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SEGAL ALGEBRAS IN COMMUTATIVE BANACH ALGEBRAS

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Dedicated to the memory of Professor Masahiro Nakamura (1919–2007)

ABSTRACT. The notion of Reiter's Segal algebra in commutative group algebras is generalized to a notion of Segal algebra in more general classes of commutative Banach algebras. Then we introduce a family of Segal algebras in commutative Banach algebras under considerations and study some properties of them.

1. Introduction. In this paper, G stands for a locally compact abelian group (LCA group) with character group $\widehat{G}$. For a commutative semi-simple Banach algebra B, Φ_B denotes the Gelfand space of B with the Gelfand topology, and $\mathcal{K}(\Phi_B)$ is the set of all compact subsets of Φ_B. The set $\mathcal{K}(\Phi_B)$ forms a directed set with respect to the inclusion order: $K_1 \leq K_2 \Leftrightarrow K_1 \subseteq K_2$ $(K_1, K_2 \in \mathcal{K}(\Phi_B))$. If $x \in B, \widehat{x}$ stands for the Gelfand transform of x. For a subset E of B, $\widehat{E} := \{\widehat{x} : x \in E\}$ and $B_c := \{x \in B : \widehat{x}$ has compact support$\}$. We denote by $(\widehat{B}, \| \ \|_{\hat{B}})$ a Banach function algebra on Φ_B which is isometrically isomorphic to $(B, \| \ \|_B)$ through the Gelfand transform. In the case $B = L^1(\widehat{G})$, $(\widehat{B}, \| \ \|_{\hat{B}})$ is the Fourier algebra on G, which will be denoted by $(\mathcal{A}(G), \| \ \|_{\mathcal{A}(G)})$, or $(\mathcal{A}, \| \ \|_{\mathcal{A}})$.

In the rest of this paper, B stands for a non-unital commutative semi-simple Banach algebra which satisfies the following conditions:

(α_B) B is regular.

(β_B) There exists a bounded approximate identity of B composed of elements in B_c.

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For such a B above, we will define Segal algebras in B, which are generalizations of Reiter's Segal algebras in $L^1(G)$.

In Section 2, definitions and results concerning Reiter's Segal algebras in $L^1(G)$ are stated briefly. In Section 3, the notion of Segal algebras in $L^1(G)$ are generalized to the notion of Segal algebras in B, and the results on Segal algebras in $L^1(G)$ stated in Section 2 are generalized to the results on Segal algebras in B. In Section 4, definitions and some basic properties of the multipliers of Segal algebras in B are stated for later use.

In Section 5 we introduce the notion "local A-functions," and in Sections 5–9, by making use of local A-functions, we introduce some classes of Segal algebras in A (which seems new even in the case of classical Segal algebras in group algebras), and study properties of them in detail.

In Section 10, as another application of local A-functions, we give Theorem 10.3, which characterizes the multiplier algebra of the smallest translation invariant Segal algebra in the Fourier algebra $\mathcal{A} = \mathcal{A}(G)$ on a non-compact LCA group G, and some new results follow from this theorem.

2. Segal algebras and normed ideals in $L^1(G)$. In this section, we state the definitions and results concerning the theory of Reiter's Segal algebras in $L^1(G)$, which are necessary to state our results later. Although Reiter's Segal algebras are defined in group algebras on locally compact groups, we restrict ourselves to the commutative cases in this paper (cf., [**17, 18**]).

2.1. Definition. A subspace $\mathcal{S}$ of $L^1(G)$ is said to be a Segal algebra if it satisfies the following conditions:

(S_0) $\mathcal{S}$ is dense in $L^1(G)$.

(S_1) $\mathcal{S}$ is a Banach space under some norm, which dominates $\|\cdot\|_1$; $\|f\|_1 \le \|f\|_{\mathcal{S}}$ $(f \in \mathcal{S})$.

(S_2) f_y is in $\mathcal{S}$ and $\|f\|_{\mathcal{S}} = \|f_y\|_{\mathcal{S}}$ for all $f \in \mathcal{S}$ and $y \in G$, where $f_y(x) = f(x-y)$ $(x \in G)$.

(S_3) For all $f \in \mathcal{S}$, $y \to f_y$ is a continuous map of G into $\mathcal{S}$.

Here we cite a few examples of Segal algebras from [**18**].

2.2. Examples. (1) Let $\mathcal{S} := \{f \in C(\mathbf{R}) : M(f) < \infty\}$, where $M(f) := \sum_{n\in Z} \sup_{0\le x\le 1} |f(x+n)|$. Then $\mathcal{S}$ is an ideal of $L^1(\mathbf{R})$ and $M(\cdot)$ is a complete algebra norm on $\mathcal{S}$ but not translation invariant. So, if we renorm $M(\cdot)$ by $\|\cdot\|_S$, where $\|f\|_S := \sup\{M(f_y) : y \in \mathbf{R}\}$, then $\| \|_S$ is a translation invariant norm on $\mathcal{S}$ which is equivalent to $M(\cdot)$, and $(\mathcal{S}, \| \|_S)$ becomes a Segal algebra in $L^1(\mathbf{R})$.

(2) $S_p(G)$. For each p $(1 < p < \infty)$, put

$$S_p(G) := \{f \in L^1(G) : \|f\|_p < \infty\} \qquad \|f\|_{S_p} := \|f\|_1 + \|f\|_p.$$

Then $(S_p(G), \| \|_{S_p})$ is a Segal algebra in $L^1(G)$.

(3) $A_{\mu,p}(G), A_p(G)$. Let μ be an unbounded positive Radon measure on $\widehat{G}$. For each p $(1 \le p < \infty)$, put

$$A_{\mu,p}(G) := \{f \in L^1(G) : \widehat{f} \in L^p(\mu)\}, \qquad \|f\|_{\mu,p} := \|f\|_1 + \|\widehat{f}\|_{L^p(\mu)}.$$

Then $(A_{\mu,p}(G), \| \|_{\mu,p})$ is a Segal algebra in $L^1(G)$. In particular, in case μ is a Haar measure $m_{\widehat{G}}$ of $\widehat{G}$, we denote this Segal algebra by $(A_p(G), \| \|_{A_p})$ instead of $(A_{m_{\widehat{G}},p}(G), \| \|_{m_{\widehat{G}},p})$ for simplicity.

Cigler [**3**] introduced the notion of *normed ideal* in $L^1(G)$, which is a generalization of the notion of the Segal algebra in $L^1(G)$, and gave necessary and sufficient conditions for a normed ideal to be a Segal algebra. Also, Dunford [**5**] and Riemersma [**19**] gave alternative necessary and sufficient conditions for a normed ideal to be a Segal algebra.

2.3. Definition (cf., [**3**]). Let $\mathcal{N}$ be a linear subspace of $L^1(G)$. $\mathcal{N}$ is called a normed ideal in $L^1(G)$ if $\mathcal{N}$ satisfies the following conditions:

(a) $\mathcal{N}$ is a dense ideal in $L^1(G)$,

(b) $\mathcal{N}$ is a Banach space under some norm $\| \|_{\mathcal{N}}$ such that

$$\|f\|_1 \le \|f\|_{\mathcal{N}} \quad (f \in \mathcal{N}),$$
$$\|f * g\|_{\mathcal{N}} \le \|f\|_1 \|g\|_{\mathcal{N}} \quad (f \in L^1(G),\ g \in \mathcal{N}).$$

Next we state fundamental properties of normed ideals and Segal algebras in $L^1(G)$.

2.4. Lemma (cf., [**3**]). *Let $\mathcal{N}$ be a normed ideal in $L^1(G)$. Then the following conditions hold.*

(i) *If U is a neighborhood of $\gamma_0 \in \widehat{G}$, then there is an $f \in \mathcal{N}$ such that* $\operatorname{supp} \widehat{f} \subset U$ *and $\widehat{f}(\gamma) = 1$ for every γ in a neighborhood of γ_0.*

(ii) *If $K, U \subset \widehat{G}$ such that K is compact and U is open with $K \subset U$, then there is an $e \in \mathcal{N}$ such that $\widehat{e}(\gamma) = 1$ $(\gamma \in K)$ and* $\operatorname{supp} \widehat{e} \subset U$.

(iii) *$L^1(G)_c$ is contained in $\mathcal{N}$, where $L^1(G)_c := \{f \in L^1(G) :$* $\operatorname{supp} \widehat{f}$ is compact$\}$.

2.5. Theorem A ([**3, 5, 19**]). *For a normed ideal $\mathcal{N}$ in $L^1(G)$, the following conditions are equivalent*:

(a) *$\mathcal{N}$ is a Segal algebra.*

(b) *For any closed ideal $\mathcal{J}$ in $\mathcal{N}$, there is a closed ideal $\mathcal{I}$ in $L^1(G)$ such that $\mathcal{J} = \mathcal{I} \cap \mathcal{N}$.*

(c) *$\mathcal{N} = \mathcal{N}_0$, where $\mathcal{N}_0$ is the norm closure of $L^1(G)_c$ in $\mathcal{N}$.*

(d) *$\mathcal{N}$ has approximate units, that is, for all $f \in \mathcal{N}$ and for all $\varepsilon > 0$, there exists*

$$e \in \mathcal{N} \quad \text{such that } \|f - f * e\|_{\mathcal{N}} < \varepsilon.$$

(e) $\mathcal{N} = \{f * g : f \in L^1(G), g \in \mathcal{N}\}$.

2.6. Theorem B (cf., [**17, 18**]). *Let $\mathcal{S}$ be a Segal algebra in $L^1(G)$.*

(i) *The ideal theory of $\mathcal{S}$ is the same as that of $L^1(G)$. More precisely, if $\mathcal{I}$ is a closed ideal of $L^1(G)$, then $\mathcal{I} \cap \mathcal{S}$ is a closed ideal of $\mathcal{S}$, and conversely each closed ideal of $\mathcal{S}$ is of this form for a unique closed ideal $\mathcal{I}$ of $L^1(G)$.*

(ii) *$\widehat{G}$ and $\Phi_{\mathcal{S}}$ are homeomorphic to each other. More precisely, the map: $\widehat{G} \cong \Phi_{L^1(G)} \to \Phi_{\mathcal{S}} : \varphi \to \varphi|_{\mathcal{S}}$ is a homeomorphism.*

2.7. Theorem C (cf., [**22**]). (i) *If $\mathcal{S}$ is a Segal algebra in $L^1(G)$, and if $\{e_\lambda\}_{\lambda\in\Lambda}$ is a bounded approximate identity of $L^1(G)$ composed of elements in $L^1(G)_c$, then $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of $\mathcal{S}$ which is bounded with respect to the norm defined by*

$$\|f\|_{\mathrm{op}} := \sup\{\|f * g\|_{\mathcal{S}} : g \in \mathcal{S}, \|g\|_{\mathcal{S}} \le 1\} \quad (f \in \mathcal{S}).$$

(ii) *If a Segal algebra $\mathcal{S}$ has a bounded approximate identity, then $\mathcal{S} = L^1(G)$ holds.*

2.8. Theorem D (cf., [**22**]). *If $(\mathcal{S}_1, \| \ \|_{\mathcal{S}_1})$ and $(\mathcal{S}_2, \| \ \|_{\mathcal{S}_2})$ are Segal algebras in $L^1(G)$, then $\mathcal{S} := \mathcal{S}_1 \cap \mathcal{S}_2$ is a Segal algebra in $L^1(G)$ with respect to the norm $\| \ \|_{\mathcal{S}} = \| \ \|_{\mathcal{S}_1} + \| \ \|_{\mathcal{S}_2}$.*

3. Definitions and fundamental properties of Segal algebras in B. Burnham [**2**] defined abstract Segal algebras (ASA) in general Banach algebras, which are generalizations of Cigler's normed ideals in group algebras. In this section, we will define "Segal algebras in B," which are generalizations of Reiter's Segal algebras in $L^1(G)$.

