i, g_i)_{1\leq i\leq m}) = \sum_{i=1}^{m}(c_i, g_i)(b_i, f_i)$. But $\mathrm{Ker}\,(u) \subseteq (M \propto E)^m$ by [**12**, Lemma 4.43, page 134]. Hence,

$$\begin{aligned}\mathrm{Ker}\,(u) &= \left\{(c_i, g_i)_{1\leq i\leq m} \in R^m / \sum_{i=1}^{m}(c_i, g_i)(b_i, f_i) = 0\right\}\\ &= \left\{(c_i, g_i)_{1\leq i\leq m} \in R^m / \sum_{i=1}^{m} c_i b_i = 0\right\}\\ &= V \propto E^m\end{aligned}$$

where $V := \{(c_i)_{1\leq i\leq m} \in A^m / \sum_{i=1}^{m} c_i b_i = 0\}$. Also, we have the exact sequence of R-modules

$$0 \longrightarrow V \propto E^m \longrightarrow R^m \xrightarrow{w} J_0 \propto 0 \longrightarrow 0,$$

where $w((c_i, g_i)_{1\leq i\leq m}) = \sum_{i=1}^{m}(c_i, g_i)(b_i, 0)$. Therefore, we may assume that $J = J_0 \propto 0$, and so $I = I_0 \propto 0$ (since $I \subseteq J$). On the other hand, V is a finitely generated A-module since $V \propto E^m (= \mathrm{Ker}\,(u))$ is a finitely generated R-module (by the above exact sequence since J is finitely presented) and so J_0 is a finitely presented ideal of A by the exact sequence of A-modules

$$0 \longrightarrow V \longrightarrow A^m \xrightarrow{v} J_0 \longrightarrow 0,$$

where $v((c_i)_{1\leq i\leq m}) = \sum_{i=1}^{m} c_i b_i$. Hence, I_0 is a finitely presented ideal of A since A is weak coherent and $I_0 \subseteq J_0$. Therefore, by the same reasoning as for J, we may show that I is a finitely presented ideal of R, and this completes the proof of Theorem 2.6. $\square$

Now, we are able to give a new class of a non-coherent weak coherent rings.

Example 2.7. Let (A, M) be a local coherent ring with non-finitely generated maximal ideal M, E an A-module such that $ME = 0$, $R := A \propto E$ the trivial ring extension of A by E and $I := 0 \propto E$. Then:

1) R is a weak coherent ring by Theorem 2.6.

2) R is not a coherent ring by [**9**, Theorem 2.6 (2)] since M is not finitely generated.

The localization of a weak coherent ring is not always a weak coherent ring as the following example shows.

Example 2.8. Let $A := \mathbf{Z}_2 + X\mathbf{R}[[X]]$ be a local ring with maximal ideal $M = 2\mathbf{Z}_2 + X\mathbf{R}[[X]]$, E an A/M-vector space with infinite rank and $R := A \propto E$ the trivial ring extension of A by E. Let S be the multiplicative subset of R given by $S := \{(2, 0)^n \ / \ n \in \mathbf{N}\}$ and S_0 the multiplicative subset of A given by $S_0 := \{2^n \ / \ n \in \mathbf{N}\}$. Then:

1) R is a weak coherent ring.

2) $S^{-1}R$ is not a weak coherent ring.

Proof. 1) Clear by Example 2.1.

2) Since $2E \subseteq ME = 0$ and $2 \in S_0$, then $S_0^{-1}E = 0$. Thus, $S^{-1}(0 \propto E) = 0$, and so $S^{-1}R = \{[(a, 0)/(s, 0)]/a \in A \text{ and } s \in S_0\}$, which is clearly isomorphic to a ring $S_0^{-1}A$. But $S_0^{-1}A = S_0^{-1}\mathbf{Z}_2 + X\mathbf{R}[[X]] = \mathbf{Q} + X\mathbf{R}[[X]]$, which is not a coherent domain by [**5**, Theorem 5.2.3]. Therefore, $S^{-1}R$ is not a weak coherent ring by Proposition 2.2, as desired. $\square$

We know that a coherent ring is weak coherent and strong 2-coherent. The following two examples show that the class of weakly finite conductor rings and the class of 2-coherent rings are not comparable.

Example 2.9. Let R be a non-coherent strong 2-coherent domain (see for example [**10**, Theorem 3.1]). Then, R is not a weak coherent domain (by Proposition 2.2, since R is not a coherent domain).

Example 2.10. Let (A, M) be a local coherent domain with non-finitely generated maximal ideal M, and let $R := A \propto (A/M)$ be the trivial ring extension of A by A/M. Then:

1) R is a weak coherent ring by Theorem 2.6.

2) R is not a strong 2-coherent ring by [**8**, Theorem 3.1].

Next, we study the transfer of the weak coherent property to finite direct products.

Proposition 2.11. *Let $(R_i)_{i=1,\dots,n}$ be a family of rings. Then, $\prod_{i=1}^n R_i$ is a weak coherent ring if and only if R_i is a weak coherent ring for each $i = 1, \dots, n$.*

Proof. By induction on n, it suffices to prove the assertion for $n=2$. Since an ideal of $R_1 \prod R_2$ is of the form $I_1 \prod I_2$ where I_i is an ideal of R_i for $i=1,2$ the conclusion follows easily from [**10**, Lemma 2.5(1)]. $\square$

Now, we are able to give a new class of a non-coherent weak coherent rings.

Example 2.12. Let R_1 be a non-coherent weak coherent ring, R_2 a coherent ring and $R = R_1 \prod R_2$. Then:

1) R is a weak coherent ring by Proposition 2.11.

2) R is not a coherent ring by [**5**, Theorem 2.4.3].

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Department of Mathematics and Computer Sciences, Faculty of Science Ain Chock, University Hassan II, Casablanca, Morocco
Email address: cbakkari@hotmail.com

Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco
Email address: mahdou@hotmail.com