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ON EXTENDING ABHYANKAR'S TWO POINT LEMMA TO POSITIVE WEIGHTS

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ABSTRACT. In this paper we answer the question of whether we can extend the current proof of Abhyankar's two point lemma to any positive weight in the negative. This is done by constructing suitable w-homogeneous polynomial pairs of positive weight that behave the same way under the Jacobian as their negatively weighted counterparts, but they do not possess the same pairwise properties.

1. Introduction. The motivation for this work comes from the Jacobian conjecture.

Conjecture 1.1 (Jacobian conjecture). *If $f, g \in k[X, Y]$ and*

$$J(f,g) = \begin{vmatrix} f_X & f_Y \\ g_X & g_Y \end{vmatrix} = f_X g_Y - f_Y g_X = \theta,$$

then (f, g) is an automorphic pair, i.e., $k[f, g] = k[X, Y]$.

Of course, k is a field of characteristic zero, and θ is used to represent a nonzero constant in k. This conjecture has many different incarnations, including an obvious generalization in $k[X_1, X_2, \ldots, X_n]$. A nice introduction to these things can be found in Van den Essen [**12**].

The main technique used in this paper is the theory of weights, which is really an algebrization of the Newton polygon. This approach, in conjunction with the Jacobian, was started by Magnus [**7**] in 1955 and later generalized by Abhyankar in the Jacobian lectures he gave at Purdue in 1971. These have been wonderfully preserved in the lecture notes of van der Put and Heinzer which were later published by Abhyankar with an update by Sathaye in [**2**]. This method gained

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momentum near the end of the 70's and through the 80's in the work of Abhyankar [**1**], Appelgate-Onishi [**6**], Nagata [**8**] [**9**], Nakai-Baba [**10**], and Oka [**11**].

More recently, Abhyankar revisited some of his work after realizing a certain case of this conjecture he had proved was never published. He published this previously omitted case, along with some of this revisited work in [**3**], [**4**], and [**5**]. The results here follow mainly from the second of these publications in which he proves the following:

Theorem 1 (Two point lemma). *If w is negative and $J(f,g) = \not\theta$, then f has at most two w-points at infinity.*

One corollary to this theorem is that the conjecture is true if the greatest common divisor of the degrees of f and g is less than or equal to eight ([**2**], [**9**], or [**4**] and [**5**]). It can also be shown that the Jacobian conjecture will follow if the theorem holds for positive weights. This is due in part to the following corollary of the automorphism theorem [**1**].

Corollary 1. *If (x,y) is a positively w-automorphic pair in $k[X,Y]$, then $\{x,y\} = \{\alpha X, \gamma Y\}$, where $\alpha, \gamma \in k^{\times}$.*

This leads naturally to the question of whether or not we can extend Abhyankar's proof of theorem 1 to include some positive weights. The answer we give in this paper is we cannot. In particular, there are two lemmas used in his proof that do not hold when the weight is positive.

Lemma 1. *If w is negative and $J(F,G) = \not\theta F$ with F and G w-homogeneous polynomials, then F has at most two w-points at infinity.*

Lemma 2. *If w is negative and $J(F,G) = \not\theta F^2$ with F and G w-homogeneous polynomials, then $G/F \in k[X,Y]$.*

The fact that these two lemmas do not hold for any positive weight is shown in Proposition 3.1 and Proposition 3.2 in Section 3. This is done by generalizing the following weighted homogeneous polynomial pairs due to Abhyankar [**4**], Nagata [**9**], and Cassou-Nogues [**5**] to any positive weight:

$$(X(XY^3+1)^2, XY(XY^3+1)) \text{ with } w=(3,-1), \tag{1}$$

$$(Y(X^3Y+1)^2, XY(X^3Y+1)) \text{ with } w=(1,-3), \tag{2}$$

and

$$(X(X^2Y-1)^2(X^2Y-2), (X^2Y-1)^3(3X^2Y-7)) \tag{3}$$

with $w=(1,-2)$. These pairs were originally constructed as counterexamples to these two lemmas.