3.1. Definition (cf., [**7**]). An ideal $\mathcal{N}$ in B is called a Banach ideal if $\mathcal{N}$ satisfies the following two conditions.

(a) $\mathcal{N}$ is a Banach space under some norm $\| \ \|_{\mathcal{N}}$ which dominates the original norm: $\|a\|_B \leq \|a\|_{\mathcal{N}}$ $(a \in \mathcal{N})$.

(b) $\|ax\|_{\mathcal{N}} \leq \|a\|_B \|x\|_{\mathcal{N}}$ $(a \in B, x \in \mathcal{N})$.

3.2. Definition (cf., [**3**]). A Banach ideal $(\mathcal{N}, \| \ \|_{\mathcal{N}})$ in B is called a Segal algebra in B if $\mathcal{N}$ satisfies the following properties.

(i) $\mathcal{N}$ is dense in B,

(ii) $\mathcal{N}$ has approximate units, that is, $\mathcal{N}$ satisfies for all $x \in \mathcal{N}$ and for all $\varepsilon > 0$, there exists

$$e \in \mathcal{N} \quad \text{such that } \|x - xe\|_{\mathcal{N}} < \varepsilon.$$

One will see immediately that an abstract Segal algebra in B is a Segal algebra in B if and only if it possesses approximate units (cf., Burnham [**2**]).

3.3. Examples. The following are examples of Banach algebras B satisfying the conditions (α_B) and (β_B).

(1) Group algebras $L^1(G)$ of non-discrete LCA groups G.

(2) Beurling algebras $L^1_\omega(G)$ on a non-discrete LCA groups G with

a weight function ω satisfying the Beurling-Domar condition (cf., [**4, 18**]).

(3) Lipschitz algebra $\mathrm{Lip}_1^0(\mathbf{R})$ (cf., [**14**]).

(4) Commutative C^*-algebras $C_0(X)$ on non-compact locally compact Hausdorff spaces X.

3.4. Lemma ([**3, 18**]). *Suppose that $\mathcal{N}$ is a dense Banach ideal in B. Then the following hold.*

(i) *If U is a neighborhood of $\varphi_0 \in \Phi_B$, then there is an $e \in \mathcal{N}$ such that $\widehat{e}(\varphi) = 1$ for all φ in a neighborhood of φ_0 and* $\operatorname{supp} \widehat{e} \subset U$.

(ii) *If $K, U \subset \Phi_B$ such that K is compact and U is open with $K \subset U$, then there is an $e_K \in \mathcal{N}$ such that $\widehat{e_K}(\varphi) = 1$ $(\varphi \in K)$ and* $\operatorname{supp} \widehat{e_K} \subset U$.

(iii) $B_c \subset \mathcal{N}$.

Proof. (i) Since $\mathcal{N}$ is dense in B, there exists an $x \in \mathcal{N}$ such that $\widehat{x}(\varphi_0) \neq 0$. Choose a $y \in B$ such that $\widehat{y}(\varphi_0) \neq 0$ with $\operatorname{supp} \widehat{y} \subset U$. We can choose a $z \in B$ such that $\widehat{z}(\varphi) = 1/(\widehat{x}(\varphi)\widehat{y}(\varphi))$ for all φ in a neighborhood of φ_0 since B is regular. Letting $e = xyz \in \mathcal{N}$, it is easy to see that e is a desired element.

(ii) For each $\varphi \in K$, there exists an $a_\varphi \in \mathcal{N}$ and a neighborhood V_φ of φ such that $\operatorname{supp} \widehat{a_\varphi} \subset U$ and $\widehat{a_\varphi} = 1$ on V_φ by (i). We can choose a finite number of elements $\varphi_1, \dots, \varphi_n \in K$ such that $\cup_{i=1}^n V_{\varphi_i} \supset K$. Then, if we define $e_K \in \mathcal{N}$ by $1 - e_K = (1 - a_{\varphi_1}) \cdots (1 - a_{\varphi_n})$, it is easy to see that e_K is a desired element.

(iii) Let $x \in B_c$ be arbitrary, and put $K := \operatorname{supp} \widehat{x}$. Then, by (ii), there is an $e \in \mathcal{N}$ such that $\widehat{e} = 1$ on K; hence, $x = xe \in \mathcal{N}$. Thus, B_c is contained in $\mathcal{N}$. □

Under the above definition of Segal algebras in B, all the theorems (A, B, C and D) of the previous section are also valid for Segal algebras in B. Although the proofs are similar to those in the case of Reiter's Segal algebras in $L^1(G)$, we show them for the sake of completeness.

3.5. Theorem A′ (cf., [**2, 3, 5**]). *Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a bounded approximate identity of B composed of elements in B_c. If $\mathcal{N}$ is a dense Banach ideal in B, the following five conditions are equivalent*:

(a) *$\mathcal{N}$ is a Segal algebra in B.*

(b) *For any closed ideal $\mathcal{J}$ of $\mathcal{N}$, there exists a closed ideal $\mathcal{I}$ of B such that $\mathcal{J} = \mathcal{I} \cap \mathcal{N}$.*

(c) *$\mathcal{N} = \mathcal{N}_0$, where $\mathcal{N}_0$ is the norm closure of B_c in $\mathcal{N}$.*

(d) *$\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of $\mathcal{N}$.*

(e) *$\mathcal{N} = \{ax : a \in B, x \in \mathcal{N}\}$.*

Proof. (a) $\Rightarrow$ (b). Let $\mathcal{J}$ be a closed ideal of $\mathcal{N}$. One can prove easily by a routine argument that the closure $\overline{\mathcal{J}}$ of $\mathcal{J}$ in B is a closed ideal in B, and we omit the proof. For each $x \in \overline{\mathcal{J}} \cap \mathcal{N}$ and $\varepsilon > 0$, there exists an $e \in \mathcal{N}$ such that $\|x - xe\|_{\mathcal{N}} < \varepsilon/2$. Choose a $y \in \mathcal{J}$ such that $\|x - y\|_B \le \varepsilon/(2\|e\|_{\mathcal{N}})$. Then we have

$$\begin{aligned}\|x - ye\|_{\mathcal{N}} &\le \|x - xe\|_{\mathcal{N}} + \|xe - ye\|_{\mathcal{N}}\\ &\le \varepsilon/2 + \|x - y\|_B\|e\|_{\mathcal{N}}\\ &\le \varepsilon/2 + \frac{\varepsilon}{2\|e\|_{\mathcal{N}}}\|e\|_{\mathcal{N}}\\ &= \varepsilon.\end{aligned}$$

The facts $ye \in \mathcal{J}$ and that $\mathcal{J}$ is closed in $\mathcal{N}$ yield $x \in \mathcal{J}$, which implies $\overline{\mathcal{J}} \cap \mathcal{N} \subseteq \mathcal{J}$. Since the reverse inclusion is obvious, we have $\mathcal{J} = \overline{\mathcal{J}} \cap \mathcal{N}$.

(b) $\Rightarrow$ (c). Since $\mathcal{N}_0$ is a closed ideal of $\mathcal{N}$, there exists by (b) a closed ideal $\mathcal{I}$ of B such that $\mathcal{N}_0 = \mathcal{I} \cap \mathcal{N}$. Since $\mathcal{N}_0$ contains B_c, $\mathcal{N}_0$ and hence $\mathcal{I}$ is dense in B, which implies that $\mathcal{I} = B$. Therefore, $\mathcal{N}_0 = \mathcal{I} \cap \mathcal{N} = B \cap \mathcal{N} = \mathcal{N}$.

(c) $\Rightarrow$ (d). Let $x \in \mathcal{N}$ and $0 < \varepsilon (\le 1)$ be arbitrary. Then there is an $x_\varepsilon \in B_c$ such that $\|x - x_\varepsilon\|_{\mathcal{N}} < \varepsilon/(2(C_0 + 1))$, where C_0 is a bound of $\{e_\lambda\}_{\lambda\in\Lambda}$. Choose an $e \in \mathcal{N}$ and a $\lambda_0 \in \Lambda$ such that $x_\varepsilon e = x_\varepsilon$ and

$$\|e_\lambda e - e\|_B < \frac{\varepsilon}{2(\|x_\varepsilon\|_{\mathcal{N}} + 1)} \quad (\lambda \ge \lambda_0).$$

Then we have

$$\begin{aligned}\|e_\lambda x_\varepsilon - x_\varepsilon\|_{\mathcal{N}} &= \|e_\lambda x_\varepsilon e - x_\varepsilon e\|_{\mathcal{N}}\\ &\le \|e_\lambda e - e\|_B\|x_\varepsilon\|_{\mathcal{N}}\\ &\le \frac{\varepsilon}{2(\|x_\varepsilon\|_{\mathcal{N}} + 1)}\|x_\varepsilon\|_{\mathcal{N}}\\ &\le \frac{\varepsilon}{2} \quad (\lambda \ge \lambda_0),\end{aligned}$$

and hence we have

$$\begin{aligned}\|e_\lambda x - x\|_{\mathcal{N}} &= \|e_\lambda(x - x_\varepsilon) + (e_\lambda x_\varepsilon - x_\varepsilon) + (x_\varepsilon - x)\|_{\mathcal{N}} \\ &\le \|e_\lambda\|_B \|x - x_\varepsilon\|_{\mathcal{N}} + \|e_\lambda x_\varepsilon - x_\varepsilon\|_{\mathcal{N}} + \|x_\varepsilon - x\|_{\mathcal{N}} \\ &\le (C_0 + 1)\frac{\varepsilon}{2(C_0+1)} + \varepsilon/2 = \varepsilon \quad (\lambda \ge \lambda_0).\end{aligned}$$

Thus, we get that $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of $\mathcal{N}$.

(d) $\Rightarrow$ (e). Suppose (d). Then an application of the Cohen factorization theorem yields (e).

(e) $\Rightarrow$ (a). Let $x \in \mathcal{N}$ be arbitrary. By (e), there exist an $a \in B$ and a $y \in \mathcal{N}$ such that $x = ay$. For each $\varepsilon > 0$, there exists a $\lambda \in \Lambda$ such that $\|a - ae_\lambda\|_B < \varepsilon/\|y\|_{\mathcal{N}}$. Then we have

$$\|e_\lambda x - x\|_{\mathcal{N}} = \|e_\lambda ay - ay\|_{\mathcal{N}} \le \|y\|_{\mathcal{N}}\|a - ae_\lambda\|_B \le \|y\|_{\mathcal{N}}(\varepsilon/\|y\|_{\mathcal{N}}) = \varepsilon.$$

Since $\{e_\lambda\} \subseteq B_c \subseteq \mathcal{N}$, $\mathcal{N}$ has approximate units, and hence (a) holds. □

3.6. Theorem B′. *Let $\mathcal{S}$ be a Segal algebra in B.*

(i) *The ideal theory of $\mathcal{S}$ is the same as that of B. More precisely, if $\mathcal{I}$ is a closed ideal of B, then $\mathcal{I}\cap\mathcal{S}$ is a closed ideal of $\mathcal{S}$, and conversely each closed ideal of $\mathcal{S}$ is of this form for a unique closed ideal $\mathcal{I}$ of B.*

(ii) *Φ_B and $\Phi_{\mathcal{S}}$ are homeomorphic to each other. More precisely, the map: $\Phi_B \to \Phi_{\mathcal{S}} : \varphi \to \varphi|_{\mathcal{S}}$ is a homeomorphism.*

For proofs of this theorem, we refer to [**2, 6**].

3.7. *Remark.* Theorem B′ (i) does not necessarily hold for abstract Segal algebras. In fact, $\mathcal{N} := L^\infty(\mathbf{T})$ is an abstract Segal algebra (which is the same as a normed ideal) in $B = L^1(\mathbf{T})$ for the circle group $\mathbf{T}$. But the closed ideal $C(\mathbf{T})$ of $\mathcal{N}$ cannot be represented in the form $\mathcal{N}\cap\mathcal{I}$ with any closed ideal $\mathcal{I}$ of B (cf., [**3**, page 277]).

3.8. Theorem C′. (i) *Let $\mathcal{S}$ be a Segal algebra in B, and let $\{e_\lambda\}_{\lambda\in\Lambda}$ be a bounded approximate identity of B composed of elements in B_c. Then $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of $\mathcal{S}$ which is bounded with respect to the multiplication operator norm*

$$\|x\|_{\mathrm{op}} := \sup\{\|xy\|_{\mathcal{S}} : y \in \mathcal{S}, \|y\|_{\mathcal{S}} \le 1\} \quad (x \in \mathcal{S}).$$

(ii) *If a Segal algebra $\mathcal{S}$ in B has a bounded approximate identity, then $\mathcal{S} = B$ holds.*

Proof. (i) $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of $\mathcal{S}$ by Theorem A$'$. It is bounded with respect to the multiplication operator norm since, for each $\lambda_0 \in \Lambda$,

$$\begin{aligned} \|e_{\lambda_0}\|_{\mathrm{op}} &= \sup_{x\in\mathcal{S}, \|x\|_{\mathcal{S}}\le 1} \|e_{\lambda_0}x\|_{\mathcal{S}} \\ &\le \sup_{x\in\mathcal{S}, \|x\|_{\mathcal{S}}\le 1} \|e_{\lambda_0}\|_B \|x\|_{\mathcal{S}} \\ &\le \sup_{\lambda\in\Lambda} \|e_\lambda\|_B < \infty. \end{aligned}$$

(ii) For the proof, we refer to [**2**]. □

3.9. Theorem D$'$. *If $(\mathcal{S}_1, \| \ \|_{\mathcal{S}_1})$ and $(\mathcal{S}_2, \| \ \|_{\mathcal{S}_2})$ are Segal algebras in B, then $\mathcal{S} := \mathcal{S}_1 \cap \mathcal{S}_2$ is a Segal algebra in B with respect to the norm $\| \ \|_{\mathcal{S}} = \| \ \|_{\mathcal{S}_1} + \| \ \|_{\mathcal{S}_2}$.*

Proof. It is easy to see that $(\mathcal{S}, \| \ \|_{\mathcal{S}})$ is a dense Banach ideal in B, and we omit a proof. If we take a bounded approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ of B which is composed of elements in B_c by the condition (β_B), then by Theorem A$'$, $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of $(\mathcal{S}_i, \| \ \|_{\mathcal{S}})$, $i = 1, 2$. Let $x \in \mathcal{S}$ and $\varepsilon > 0$ be arbitrary, and choose λ_i $(i = 1, 2)$ such that $\|x - e_\lambda x\|_{\mathcal{S}_i} \le \varepsilon/2$ $(\lambda \ge \lambda_i)$ for $i = 1, 2$. Therefore, if we take a $\lambda_3 \in \Lambda$ such that $\lambda_3 \ge \lambda_i$ $(i = 1, 2)$, then

$$\begin{aligned} \|x - xe_\lambda\|_{\mathcal{S}} &= \|x - xe_\lambda\|_{\mathcal{S}_1} + \|x - xe_\lambda\|_{\mathcal{S}_2} \\ &\le \varepsilon/2 + \varepsilon/2 = \varepsilon \quad (\lambda \ge \lambda_3). \end{aligned}$$

Thus, $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of $(\mathcal{S}, \| \ \|_{\mathcal{S}})$. Hence, the assertion of the theorem follows from Theorem A$'$. □

If $\mathcal{S}$ is a Segal algebra in B, we can identify $\Phi_{\mathcal{S}}$ with Φ_B by the homeomorphism $\Phi_B \to \Phi_{\mathcal{S}} : \varphi \to \varphi|_{\mathcal{S}}$ (cf., Theorem B$'$). By this identification, the Gelfand transform of an element $x \in \mathcal{S}$ is just equal to the Gelfand transform of x as an element of B.

4. Multiplier algebras of Segal algebras in B.

4.1. Definition. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Segal algebras in B. A map T of $\mathcal{S}_1$ to $\mathcal{S}_2$ is called a multiplier of $\mathcal{S}_1$ to $\mathcal{S}_2$ if T is a bounded linear map satisfying $(Tx)y = x(Ty)$ $(x, y \in \mathcal{S}_1)$. The set of all multipliers of $\mathcal{S}_1$ to $\mathcal{S}_2$ is denoted by $M(\mathcal{S}_1, \mathcal{S}_2)$, and $M(\mathcal{S}_1, \mathcal{S}_1)$ will simply be denoted by $M(\mathcal{S}_1)$.

4.2. Lemma (cf., [**16**, Theorem 1.2.2]). *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Segal algebras in B, and let T be a linear map of $\mathcal{S}_1$ to $\mathcal{S}_2$. Then the following conditions are equivalent*:

(a) $T \in M(\mathcal{S}_1, \mathcal{S}_2)$.

(b) *There exists a unique continuous function σ on Φ_B satisfying $\widehat{Tx} = \widehat{x}\sigma$ $(x \in \mathcal{S}_1)$.*

Furthermore, if $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy the conditions $\mathcal{S}_2 \subseteq \mathcal{S}_1$ and $\|x\|_{\mathcal{S}_1} \leq \|x\|_{\mathcal{S}_2}$ $(x \in \mathcal{S}_2)$, then (b) *is equivalent to the following* (b$'$).

(b$'$) *There exists a unique bounded continuous function σ on Φ_B satisfying $\widehat{Tx} = \widehat{x}\sigma$ $(x \in \mathcal{S}_1)$.*

Proof. (a) $\Rightarrow$ (b). Let $\varphi \in \Phi_B$ and $x, y \in \mathcal{S}_1$ be such that $\widehat{x}(\varphi) \neq 0$, $\widehat{y}(\varphi) \neq 0$. Since $(Tx)y = x(Ty)$, it follows that

$$\frac{\widehat{Tx}(\varphi)}{\widehat{x}(\varphi)} = \frac{\widehat{Ty}(\varphi)}{\widehat{y}(\varphi)}.$$

For each $\varphi \in \Phi_B$, choose $x \in \mathcal{S}_1$ such that $\widehat{x}(\varphi) \neq 0$, and define

$$\sigma(\varphi) := \widehat{Tx}(\varphi)/\widehat{x}(\varphi).$$

The preceding equation shows that the definition is independent of x and hence σ is a well-defined continuous function on Φ_B. Moreover, if $\widehat{x}(\varphi) = 0$ and $\widehat{y}(\varphi) \neq 0$, then $\widehat{Tx}(\varphi)\widehat{y}(\varphi) = \widehat{x}(\varphi)\widehat{Ty}(\varphi) = 0$ implies that $\widehat{Tx}(\varphi) = 0$. Hence, the equation $\widehat{Tx}(\varphi) = \sigma(\varphi)\widehat{x}(\varphi)$ is valid for all $\varphi \in \Phi_B$ and $x \in \mathcal{S}_1$. If τ is a continuous function on Φ_B for which $\widehat{Tx} = \tau\widehat{x}$ $(x \in \mathcal{S}_1)$, then $[\sigma(\varphi) - \tau(\varphi)]\widehat{x}(\varphi) = 0$ for all $x \in \mathcal{S}_1$ and $\varphi \in \Phi_B$. This implies that $\sigma(\varphi) = \tau(\varphi)$ $(\varphi \in \Phi_B)$. Thus σ is unique.

(b) $\Rightarrow$ (a). Since σ is a continuous function on Φ_B such that $\widehat{Tx} = \sigma\widehat{x}$ ($x \in \mathcal{S}_1$), it is easy to see that T satisfies the equation $(Tx)y = x(Ty)$ ($x, y \in \mathcal{S}_1$). The boundedness of T is an easy consequence of the routine argument applying the closed graph theorem.

Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy $\mathcal{S}_2 \subseteq \mathcal{S}_1$ and $\|x\|_{\mathcal{S}_1} \leq \|x\|_{\mathcal{S}_2}$ ($x \in \mathcal{S}_2$). We set $K_{\varphi,i} := \sup\{|\widehat{x}(\varphi)| : x \in \mathcal{S}_i, \|x\|_{\mathcal{S}_i} \leq 1\}$ for each $\varphi \in \Phi_B$ and $i = 1, 2$. Then we have $0 < K_{\varphi,2} \leq K_{\varphi,1} \leq 1$ ($\varphi \in \Phi_B$). For each $x \in \mathcal{S}_1$ and $\varphi \in \Phi_B$,

$$|\sigma(\varphi)\widehat{x}(\varphi)| = |\widehat{Tx}(\varphi)| \leq K_{\varphi,2}\|Tx\|_{\mathcal{S}_2} \leq K_{\varphi,2}\|T\|\|x\|_{\mathcal{S}_1}.$$

In particular, restricting our attention to only those $x \in \mathcal{S}_1$ such that $\|x\|_{\mathcal{S}_1} \leq 1$, we obtain

$$\begin{aligned} |\sigma(\varphi)| &\leq \inf\{K_{\varphi,2}\|T\|/|\widehat{x}(\varphi)| : \widehat{x}(\varphi) \neq 0, \|x\|_{\mathcal{S}_1} \leq 1\} \\ &= K_{\varphi,2}\|T\|/\sup\{|\widehat{x}(\varphi)| : \widehat{x}(\varphi) \neq 0, \|x\|_{\mathcal{S}_1} \leq 1\} \\ &= \frac{K_{\varphi,2}}{K_{\varphi,1}}\|T\| \leq \|T\|. \end{aligned}$$

Thus, $\|\sigma\|_\infty \leq \|T\| < \infty$ follows. This implies (b) $\Rightarrow$ (b)$'$. The converse is trivial. □

4.3. Definition. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Segal algebras in B. For each $T \in M(\mathcal{S}_1, \mathcal{S}_2)$ there exists a unique continuous function σ on Φ_B such that $\widehat{Tx} = \sigma\widehat{x}$ ($x \in \mathcal{S}_1$) by Lemma 4.2. We denote this σ by $\widehat{T}$. The space of all $\widehat{T}$ of $T \in M(\mathcal{S}_1, \mathcal{S}_2)$ will be denoted by $\widehat{M}(\mathcal{S}_1, \mathcal{S}_2)$.

It is easy to see that the map $T \to \widehat{T}$ is a bijection of $M(\mathcal{S}_1, \mathcal{S}_2)$ to $\widehat{M}(\mathcal{S}_1, \mathcal{S}_2)$.

4.4. Lemma [**1**]. *Suppose $\{e_\lambda\}_{\lambda\in\Lambda}$ is a bounded approximate identity of B composed of elements in B_c such that $\sup_{\lambda\in\Lambda}\|e_\lambda\|_B < C_0$. For every $K \in \mathcal{K}(\Phi_B)$, we can choose an $e_K \in B_c$ satisfying $\widehat{e_K}(\varphi) = 1$ ($\varphi \in K$) so that $\sup_{K\in\mathcal{K}(\Phi_B)}\|e_K\|_B < C_0$.*

In particular, $\{e_K\}_{K\in\mathcal{K}(\Phi_B)}$ is a bounded approximate identity of B since B is Tauberian.