2. Weights and the Jacobian. We will adopt the following notation in order to simplify the examples and computations given in the next section. For any nonzero $f \in k[X,Y]$, we can express $f=\sum \alpha_{ij}X^iY^j$ as

$$f=\sum \alpha_P Z^P, \quad \text{where } P=(i,j), \alpha_P=\alpha_{ij}, \text{ and } Z^P=X^iY^j. \tag{4}$$

Any such polynomial can be associated to a polygon with vertices in $\mathbf{N}\times\mathbf{N}$ by defining the support of f to be $\operatorname{Supp}(f)=\{P|a_P\neq 0\}$ and then taking the Abhyankar polygon of f to be the convex hull of the $\operatorname{Supp}(f)$.

This can be algebraicized by defining a weight to be an element of the set $\mathbf{W}=\{w=(w_1,w_2)\in(\mathbf{Z}\times\mathbf{Z})^\times \mid \gcd(w_1,w_2)=1\}$. Then, for any $w\in\mathbf{W}$, we define the w-degree of f to be

$$\deg_w(f)=\max\{P\cdot w \mid P\in\operatorname{Supp}(f)\}, \tag{5}$$

and the w-degree form of f to be

$$F=\sum_{\{P\in\operatorname{Supp}(f)|P\cdot w=\deg_w(f)\}} a_P Z^P. \tag{6}$$

f is said to be w-homogeneous if $f=F$. Observe that the w-degree forms of f correspond to the sides and vertices of the Abhyankar polygon and that the slope of these sides can be algebraicized by defining a weight $w\in\mathbf{W}$ with $w_1\neq 0$ to be negative if $-w_2/w_1<0$ and positive if $-w_2/w_1>0$.

For a Jacobian pair, i.e., a pair of polynomials (f,g) such that $J(f,g)=\theta$, not every weight needs to be considered. In fact, we only need to consider the set of standard weights given by $W=\{w\in$

$\mathbf{W}|w_1 > 0\}$. For $w \in W$, f is said to have at most two w-points at infinity if $F \mathrel{=\!\!\!\!\theta}\, x^i y^j$, where $i \in \mathbf{N}^\times$, $j \in \mathbf{N}$, and (x, y) is a w-automorphic pair, i.e., $x, y \in k[X, Y]$ are w-homogeneous polynomials with $k[x, y] = k[X, Y]$. The statements of Theorem 1, Corollary 1, Lemma 1 and Lemma 2 should now be clear.

In order to further simplify the Jacobian computations of the next section, we define the wedge product of two points $P = (a, b), Q = (c, d) \in \mathbf{N} \times \mathbf{N}$ and two nonzero polynomials $f = \sum \alpha_P Z^P, g = \sum \beta_Q Z^Q \in k[X, Y]$ as follows: the wedge product of two points is $P \wedge Q = ad - bc$, and the wedge product of two nonzero polynomials is

$$f \wedge g = \sum \alpha_P \beta_Q (Z^P \wedge Z^Q), \tag{7}$$

where $Z^P \wedge Z^Q = (P \wedge Q) Z^{P+Q}$. This gives an equivalent way to compute the Jacobian of two polynomials which has the same algebraic properties as the Jacobian. This clearly follows from (7) and the identity

$$J(f, g) = \frac{f \wedge g}{Z^I}, \quad \text{where } I = (1, 1). \tag{8}$$

3. Abhyankar's two point lemma and positive weights. We begin here with two observations that clearly follow from (5), (6) and (7). The first is that the following polynomial pair is w-homogeneous when $w \in W$ is positive:

$$(F, G) = (Z^P f(T), Z^Q g(T)), \tag{9}$$

where $P = (a, b)$, $Q = (c, d) \in \mathbf{N} \times \mathbf{N}$, $f, g \in k[X]^\times$ and $T = Z^{w_\perp}$ with $w_\perp = (-w_2, w_1)$. The second observation is that the wedge product of such a pair can be expressed as

$$F \wedge G = [(P \wedge Q) fg + (P \wedge w_\perp) T f g' + (w_\perp \wedge Q) T f' g]\, Z^{P+Q}. \tag{10}$$