Proof. Put $\varepsilon = C_0 - \sup_{\lambda\in\Lambda}\|e_\lambda\|_B$. For any $K \in \mathcal{K}(\Phi_B)$, there exist a $u_K \in B_c$ and a $\lambda_0 \in \Lambda$ such that $\widehat{u_K} = 1$ on K and $\|u_K - u_K e_{\lambda_0}\|_B <$

$\varepsilon/2$ by (α_B) and (β_B). Put $e_K := e_{\lambda_0} + u_K - u_K e_{\lambda_0} \in B_c$. Then $\widehat{e_K}(\varphi) = \widehat{e_{\lambda_0}}(\varphi) + \widehat{u_K}(\varphi) - \widehat{u_K}(\varphi)\widehat{e_{\lambda_0}}(\varphi) = 1$ $(\varphi \in K)$ and

$$\|e_K\|_B \leq \|e_{\lambda_0}\|_B + \|u_K - u_K e_{\lambda_0}\|_B \leq \sup_{\lambda\in\Lambda} \|e_\lambda\|_B + \varepsilon/2 = C_0 - \varepsilon/2 < C_0$$

hold. Thus, we get $\sup_{K\in\mathcal{K}(\Phi_B)} \|e_K\|_B < C_0$. □

4.5. Proposition. *Suppose that B has a bounded approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ composed of elements in B_c such that $\sup_{\lambda\in\Lambda} \|e_\lambda\|_B < C_0$. Then, for any Segal algebra $\mathcal{S}$ in B, we have*:

(i) $Tx \in \mathcal{S}$, $\|Tx\|_{\mathcal{S}} \leq C_0\|T\|\|x\|_{\mathcal{S}}$ $(x \in \mathcal{S},\ T \in M(B))$.

(ii) $\widehat{M}(B) \subseteq \widehat{M}(\mathcal{S})$.

Proof. (i) Let $T \in M(B), x \in B_c$ and $\varepsilon > 0$ be arbitrary. Put $K_x := \operatorname{supp} \widehat{x}$. Then, by Lemma 4.4, there exists an $e_{K_x} \in B_c$ such that $\|e_{K_x}\|_B < C_0$ and $\widehat{e_{K_x}} = 1$ on K_x and

$$\begin{aligned}\|Tx\|_{\mathcal{S}} &= \|T(e_{K_x}x)\|_{\mathcal{S}} = \|(Te_{K_x})x\|_{\mathcal{S}} \leq \|Te_{K_x}\|_B\|x\|_{\mathcal{S}} \\ &\leq \|e_{K_x}\|_B\|T\|\|x\|_{\mathcal{S}} \leq C_0\|T\|\|x\|_{\mathcal{S}}.\end{aligned}$$

Therefore, $T|_{B_c}$ is a bounded linear operator of B_c to B_c with respect to the norm $\| \ \|_{\mathcal{S}}$. Since B_c is dense in $\mathcal{S}$, we can conclude that $T|_{\mathcal{S}}$ is a bounded linear operator of $\mathcal{S}$ of norm at most $C_0\|T\|$.

(ii) If $\widehat{T} \in \widehat{M}(B)$, we have $\widehat{T}\widehat{x} \in \widehat{\mathcal{S}}$ $(x \in \mathcal{S})$ by (i). Therefore, $\widehat{T} \in \widehat{M}(\mathcal{S})$ by Lemma 4.2. □

In the rest of this paper, A stands for a regular Banach function algebra on a locally compact, non-compact Hausdorff space X satisfying the following conditions:

(α_A) A is natural in the sense that any non-zero complex homomorphism φ of A is represented in the form $\varphi(f) = f(x)$ $(f \in A)$ by a unique element $x \in X$.

(β_A) A has a bounded approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ satisfying $e_\lambda \in A_c$ $(\lambda \in \Lambda)$ and $\sup_{\lambda\in\Lambda} \|e_\lambda\|_A < C_0$.

By Lemma 4.4, (β_A) can be replaced by

(β'_A) A is has a bounded approximate identity $\{e_K\}_{K\in\mathcal{K}(X)}$ satisfying: $e_K \in A_c$ with $e_K(x) = 1$ $(x \in K)$, $K \in \mathcal{K}(X)$, and $\sup_{K\in\mathcal{K}(X)} \|e_K\|_A < C_0$.

Obviously, the Banach function algebra $(\widehat{B}, \| \ \|_{\hat{B}})$ on Φ_B is of this type. Under these circumstances, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are Segal algebras in A, the multiplier space of $\mathcal{S}_1$ to $\mathcal{S}_2$ is $M(\mathcal{S}_1, \mathcal{S}_2) = \{\tau \in C(X) : f\tau \in \mathcal{S}_2$ $(f \in \mathcal{S}_1)\}$ with the operator norm $\| \ \|_{M(\mathcal{S}_1,\mathcal{S}_2)}$. Analogously, the multiplier algebra of a Segal algebra $\mathcal{S}$ is $M(\mathcal{S}) = \{\tau \in C_b(X) : f\tau \in \mathcal{S}$ $(f \in \mathcal{S})\}$ with the operator norm $\| \ \|_{M(\mathcal{S})}$.

In the following sections, we introduce new classes of Segal algebras in A and investigate their properties in detail.

5. Segal algebras induced by local A-functions. I.

5.1. Definition. Let σ be a complex-valued continuous function on X satisfying $f\sigma \in A$ for all $f \in A_c$, where $A_c = \{f \in A :$ $\operatorname{supp} f$ is compact in $X\}$. We call such a σ a local A-function, and the set of local A-functions is denoted by A_{loc}.

We remark that the terminology *local A-function* is proper since we will see in Proposition 7.2 that "$\sigma \in C(X)$ belongs to A_{loc} if and only if for every $x \in X$ there exists an $f \in A$ such that $\sigma = f$ on a neighborhood of x."

5.2. Examples. (i) If $\mathcal{S}$ is a Segal algebra in A, each $\sigma \in M(\mathcal{S}, A)$ is a continuous function on X satisfying $\sigma f \in A$ $(f \in \mathcal{S})$. Since $A_c \subseteq \mathcal{S}$, it follows that σ is a local A-function. In the same way, we have that every $\sigma \in M(\mathcal{S})$ is a bounded local A-function by Lemma 4.2.

Conversely, it turns out that every local A-function belongs to $M(\mathcal{S}, A)$ for some Segal algebra $\mathcal{S}$ in A (Theorem 5.4 (ii)). Also, every bounded local A-function belongs to $M(\mathcal{S})$ for some Segal algebra $\mathcal{S}$ in A (Corollary 6.3).

(ii) If $f \in A$ and $f(x) \neq 0$ for all $x \in X$, then $f^{-1} \in A_{\mathrm{loc}}$. Indeed, for every $x \in X$ there is a $g \in A$ such that $f^{-1} = g$ on a neighborhood of x. Hence $f^{-1} \in A_{\mathrm{loc}}$ follows from Proposition 7.2.

(iii) If X is a disjoint union of a family of open compact subsets $\{X_\lambda : \lambda \in \Lambda\}$ of X, then any function on X which is constant on each X_λ is a local A-function. In fact, this follows from Proposition 7.2 since A possesses local units (see the condition (β'_A)).

As a special case, when X is discrete, any complex function on X is a local A-function.

5.3. Definition. For each complex continuous function τ on X and a non-negative integer n, we put

$$A_{\tau(n)} := \{f \in A : f\tau^k \in A \quad (0 \le k \le n)\}$$
$$\|f\|_{\tau(n)} := \sum_{k=0}^{n} \|f\tau^k\|_A \quad (f \in A_{\tau(n)}).$$

Note that $(A_{\tau(0)}, \| \ \|_{\tau(0)})$ is nothing but $(A, \| \ \|_A)$.

5.4. Theorem. *Suppose* $\tau \in A_{\mathrm{loc}}$. *Then we have*:

(i) $\tau \in M(A)$ *if and only if* $A_{\tau(1)} = A$.

(ii) *For each positive integer* n, $(A_{\tau(n)}, \| \ \|_{\tau(n)})$ *is a Segal algebra in* A *and* $\tau \in M(A_{\tau(n)}, A)$.

Proof. (i) Suppose $\tau \in M(A)$. Then $f\tau \in A$ $(f \in A)$, and hence $A_{\tau(1)} = A$. Conversely, if $A_{\tau(1)} = A$, we have $f\tau \in A$ $(f \in A)$ and $\tau \in M(A)$ follows.

(ii) It is easy to see that $A_{\tau(n)}$ is a linear subspace of A and $\| \ \|_{\tau(n)}$ is a norm on $A_{\tau(n)}$. For each $g \in A$ and $f \in A_{\tau(n)}$, we have $(gf)\tau^k = g(f\tau^k) \in A$ $(k = 1, \dots, n)$ with $\|gf\|_{\tau(n)} = \sum_{k=0}^{n} \|(gf)\tau^k\|_A \le \|g\|_A \|f\|_{\tau(n)}$. That $A_{\tau(n)}$ is dense in A follows from the facts that $A_c \subseteq A_{\tau(n)}$ and A satisfies (β_A).

Next we will show that $\| \ \|_{\tau(n)}$ is complete. To see this, let $\{f_i\}_{i=1}^{\infty}$ be a Cauchy sequence in $A_{\tau(n)}$. Then $\lim_{i,j\to\infty} \|f_i - f_j\|_{\tau(n)} = 0$ implies $\lim_{i,j\to\infty} \|f_i\tau^k - f_j\tau^k\|_A = 0$ for $k = 0, \dots, n$, and hence there exist an $f \in A$ and a $g_k \in A(1 \le k \le n)$ such that $\lim_{i\to\infty} \|f_i - f\|_A = \lim_{i\to\infty} \|f_i\tau^k - g_k\|_A = 0$ $(k = 1, \dots, n)$. Since $\lim_{i\to\infty} f_i(x) = f(x)$ and $\lim_{i\to\infty} f_i(x)\tau^k(x) = g_k(x)$ $(x \in X)$, it follows that $f\tau^k = g_k \in A$

$(k = 1, \dots, n)$, and hence $f \in A_{\tau(n)}$. Therefore,

$$\begin{aligned}\lim_{i\to\infty} \|f_i - f\|_{\tau(n)} &= \lim_{i\to\infty} \sum_{k=0}^{n} \|f_i\tau^k - f\tau^k\|_A \\ &= \lim_{i\to\infty} \sum_{k=0}^{n} \|f_i\tau^k - g_k\|_A = 0.\end{aligned}$$

Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be a bounded approximate identity of A composed of elements of A_c. We show that $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of $A_{\tau(n)}$. Let $f \in A_{\tau(n)}$ be arbitrary. Since $\|e_\lambda f - f\|_{\tau(n)} = \sum_{k=0}^{n} \|e_\lambda(f\tau^k) - f\tau^k\|_A$, we obtain the desired result by taking the limit with respect to $\lambda \in \Lambda$.

We have shown that $(A_{\tau(n)}, \|\ \|_{\tau(n)})$ is a dense Banach ideal with an approximate identity, which implies the first assertion of (ii). That $\tau \in M(A_{\tau(n)}, A)$ follows from $\tau f \in A$ $(f \in A_{\tau(n)})$. □

5.5. Theorem. *For $f \in A$ such that* $\operatorname{supp} f$ *is σ-compact but not compact, $f \notin \cap\{A_{\tau(1)} : \tau \in A_{\mathrm{loc}}\}$ holds.*

Proof. Since $\operatorname{supp} f$ is σ-compact, there exists a sequence $\{K_n\}$ of compact subsets of X such that $\operatorname{supp} f = \cup_{n=1}^{\infty} K_n$. Let U_1 be a relatively compact open subset of X such that $K_1 \subseteq U_1$. Since $\operatorname{supp} f$ is not compact, we can choose an $n_1 \in \mathbf{N}$ $(1 < n_1)$ such that $\varnothing \neq \cup_{i=1}^{n_1} K_i \setminus \overline{U_1}$. Let U_2 be a relatively compact open subset of X such that $\overline{U_1} \cup (\cup_{i=1}^{n_1} K_i) \subseteq U_2$. Suppose that we have chosen $\{n_k\}_{k=1}^{N}$ and $\{U_k\}_{k=1}^{N+1}$ such that

$$1 < n_1 < \cdots < n_N,$$
$$\varnothing \neq \cup_{i=1}^{n_k} K_i \setminus \overline{U_k},$$

and

$$\overline{U_k} \cup \left(\cup_{i=1}^{n_k} K_i \right) \subseteq U_{k+1} \quad (1 \leq k \leq N).$$

Choose an $n_{N+1} \in \mathbf{N}(n_N < n_{N+1})$ and a relatively compact open subset U_{N+2} of X so that

$$\varnothing \neq \cup_{i=1}^{n_{N+1}} K_i \setminus \overline{U_{N+1}} \quad \text{and} \quad \overline{U_{N+1}} \cup (\cup_{i=1}^{n_{N+1}} K_i) \subseteq U_{N+2}.$$

By induction on n, we have an increasing sequence $\{U_n\}$ of relatively compact open subsets of X which constitutes an open covering of $\operatorname{supp} f$ such that

$$\operatorname{supp} f \cap \overline{U_1} \subsetneqq \operatorname{supp} f \cap U_2 \subseteq \operatorname{supp} f \cap \overline{U_2} \subsetneqq \operatorname{supp} f \cap U_3 \subseteq \cdots .$$

For each n, take an element $y_n \in \operatorname{supp} f \cap (U_{n+1} \setminus \overline{U_n})$. Since $\{x \in X : f(x) \neq 0\}$ is dense in $\operatorname{supp} f$ and $\operatorname{supp} f \cap (U_{n+1} \setminus \overline{U_n})$ is an open neighborhood of $y_n \in \operatorname{supp} f$, we can find an element $z_n \in \operatorname{supp} f \cap (U_{n+1} \setminus \overline{U_n})$ such that $f(z_n) \neq 0$, and put

$$G_n = (U_{n+1} \setminus \overline{U_n}) \cap \{x \in X : f(x) \neq 0\},$$

and we can choose an $f_n \in A$ so that $f_n(z_n) = 1/f(z_n)$ and $\operatorname{supp} f_n \subseteq G_n$ by Lemma 3.4.

We next consider a complex function τ on X defined by

$$\tau(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in X).$$

Since $G_1, G_2, G_3, \cdots$ are mutually disjoint, τ is well defined. Here we assert that $\operatorname{supp} \tau = \cup_{n=1}^{\infty} \operatorname{supp} f_n$. In fact, for each $n \in \mathbf{N}$, we have

$$\tau(x) = \sum_{i=1}^{n} f_i(x) \quad (x \in U_{n+1}), \tag{1}$$

and hence $\operatorname{supp} \tau \cap U_{n+1} = \cup_{i=1}^{n} \operatorname{supp} f_i$. From this, it follows that $\operatorname{supp} \tau \cap (\cup_{n=1}^{\infty} U_{n+1}) = \cup_{i=1}^{\infty} \operatorname{supp} f_i$ and, with the relation $\operatorname{supp} \tau \subseteq \operatorname{supp} f \subseteq \cup_{n=1}^{\infty} U_{n+1}$, we have $\operatorname{supp} \tau = \cup_{i=1}^{\infty} \operatorname{supp} f_i$.

To see $\tau \in C(X)$, it suffices to show that τ is continuous at all points of $\operatorname{supp} \tau$. If $x \in \operatorname{supp} \tau$, we have $x \in \operatorname{supp} f_n$ for some n from above. Since $\operatorname{supp} f_n \subset G_n$ and $\tau = f_n$ on G_n, τ is continuous at x.

If $g \in A_c$, $\operatorname{supp} g \cap \operatorname{supp} f$ is compact. Since $\{U_n\}_{n=1}^{\infty}$ is an open covering of $\operatorname{supp} f$, there is a positive integer n such that $\operatorname{supp} g \cap \operatorname{supp} f \subseteq U_{n+1}$. Then $\operatorname{supp} g \subseteq (U_{n+1} \cap \operatorname{supp} f) \cup (X \setminus \operatorname{supp} f)$ and $\tau = \sum_{k=1}^{n} f_k$ on $U_{n+1} \cup (X \setminus \operatorname{supp} f)$ by (1). This implies that $\tau g = (\sum_{k=1}^{n} f_k) g \in A$, and hence $\tau \in A_{\mathrm{loc}}$.

Finally we show $f \notin A_{\tau(1)}$. Suppose on the contrary that $f \in A_{\tau(1)}$. Then $\tau f \in A \subseteq C_0(X)$, and hence there exists a compact subset K of X such that $|\tau(x)f(x)| < 1/2$ $(x \in X \setminus K)$. Since $\{U_n \cup (X \setminus \operatorname{supp} f) : n = 1, 2, 3, \dots\}$ is an increasing sequence of open subsets of X which constitutes a covering of X, there exists an $n_0 \in \mathbf{N}$ such that $K \subseteq U_{n_0} \cup (X \setminus \operatorname{supp} f)$. Since $z_{n_0} \notin U_{n_0} \cup (X \setminus \operatorname{supp} f)$, $|\tau(z_{n_0})f(z_{n_0})| < 1/2$ must hold. On the other hand, $|\tau(z_{n_0})f(z_{n_0})| = |f_{n_0}(z_{n_0})f(z_{n_0})| = 1$ holds by the definition of τ. Thus, we arrive at a contradiction. Therefore, $f \notin A_{\tau(1)}$. □

5.6. Corollary. *If X is a disjoint union of σ-compact open closed subsets of X, then*

$$A_c = \cap\{A_{\tau(1)} : \tau \in A_{\mathrm{loc}}\}$$

holds.

Proof. Let $f \in A \setminus A_c$ be arbitrary. To prove the corollary, it suffices to show that $f \notin \cap\{A_{\tau(1)} : \tau \in A_{\mathrm{loc}}\}$. Suppose that X is a disjoint union of a family $\{X_\lambda\}_{\lambda \in \Lambda}$ of σ-compact open closed subsets of X. For each positive integer n, $\{x \in X : |f(x)| \geq 1/n\}$ is compact and hence covered by a union of a finite subfamily of $\{X_\lambda\}_{\lambda\in\Lambda}$. Therefore, there is a countable subfamily $\{X_{\lambda_n}\}_{n=1}^{\infty}$ such that $\cup_{n=1}^{\infty} X_{\lambda_n} \supseteq \{x \in X : f(x) \neq 0\}$. In this case, each X_{λ_n} is σ-compact and $\cup_{n=1}^{\infty} X_{\lambda_n}$ is closed in X by the assumption on $\{X_\lambda\}_{\lambda\in\Lambda}$. From this, it follows that $\operatorname{supp} f$ is a closed subset of a σ-compact set $\cup_{n=1}^{\infty} X_{\lambda_n}$. Thus, $\operatorname{supp} f$ is σ-compact since a closed subset of a σ-compact set is also σ-compact. Therefore, $f \notin \cap\{A_{\tau(1)} : \tau \in A_{\mathrm{loc}}\}$ from Theorem 5.5. □

5.7. *Remark.* Corollary 5.6 is applicable for A with discrete X and for A which is the Fourier algebra on a non-compact LCA group.

5.8. Proposition. *For a Segal algebra $\mathcal{S}$ in A, the following are equivalent.*

(a) $M(\mathcal{S}, A) = M(A)$.

(b) *There is no $\tau \in A_{\mathrm{loc}}$ satisfying $\mathcal{S} \subset A_{\tau(1)} \subsetneqq A$.*

Proof. (a) $\Rightarrow$ (b). If (b) does not hold, there is a $\tau \in A_{\mathrm{loc}}$ satisfying $\mathcal{S} \subset A_{\tau(1)} \subsetneqq A$. Then $\tau \in M(\mathcal{S}, A) \setminus M(A)$ by Theorem 5.4, that is, (a) does not hold.

(b) $\Rightarrow$ (a). If (a) does not hold, there is a $\tau \in M(\mathcal{S}, A) \setminus M(A)$. Then it is easy to see by Theorem 5.4 that $\mathcal{S} \subset A_{\tau(1)} \subsetneqq A$. Thus (b) does not hold. $\square$

5.9. Corollary. *Let $\mathcal{A}$ be the Fourier algebra on a non-compact and non-discrete LCA group G. Suppose that $\mathcal{S} = \mathcal{S}_p(\widehat{G})$ $(1 < p < \infty)$ or $\mathcal{S} = \mathcal{A}_p(\widehat{G})$ $(1 \leq p < \infty)$ (cf., Example 2.2). Then there is no $\tau \in \mathcal{A}_{\mathrm{loc}}$ satisfying $\widehat{\mathcal{S}} \subseteq \mathcal{A}_{\tau(1)} \subsetneqq \mathcal{A}$.*

Proof. Since $M(\mathcal{S}, L^1(\widehat{G})) = M(L^1(\widehat{G}))$ holds (see [**16**, Corollary 3.5.1] for $\mathcal{S} = \mathcal{S}_p(\widehat{G})$ and [**7**, Theorem 3.4] for $\mathcal{S} = \mathcal{A}_p(\widehat{G})$), we have the assertion of the corollary from Proposition 5.8. $\square$

In contrast to Corollary 5.9, for $\mathcal{S} = \mathcal{A}_{\nu,p}(\widehat{G})$ of Example 2.2 (3), we have the following result.

5.10. Proposition. *Let $\mathcal{A}$ be the Fourier algebra on an infinite discrete abelian group G, and suppose that $\tau \in \mathcal{A}_{\mathrm{loc}}$ with $0 < \inf_{x\in G}\tau(x) \leq \sup_{x\in G}\tau(x) = \infty$. We define an unbounded Radon measure ν on G by $\nu := \tau m_G$, where m_G is the normalized Haar measure on G. Then we have $\widehat{\mathcal{A}}_{\nu,1}(\widehat{G}) \subseteq \mathcal{A}_{\tau(1)} \subsetneqq \mathcal{A}$.*

Proof. The first inclusion follows from

$$\begin{aligned} f \in \widehat{\mathcal{A}}_{\nu,1}(\widehat{G}) &\Longleftrightarrow \int_G |f(x)|\, d\tau(x) m_G(x) < \infty \\ &\Longrightarrow f\tau \in L^1(G) \subseteq L^2(G) \subseteq \mathcal{A} \\ &\Longrightarrow f \in \mathcal{A}_{\tau(1)}. \end{aligned}$$

Since $\tau \notin M(\mathcal{A})$, $\mathcal{A}_{\tau(1)} \subsetneqq \mathcal{A}$ follows from Theorem 5.4 (i). $\square$

6. Segal algebras induced by local A-functions. II.

6.1. Definition. Suppose $\tau \in A_{\mathrm{loc}}$, and define $A_{\tau(\infty)}$ by

$$A_{\tau(\infty)} := \Big\{ f \in A : f\tau^k \in A \ (k = 0, 1, 2, \dots),\ \sum_{k=0}^{\infty} \|f\tau^k\|_A < \infty \Big\},$$

and put

$$\|f\|_{\tau(\infty)} := \sum_{k=0}^{\infty} \|f\tau^k\|_A \ (f \in A_{\tau(\infty)}).$$

6.2. Theorem. *Let $\tau \in A_{\mathrm{loc}}$ with $\|\tau\|_\infty < 1/C_0$, where C_0 is the constant in (β_A). Then we have the following*:

(i) $(A_{\tau(\infty)}, \| \ \|_{\tau(\infty)})$ *is a Segal algebra in A.*

(ii) $\tau \in M(A_{\tau(\infty)})$ *and* $\|\tau\|_{M(A_{\tau(\infty)})} \le 1$.