Proposition 3.1. *For any positive weight $w \in W$, there exists a w-homogeneous pair (F, G) such that $J(F, G) \mathrel{=\!\!\!\!\theta}\, F$ and F does not have at most two w-points at infinity.*

Proof. In (9), take

$$(P, f, Q, g) = \begin{cases} ((a, b), h, I, 1) & \text{if } w_1 + w_2 = 0 \\ ((\alpha, 0), (T+1)^n, I, T+1) & \text{if } w_1 + w_2 > 0 \\ ((0, \alpha), (T+1)^n, I, T+1) & \text{if } w_1 + w_2 < 0, \end{cases}$$

where $a \neq b$, $h \in k[T] \backslash k$,

$$\alpha = \frac{v_1 + v_2}{\gcd(v_1 + 1, v_1 + v_2)} \quad \text{and} \quad n = \frac{v_1 + 1}{\gcd(v_1 + 1, v_1 + v_2)}$$

with $v = w$ or $w^* = (-w_2, -w_1)$ depending on whether $w_1 + w_2 > 0$ or $w_1 + w_2 < 0$, respectively. Then (F, G) is w-homogeneous by (9), and F does not have at most two w-points at infinity by Corollary 1. Also,

$$P \wedge I = \begin{cases} a - b & w_1 + w_2 = 0 \\ \alpha & w_1 + w_2 > 0 \\ -\alpha & w_1 + w_2 < 0 \end{cases} =\!\!\!\!\theta\, , \tag{11}$$

and $P \wedge I + P \wedge w_\perp + n(w_\perp \wedge I)$

$$= \begin{cases} \alpha + \alpha w_1 - n(w_1 + w_2) & \text{if } w_1 + w_2 > 0 \\ -\alpha + \alpha w_2 - n(w_1 + w_2) & \text{if } w_1 + w_2 < 0 \end{cases} = 0 \tag{12}$$

by definition. From (10), we have for $w_1 + w_2 = 0$ and $w_1 + w_2 \neq 0$ that

$$F \wedge G = \begin{cases} [(P \wedge I)h + (w_\perp \wedge I)Th']\, Z^{P+I} \\ [((P \wedge I) + (P \wedge w_\perp) + n(w_\perp \wedge I))T + (P \wedge I)]\, Z^{P+I}(T+1)^n \\ \qquad = (P \wedge I) Z^{P+I} f =\!\!\!\!\theta\, F Z^I. \end{cases}$$

This follows in the first case from $w_1 + w_2 = 0 \Rightarrow w = (1, -1)$ and in the second case from (11) and (12). Therefore, $J(F, G) =\!\!\!\!\theta\, F$ by (8). $\square$

Proposition 3.2. *For any positive weight $w \in W$ with $w \neq (1, -1)$, there exists a w-homogeneous pair (F, G) such that $J(F, G) =\!\!\!\!\theta\, F^2$ and $G/F \notin k[X, Y]$.*

Proof. In (9), take $(P, f, Q, g) = (w_\perp - I, h', (0, 0), h)$, where $h \in k[T]^\times$ with $h' \neq 0$ and $\operatorname{ord}(h) = \min\{i + j | (i, j) \in \operatorname{Supp}(h)\} = 0$. Then (F, G) is w-homogeneous by (9), and $\frac{G}{F} \notin k[X, Y]$ because the

$\operatorname{ord}(G/F) = \operatorname{ord}(G) - \operatorname{ord}(F) = 2 - (w_1 - w_2 + \operatorname{ord}(h')) < 0$. Also,

$$P \wedge w_\perp = w_\perp \wedge I \mathrel{=\!\!\!\theta} \quad \text{and} \quad P + w_\perp = 2P + I \tag{13}$$

since $w \neq (1, -1)$. From (10), we have

$$F \wedge G = ((w_\perp - I) \wedge w_\perp) Z^{P+w_\perp} h'h' = (w_\perp \wedge I) Z^{2P+I} h'^2 \mathrel{=\!\!\!\theta} F^2 Z^I$$

by (13). Therefore, $J(F, G) \mathrel{=\!\!\!\theta} F^2$ by (8).

$\square$

Interestingly, Lemma 2 is true when $w = (1, -1)$ [**3**].

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