(iii) *If $\tau \notin M(A)$, we have $A_{\tau(\infty)} \subsetneqq A$.*

Proof. (i) It is easy to see that $A_{\tau(\infty)}$ is a linear subspace of A and $\| \ \|_{\tau(\infty)}$ is a norm on $A_{\tau(\infty)}$. For each $g \in A$ and $f \in A_{\tau(\infty)}$, $(gf)\tau^k = g(f\tau^k) \in A$ $(k = 0, 1, 2, \dots)$ hold. Since

$$\begin{aligned}\sum_{k=0}^{\infty} \|(gf)\tau^k\|_A = \sum_{k=0}^{\infty} \|g(f\tau^k)\|_A &\le \sum_{k=0}^{\infty} \|g\|_A \|f\tau^k\|_A \\ &= \|g\|_A \|f\|_{\tau(\infty)} < \infty,\end{aligned}$$

we have $gf \in A_{\tau(\infty)}$ and $\|gf\|_{\tau(\infty)} \le \|g\|_A \|f\|_{\tau(\infty)}$. The space $A_{\tau(\infty)}$ is dense in A since $A_c \subseteq A_{\tau(\infty)}$. In fact, if $f \in A_c$, then $f\tau^k \in A$ $(k = 1, 2, 3, \dots)$ holds by the definition of local A-functions. Put $K := \operatorname{supp} f$. By the condition (β'_A), we have an $e_K \in A_c$ with $e_K(x) = 1$ on K and $\|e_K\|_A < C_0$. If we put $g = e_K\tau$, we have $\|g\|_\infty < 1-\varepsilon$ for some $\varepsilon > 0$. By the spectral radius formula, there is an $n_0 \in \mathbf{N}$ such that $\|g^{n_0}\|_A^{1/n_0} < 1 - \varepsilon$. Since $f\tau^k = fg^k$ $(k = 0, 1, 2, \dots)$, we have

$$\begin{aligned}\sum_{k=0}^{\infty}\|f\tau^k\|_A &= \sum_{k=0}^{\infty}\|fg^k\|_A = \sum_{j=0}^{\infty}\sum_{k=0}^{n_0-1}\|fg^{k+jn_0}\|_A \\ &\le \sum_{j=0}^{\infty}\|g^{n_0}\|_A^j\Big(\sum_{k=0}^{n_0-1}\|fg^k\|_A\Big) \\ &\le \sum_{j=0}^{\infty}((1-\varepsilon)^{n_0})^j\left(\sum_{k=0}^{n_0-1}\|fg^k\|_A\right) < \infty.\end{aligned}$$

Hence, $f \in A_{\tau(\infty)}$. To see $\|\ \|_{\tau(\infty)}$ is complete, let $\{f_n\}$ be a Cauchy sequence in $A_{\tau(\infty)}$. Then

$$\|f_n - f_m\|_{\tau(\infty)} = \sum_{k=0}^{\infty}\|f_n\tau^k - f_m\tau^k\|_A \longrightarrow 0 \quad (n, m \to \infty).$$

It follows that, for each $k (k = 0, 1, 2, \ldots)$, $\{f_n\tau^k\}$ is a Cauchy sequence in A, and so there exists a $g_k \in A$ such that $\lim_{n\to\infty}\|f_n\tau^k - g_k\|_A = 0$. Since $g_0(x) = \lim_{n\to\infty} f_n(x)$ and $g_k(x) = \lim_{n\to\infty} f_n(x)\tau^k(x) = g_0(x)\tau^k(x)$ $(x \in X)$, we get $g_0\tau^k = g_k \in A$ $(k = 1, 2, 3, \ldots)$.

Now let us show that $g_0 \in A_{\tau(\infty)}$. Since $\{f_n\}$ is a Cauchy sequence in $A_{\tau(\infty)}$, it forms a bounded set in $A_{\tau(\infty)}$, that is, there is an $M > 0$ such that $\sum_{k=0}^{\infty}\|f_n\tau^k\|_A \le M$ $(n = 0, 1, 2, \ldots)$. For each $k_0 \in \mathbf{N}$, if we choose an $n_0 \in \mathbf{N}$ so that $\sum_{k=0}^{k_0}\|g_0\tau^k - f_{n_0}\tau^k\|_A < 1$, we have

$$\sum_{k=0}^{k_0}\|g_0\tau^k\|_A \le \sum_{k=0}^{k_0}\|g_0\tau^k - f_{n_0}\tau^k\|_A + \sum_{k=0}^{k_0}\|f_{n_0}\tau^k\|_A < 1 + M,$$

which implies $\sum_{k=0}^{\infty}\|g_0\tau^k\|_A \le 1 + M$ and hence $g_0 \in A_{\tau(\infty)}$.

We claim that $\|f_n - g_0\|_{\tau(\infty)} \to 0$ $(n \to \infty)$. Given $\varepsilon > 0$, let $n_1 \in \mathbf{N}$ be such that $\|f_n - f_{n_1}\|_{\tau(\infty)} < \varepsilon/3$ $(n_1 \le n)$. Choose a $k_1 \in \mathbf{N}$ so that $\sum_{k=k_1+1}^{\infty}\|g_0\tau^k\|_A < \varepsilon/6$ and $\sum_{k=k_1+1}^{\infty}\|f_{n_1}\tau^k\|_A < \varepsilon/6$. Choose also an $n_2 \in \mathbf{N}$ $(n_2 \ge n_1)$ so that $\sum_{k=0}^{k_1}\|g_0\tau^k - f_n\tau^k\|_A < \varepsilon/3$ $(n \ge n_2)$.

Then we have

$$\begin{aligned}\|f_n - g_0\|_{\tau(\infty)} &\le \sum_{k=0}^{k_1} \|f_n\tau^k - g_0\tau^k\|_A \\ &\quad + \sum_{k=k_1+1}^{\infty} \left(\|f_n\tau^k - f_{n_1}\tau^k\|_A + \|g_0\tau^k\|_A + \|f_{n_1}\tau^k\|_A\right) \\ &\le \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{6}\varepsilon + \frac{1}{6}\varepsilon = \varepsilon \quad (n \ge n_2).\end{aligned}$$

Hence, $\lim_{n\to\infty} \|f_n - g_0\|_{\tau(\infty)} = 0$. Therefore, $\| \; \|_{\tau(\infty)}$ is complete.

Finally, let $\{e_\lambda\}_{\lambda\in\Lambda}$ be a bounded approximate identity of A composed of elements in A_c with a bound C_0 (see (β_A)). We will observe that $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of $A_{\tau(\infty)}$. Let $f \in A_{\tau(\infty)}$ and $\varepsilon > 0$ be arbitrary. Choose a positive integer n_0 such that $(C_0 + 1)\sum_{k=n_0+1}^{\infty} \|f\tau^k\|_A < \varepsilon/2$. Choose a $\lambda_0 \in \Lambda$ such that $\sum_{k=0}^{n_0} \|e_\lambda f\tau^k - f\tau^k\|_A < \varepsilon/2$ $(\lambda \ge \lambda_0)$. Then we obtain

$$\begin{aligned}\|e_\lambda f - f\|_{\tau(\infty)} &\le \sum_{k=0}^{n_0} \|e_\lambda f\tau^k - f\tau^k\|_A \\ &\quad + \sum_{k=n_0+1}^{\infty} \|e_\lambda f\tau^k\|_A + \sum_{k=n_0+1}^{\infty} \|f\tau^k\|_A \\ &\le \varepsilon/2 + (C_0+1)\sum_{k=n_0+1}^{\infty} \|f\tau^k\|_A \le \varepsilon \quad (\lambda \ge \lambda_0).\end{aligned}$$

Thus, $(A_{\tau(\infty)}, \| \; \|_{\tau(\infty)})$ is a dense Banach ideal of A with an approximate identity, which implies (i).

(ii) If $f \in A_{\tau(\infty)}$, then $f\tau \in A_{\tau(\infty)}$ and $\|f\tau\|_{\tau(\infty)} \le \|f\|_{\tau(\infty)}$. So $\tau \in M(A_{\tau(\infty)})$ and $\|\tau\|_{M(A_{\tau(\infty)})} \le 1$.

(iii) Let $\tau \notin M(A)$. Then we have $A_{\tau(1)} \subsetneqq A$ by Theorem 5.4 (i). Since $A_{\tau(\infty)} \subseteq A_{\tau(1)}$, it follows that $A_{\tau(\infty)} \subsetneqq A$. □

6.3. Corollary. *Let $\sigma \in A_{\mathrm{loc}} \cap C_b(X)$. Then $\sigma \in M(\mathcal{S})$ for some Segal algebra $\mathcal{S}$ in A.*

Proof. Put $\tau = \sigma/C$, where $C = C_0(\|\sigma\|_\infty + 1)$. Then we have $\tau \in A_{\mathrm{loc}}$ with $\|\tau\|_\infty < 1/C_0$. By Theorem 6.2 (ii), $A_{\tau(\infty)}$ is a Segal algebra in A with $\tau \in M(A_{\tau(\infty)})$, and hence $\sigma = C\tau \in M(\mathcal{S})$ with $\mathcal{S} = A_{\tau(\infty)}$. □

6.4. Theorem. *Suppose that $\mathcal{S}$ is a Segal algebra in A satisfying $M(\mathcal{S}) \supsetneqq M(A)$. If $\sigma \in M(\mathcal{S}) \setminus M(A)$, then there exists a $\tau \in A_{\mathrm{loc}}$ which satisfies*:

(i) $\|\tau\|_\infty < 1/C_0$ *and* $\mathcal{S} \subseteq A_{\tau(\infty)} \subsetneqq A$,

(ii) $\sigma \in M(A_{\tau(\infty)})$).

Proof. $\sigma \in A_{\mathrm{loc}}$ as we saw in Example 5.2 (i). Choose an $\varepsilon > 0$ so that $\|\varepsilon\sigma\|_\infty < 1/C_0$ and $\|\sigma\|_{\mathrm{op}} := \sup_{0\neq f\in\mathcal{S}} \|f\sigma\|_{\mathcal{S}}/\|f\|_{\mathcal{S}} < 1/\varepsilon$. Put $\tau = \varepsilon\sigma$. Then $\tau \notin M(A)$, and we have by Theorem 6.2 that $A_{\tau(\infty)} \subsetneqq A$ with $\tau \in M(A_{\tau(\infty)})$. Hence, $\sigma \in M(A_{\tau(\infty)})$. We claim that $\mathcal{S} \subseteq A_{\tau(\infty)}$. To see this, let $f \in \mathcal{S}$ be arbitrary. Since $f\tau^k = \varepsilon^k(f\sigma^k) \in \mathcal{S} \subseteq A$ for all $k = 0, 1, 2, \ldots,$ and

$$\begin{aligned}\sum_{k=0}^{\infty} \|f\tau^k\|_A \leq \sum_{k=0}^{\infty} \|f\tau^k\|_{\mathcal{S}} &= \sum_{k=0}^{\infty} \|f(\varepsilon^k\sigma^k)\|_{\mathcal{S}} \\ &\leq \|f\|_{\mathcal{S}} \sum_{k=0}^{\infty} \left(\|\varepsilon\sigma\|_{\mathrm{op}}\right)^k < \infty.\end{aligned}$$

It follows that $f \in A_{\tau(\infty)}$. Hence, $\mathcal{S} \subseteq A_{\tau(\infty)}$ as required. □

6.5. Proposition. *Suppose $\tau \in M(A)$ with $\|\tau\|_\infty < 1/C_0$, and let $\rho_\tau = \lim_{n\to\infty} \|\tau^n\|_{M(A)}^{1/n}$, the spectral radius of τ. Then we have the following.*

(i) $A_{\tau(\infty)}$ *coincides with A for $\rho_\tau < 1$.*

(ii) $A_{\tau(\infty)}$ *is a proper Segal algebra in A satisfying $M(A) \subsetneqq M(A_{\tau(\infty)})$ for $\rho_\tau > 1$.*

Proof. (i) By the condition on τ and the spectral radius formula, there exists an $n_0 \in \mathbf{N}$ and an ε $(0 < \varepsilon < 1)$ such that $\|\tau^{n_0}\|_{M(A)}^{1/n_0} \leq (1 - \varepsilon)$.

Then, for each $f \in A$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \|f\tau^n\|_A &= \sum_{j=0}^{\infty} \sum_{k=0}^{n_0-1} \|\tau^{n_0 j+k} f\|_A \\ &\leq \sum_{j=0}^{\infty} \|\tau^{n_0}\|_{M(A)}^{j} \sum_{k=0}^{n_0-1} \|f\tau^k\|_A \\ &\leq \sum_{j=0}^{\infty} (1-\varepsilon)^{n_0 j} \sum_{k=0}^{n_0-1} \|f\tau^k\|_A < \infty.\end{aligned}$$

Hence, $f \in A_{\tau(\infty)}$. Thus, $A_{\tau(\infty)} = A$ holds.

(ii) Suppose that $M(A) \subsetneqq M(A_{\tau(\infty)})$ does not hold. Then by Proposition 4.5 (ii), we have $M(A) = M(A_{\tau(\infty)})$. In this case two norms $\| \ \|_{M(A_{\tau(\infty)})}$ and $\| \ \|_{M(A)}$ are equivalent and $\|\tau\|_{M(A_{\tau(\infty)})} \leq 1$ by Theorem 6.2 (ii), and hence we arrive at the following contradiction: $1 < \rho_\tau = \lim_{n\to\infty}(\|\tau^n\|_{M(A)})^{1/n} = \lim_{n\to\infty}(\|\tau^n\|_{M(A_{\tau(\infty)})})^{1/n} \leq 1$. Thus, we have $M(A) \subsetneqq M(A_{\tau(\infty)})$, and this also implies $A_{\tau(\infty)} \subsetneqq A$. □

6.6. *Remarks.* Suppose that $\mathcal{A}$ is the Fourier algebra on a non-compact LCA group G. In this case we note that, for any $\varepsilon > 0$, we can set a C_0 in the condition $(\beta_A)'$ so that $1 < C_0 < 1 + \varepsilon$ by [**20**, Theorem 2.6.8].

(i) There exists a $\sigma \in M(\mathcal{A})$ such that $\rho_\sigma = 2$ and $-1 \leq \sigma(g) \leq 1$ for all $g \in G$ (cf., the proof of [**20**, Theorem 5.3.4]). Put $\tau := 2\sigma/3$. Then we have $\rho_\tau = 4/3$ and $\|\tau\|_\infty \leq 2/3$. To this τ we can apply Proposition 6.5 (ii).

(ii) There exists a $\tau \in M(\mathcal{A})$ such that $0 < \inf_{x\in G} |\tau(x)|$ and $\tau^{-1} \notin M(\mathcal{A})$ by [**20**, Theorem 5.3.4]. But, in this case, $\tau^{-1} \in \mathcal{A}_{\mathrm{loc}}$ (Example 5.2 (ii)), and by Corollary 6.3, there exists a Segal algebra $\mathcal{S}$ in $\mathcal{A}$ such that $\tau^{-1} \in M(\mathcal{S})$.

7. Characterizations of local A-functions. In the rest of this paper we study BSE-^{1} and BED-properties of commutative Banach algebras. We now give some definitions and notations for later use.

For a subset F of X, $\mathrm{span}\,(F)$ is a linear span of F in A^*, the dual space of A. Any element $p \in \mathrm{span}\,(F)$ is represented by $p =$

$\sum_{x\in F}\widehat{p}(x)\delta_x$, where $\widehat{p}(x)=0$ except for a finite number of x in F and $\delta_x(f)=f(x)$ $(f\in A, x\in X)$. For a complex-valued function τ on F, we define τp by $\tau p := \sum_{x\in F}\tau(x)\widehat{p}(x)\delta_x \in \mathrm{span}\,(F)$.

For $\sigma\in C(X)$ and a proper subset F of X, define

$$\|\sigma\|_{\mathrm{BSE}\,(\mathcal{S})} := \sup\left\{\left|\sum_{x\in X}\widehat{p}(x)\sigma(x)\right| : p\in \mathrm{span}\,(X), \|p\|_{\mathcal{S}^*}\le 1\right\},$$

$$\|\sigma\|_{\mathrm{BSE}\,(\mathcal{S}),F} := \sup\left\{\left|\sum_{x\in X}\widehat{p}(x)\sigma(x)\right| : p\in \mathrm{span}\,(X\setminus F), \|p\|_{\mathcal{S}^*}\le 1\right\}.$$

We define $C_{\mathrm{BSE}\,(\mathcal{S})}(X)$ and $C^0_{\mathrm{BSE}\,(\mathcal{S})}(X)$ by

$$C_{\mathrm{BSE}\,(\mathcal{S})}(X) := \{\sigma\in C(X) : \|\sigma\|_{\mathrm{BSE}\,(\mathcal{S})}<\infty\},$$
$$C^0_{\mathrm{BSE}\,(\mathcal{S})}(X) := \{\sigma\in C_{\mathrm{BSE}\,(\mathcal{S})}(X) : \lim_{K\in\mathcal{K}(X)}\|\sigma\|_{\mathrm{BSE}\,(\mathcal{S}),K}=0\}.$$

Then $(C_{\mathrm{BSE}\,(\mathcal{S})}(X), \|\ \|_{\mathrm{BSE}\,(\mathcal{S})})$ is a Banach function algebra on X, and $C^0_{\mathrm{BSE}\,(\mathcal{S})}(X)$ is its closed ideal. We say that $\mathcal{S}$ is BSE if $C_{\mathrm{BSE}\,(\mathcal{S})}(X) = M(\mathcal{S})$, and BED if $C^0_{\mathrm{BSE}\,(\mathcal{S})}(X)=\mathcal{S}$.

For simplicity, we write $\|\sigma\|_{\mathrm{BSE}}, \|\sigma\|_{\mathrm{BSE},F}, C_{\mathrm{BSE}}(X)$ and $C^0_{\mathrm{BSE}}(X)$ instead of $\|\sigma\|_{\mathrm{BSE}(\mathcal{S})}, \|\sigma\|_{\mathrm{BSE}(\mathcal{S}),F}, C_{\mathrm{BSE}\,(\mathcal{S})}(X)$ and $C^0_{\mathrm{BSE}\,(\mathcal{S})}(X)$, respectively, in the case of $\mathcal{S}=A$. For the details of these and related subjects we refer to [**14, 21**].

The next lemma will be applied in this section and in Section 10.

7.1. Lemma. *Let E be a non-empty closed subset of X. Put $I(E)=\{f\in A : f(x)=0\ (x\in E)\}$. Then we have the following.*

(i) $\|p\|_{(A/I(E))^*}=\|p\|_{A^*}$ $(p\in\mathrm{span}\,(E))$.

(ii) $\|\tau|_E\|_{\mathrm{BSE}\,(A/I(E))}=\|\tau\|_{\mathrm{BSE},X\setminus E}$ $(\tau\in C(X))$.

Proof. (i) Suppose $p\in\mathrm{span}\,(E)$. Then we have

$$\begin{aligned}\|p\|_{(A/I(E))^*} &= \sup\left\{\left|\sum_{x\in E}\widehat{p}(x)f(x)\right| : f\in A,\ \|f+I(E)\|_{A/I(E)}\le 1\right\}\\ &= \sup\left\{\left|\sum_{x\in E}\widehat{p}(x)f(x)\right| : f\in A,\ \|f\|_{A}\le 1\right\} = \|p\|_{A^*}.\end{aligned}$$

(ii) Suppose $\tau \in C(X)$. Then we have, using (i),

$$\begin{aligned}
&\|\tau|_E\|_{\mathrm{BSE}\,(A/I(E))} \\
&\qquad = \sup\left\{\left|\sum_{x\in E}\widehat{p}(x)\tau(x)\right| : p \in \operatorname{span}(E), \|p\|_{(A/I(E))^*} \le 1\right\} \\
&\qquad = \sup\left\{\left|\sum_{x\in E}\widehat{p}(x)\tau(x)\right| : p \in \operatorname{span}(E), \|p\|_{A^*} \le 1\right\} \\
&\qquad = \|\tau\|_{\mathrm{BSE},X\setminus E}. \qquad \square
\end{aligned}$$

7.2. Proposition. *Let τ be a complex-valued continuous function on X. The following conditions are equivalent*:

(a) $\tau \in A_{\mathrm{loc}}$.

(b) *For each positive integer n, $(A_{\tau(n)}, \|\ \|_{\tau(n)})$ is a Segal algebra in A.*

(c) *There exists a Segal algebra $\mathcal{S}$ in A such that $\tau \in M(\mathcal{S}, A)$.*

(d) *For every non-empty compact subset K of X, there is an $f \in A$ such that $\tau(x) = f(x)$ $(x \in K)$.*

(e) *For any $x \in X$, there exists an $f \in A$ such that $\tau = f$ on a neighborhood of x.*

If $\tau \in C_b(X)$, each of the above conditions is equivalent to the following (c)$'$.

(c)$'$ *There exists a Segal algebra $\mathcal{S}$ in A such that $\tau \in M(\mathcal{S})$.*

Proof. (a) $\Rightarrow$ (b). This follows from Theorem 5.4 (ii).

(b) $\Rightarrow$ (c). Suppose that (b) holds. We have $\tau \in M(A_{\tau(1)}, A)$ and (c) holds with $\mathcal{S} = A_{\tau(1)}$.

(c) $\Rightarrow$ (d). Suppose (c). For each $K \in \mathcal{K}(X)$, we have an $e_K \in A_c \subseteq \mathcal{S}$ with $e_K = 1$ on K by (β'_A). Put $f = \tau e_K$. Then $f \in A$ by (c), and $f(x) = \tau(x)e_K(x) = \tau(x)$ for all $x \in K$.

(d) $\Rightarrow$ (e). Given $x \in X$, choose a compact neighborhood U_x of x. By (d), there exists an $f \in A$ such that $\tau = f$ on U_x.

(e) $\Rightarrow$ (a). Let $f \in A_c$ be arbitrary. Then $f\tau$ belongs locally to A at any point $x \in X$ and at infinity. Since A has local units with small supports by Lemma 3.4 (i), it is easy to see that $f\tau \in A$ by applying localization lemma [**18**, Lemma 2.1.8]. Hence, $\tau \in A_{\mathrm{loc}}$.

Further, if $\tau \in C_b(X)$, we see that (a) and (c)$'$ are equivalent by Example 5.2 (i) and Corollary 6.3. □

The proof of the next lemma is almost the same as that of Lemma 1 (i) of [**21**], and we omit its proof.

7.3. Lemma. *Let $\sigma_1, \sigma_2 \in C(X)$ and F be a proper subset of X such that $\|\sigma_i\|_{\mathrm{BSE},F} < \infty$ $(i = 1, 2)$. Then we have*

$$\|\sigma_1\sigma_2\|_{\mathrm{BSE},F} \le \|\sigma_1\|_{\mathrm{BSE},F}\|\sigma_2\|_{\mathrm{BSE},F}.$$

7.4. Theorem. *Let $\mathcal{A}$ be the Fourier algebra on a non-compact* LCA *group G. For any continuous function σ on G, the following are equivalent*:

(a) $\sigma \in \mathcal{A}_{\mathrm{loc}}$.

(b) $\|\sigma\|_{\mathrm{BSE},G\setminus K} < \infty$ $(\varnothing \neq K \in \mathcal{K}(G))$.

(c) *For every $x \in G$, there is a compact neighborhood $\overline{V}$ of x such that $\|\sigma\|_{\mathrm{BSE},G\setminus\overline{V}} < \infty$.*

Proof. (a) $\Rightarrow$ (b). Let K be a non-empty compact subset of G. Then there is an $e_K \in \mathcal{A}_c$ such that $e_K = 1$ on K. Suppose that $\sigma \in \mathcal{A}_{\mathrm{loc}}$. Then $\sigma e_K \in \mathcal{A}$ by the definition of local $\mathcal{A}$-functions, and hence

$$\begin{aligned}\|\sigma\|_{BSE,G\setminus K} &= \sup\left\{\left|\sum_{x\in K}\widehat{p}(x)\sigma(x)\right| : p \in \mathrm{span}\,(K), \|p\|_{\mathcal{A}^*} \le 1\right\}\\ &\le \sup\left\{\left|\sum_{x\in G}\hat{p}(x)\sigma(x)e_K(x)\right| : p \in \mathrm{span}\,(G), \|p\|_{\mathcal{A}^*} \le 1\right\}\\ &= \|\sigma e_K\|_{\mathrm{BSE}} < \infty.\end{aligned}$$

(b) $\Rightarrow$ (c). Trivial.

(c) $\Rightarrow$ (a). Let $x \in G$ be given arbitrarily. By (c) there is a compact neighborhood $\overline{V}$ of x such that $\|\sigma\|_{\mathrm{BSE},G\setminus\overline{V}} < \infty$. By Lemma 7.1, we have $\|\sigma|_{\overline{V}}\|_{\mathrm{BSE}\,(\mathcal{A}/I(\overline{V}))} = \|\sigma\|_{\mathrm{BSE},G\setminus\overline{V}} < \infty$, and by [**14**, Theorem 5.2] there is an $f \in \mathcal{A}$ such that $f = \sigma$ on $\overline{V}$. Then (a) follows from Proposition 7.2. $\square$

8. Segal algebras induced by local A-functions. III. In this section, we introduce a notion "rank n $(0 \le n \le \infty)$ local A-function." That $\tau \in A_{\mathrm{loc}}$ is of rank n will give useful information on $A_{\tau(n)}$ and $M(A_{\tau(n)})$.

8.1. Definition. For $\tau \in A_{\mathrm{loc}}$, we say τ is a rank 0 local A-function if $A = A_{\tau(1)}$, and a rank n $(1 \le n)$ local A-function if

$$A \supsetneqq \cdots \supsetneqq A_{\tau(n)} = A_{\tau(n+1)}.$$

Further, if τ is not a local A-function of finite rank, that is, $A_{\tau(k)} \supsetneqq A_{\tau(k+1)}$ for all $k = 0, 1, 2, \ldots$, then we call τ a rank ∞ local A-function.

We remark that Theorem 5.4 (i) implies that $\tau \in A_{\mathrm{loc}}$ is a rank 0 local A-function if and only if $\tau \in M(A)$. Here, we introduce the following notation:

$$A_{\mathrm{loc}}^{n} := \{\tau \in A_{\mathrm{loc}} : \tau \text{ is a rank } n \text{ local } A\text{-function}\}$$

for each n $(0 \le n \le \infty)$. In this case, we have a disjoint union representation of A_{loc}: $A_{\mathrm{loc}} = \cup_{k=0}^{\infty} A_{\mathrm{loc}}^{k} \cup A_{\mathrm{loc}}^{\infty}$.

8.2. Proposition. *Suppose $\tau \in A_{\mathrm{loc}}$.*

(i) *If τ is a rank n $(0 \le n < \infty)$ local A-function, then we have $A_{\tau(n)} = A_{\tau(n+1)} = A_{\tau(n+2)} = \cdots$, and $\tau \in M(A_{\tau(n)})$. In particular, we have $\|\tau\|_\infty < \infty$ by Lemma* 4.2.

(ii) *If $\|\tau\|_\infty = \infty$, then τ is a rank ∞ local A-function.*

Proof. (i) Let τ be a rank n local A-function. If $f \in A_{\tau(n+1)}$, then we have $f\tau \in A_{\tau(n)} = A_{\tau(n+1)}$, and hence $f \in A_{\tau(n+2)}$. This implies $A_{\tau(n+1)} \subseteq A_{\tau(n+2)}$. Since $A_{\tau(n+2)} \subseteq A_{\tau(n)}$ is obvious, the equality $A_{\tau(n+1)} = A_{\tau(n+2)}$ holds. In the same way, we have $A_{\tau(n+2)} = A_{\tau(n+3)} = A_{\tau(n+4)} = \cdots$.

Moreover, since $f\tau \in A_{\tau(n)}$ $(f \in A_{\tau(n)})$, it follows that τ is an element of $M(A_{\tau(n)})$.

(ii) Since a finite rank local A-function must be a bounded function on X, (ii) follows from (i). $\square$

In the rest of this section, we consider the following problems.

(i) For a given natural number n, are there rank n local A-functions?

(ii) For a given natural number n, how can we construct a rank n local A-function?

Put $\mathrm{Lip}_1(\mathbf{R}) = \{f \in C_b(\mathbf{R}) : \rho(f) < \infty\}$ and $\mathrm{Lip}_1^0(\mathbf{R}) = \{f \in \mathrm{Lip}_1(\mathbf{R}) \cap C_0(\mathbf{R}) : \lim_{M\to\infty} \rho_M(f) = 0\}$, where

$$\rho(f) = \sup\left\{\left|\frac{f(y)-f(x)}{y-x}\right| : -\infty < x < y < \infty\right\},$$

$$\rho_M(f) = \sup\left\{\left|\frac{f(y)-f(x)}{y-x}\right| : x \neq y, |x|, |y| \geq M\right\}.$$

Then $\mathrm{Lip}_1^0(\mathbf{R})$ is a regular semi-simple Banach function algebra under the usual pointwise addition, multiplication, scalar product and the norm $\|f\|_{\mathrm{Lip}_1} = \|f\|_\infty + \rho(f)$. We can easily show with a routine argument that the Gelfand space of $\mathrm{Lip}_1^0(\mathbf{R})$ is naturally identified with $\mathbf{R}$, and the Gelfand transform of $\mathrm{Lip}_1^0(\mathbf{R})$ is the identity mapping. Moreover, $\mathrm{Lip}_1^0(\mathbf{R})$ satisfies the conditions (α_A) and (β_A) (cf., [**14**]). $\square$

8.3. Theorem. *Suppose $A = \mathrm{Lip}_1^0(\mathbf{R})$. Then we have the following.*

(i) $M(A) = \mathrm{Lip}_1(\mathbf{R})$.

(ii) $\varnothing \neq A_{\mathrm{loc}} \cap C_b(\mathbf{R}) \setminus M(A) = A^1_{\mathrm{loc}}$.

Proof. (i) The inclusion $(C_{\mathrm{BSE}}(\mathbf{R}) =) M(A) \subseteq \mathrm{Lip}_1(\mathbf{R})$ holds due to the proof of Theorem 5.9 in [**14**]. To prove the reverse inclusion, let $f \in A$ and $g \in \mathrm{Lip}_1(\mathbf{R})$ be arbitrary. Then $fg \in C_0(\mathbf{R})$ and

$$\begin{aligned}\rho(fg) &= \sup\left\{\left|\frac{f(y)g(y)-f(x)g(x)}{y-x}\right| : -\infty < x < y < \infty\right\}\\ &\leq \sup\left\{|f(y)|\left|\frac{g(y)-g(x)}{y-x}\right|\right.\\ &\qquad \left. + \left|\frac{f(y)-f(x)}{y-x}\right||g(x)| : -\infty < x < y < \infty\right\}\\ &\leq \|f\|_\infty\rho(g) + \rho(f)\|g\|_\infty < \infty,\end{aligned}$$

and

$$\begin{aligned}\rho_M(fg) \le \sup\bigg\{&|f(y)|\left|\frac{g(y)-g(x)}{y-x}\right| \\ &+\left|\frac{f(y)-f(x)}{y-x}\right||g(x)| : x\neq y, |x|,|y|\ge M\bigg\} \\ &\le \sup_{|y|\ge M}|f(y)|\rho(g)+\rho_M(f)\|g\|_\infty \longrightarrow 0 \quad (M\to\infty).\end{aligned}$$

Therefore, $fg \in A$. Hence, $g\in M(A)$.

(ii) Let σ be a continuously differentiable function on $\mathbf{R}$ satisfying $\sigma(x) = |x|\sin x^2/(1+|x|)^{3/2}$ ($|x|\ge 1$) and $|\sigma'(x)|\le 1$ ($|x|\le 1$). Then it is easy to see that $\sigma\in A_{\mathrm{loc}}\cap C_b(\mathbf{R})$. Further, since

$$|\sigma'(x)| = \left|\frac{\frac{3}{2}(1+|x|)^{1/2}(|x|\sin x^2)-(1+|x|)^{3/2}(\sin x^2+2x^2\cos x^2)}{(1+|x|)^3}\right| \qquad \text{if } |x|\ge 1,$$

we have $\sigma'\notin C_b(\mathbf{R})$. Hence $\sigma\notin \mathrm{Lip}_1(\mathbf{R}) = M(A)$. Thus, we have $\sigma\in A_{\mathrm{loc}}\cap C_b(\mathbf{R})\setminus M(A)$.

To show the equality in (ii), let $\tau\in A_{\mathrm{loc}}\cap C_b(\mathbf{R})\setminus M(A)$ and $f\in A_{\tau(1)}$ be arbitrary. Then $f\tau\in A$ and

$$\begin{aligned}\rho(f\tau) &= \sup\left\{\left|\frac{(f\tau)(y)-(f\tau)(x)}{y-x}\right| : -\infty<x<y<\infty\right\} \\ &= \sup\bigg\{\left|f(y)\frac{\tau(y)-\tau(x)}{y-x}+\frac{f(y)-f(x)}{y-x}\tau(x)\right| : \\ &\qquad -\infty<x<y<\infty\bigg\}<\infty.\end{aligned} \tag{2}$$

Since $f\in A$ and $\tau\in C_b(\mathbf{R})$, $\sup\{|[f(y)-f(x)]/(y-x)\tau(x)| : -\infty < x<y<\infty\}<\infty$ follows, and with (2), we get

$$\sup\left\{\left|f(y)\frac{\tau(y)-\tau(x)}{y-x}\right| : -\infty<x<y<\infty\right\}<\infty. \tag{3}$$

It follows from (2) and (3) that

$$\rho(f\tau^2) = \sup\left\{\left|\frac{(f\tau^2)(y)-(f\tau^2)(x)}{y-x}\right| : -\infty < x < y < \infty\right\} \tag{4}$$
$$= \sup\left\{\left|\frac{(f\tau)(y)-(f\tau)(x)}{y-x}\tau(y) + \left(f(x)\frac{\tau(y)-\tau(x)}{y-x}\right)\tau(x)\right| : -\infty < x < y < \infty\right\} < \infty.$$

Furthermore, as $f\tau \in A$, we have

$$\rho_M(f\tau) = \sup\left\{\left|\frac{(f\tau)(y)-(f\tau)(x)}{y-x}\right| : x \neq y,\ |x|,|y| \geq M\right\}$$
$$= \sup\left\{\left|f(y)\frac{\tau(y)-\tau(x)}{y-x} + \frac{f(y)-f(x)}{y-x}\tau(x)\right| : x \neq y, |x|,|y| \geq M\right\} \longrightarrow 0 \quad (M \to \infty). \tag{5}$$

Since $\sup\{|[f(y)-f(x)]/(y-x)\tau(x)| : x \neq y, |x|,|y| \geq M\} \to 0$ $(M \to \infty)$, we have by (5) that

$$\sup\left\{\left|f(y)\frac{\tau(y)-\tau(x)}{y-x}\right| : x \neq y, |x|,|y| \geq M\right\} \longrightarrow 0 \quad (M \to \infty). \tag{6}$$

It follows from (5) and (6) that

$$\rho_M(f\tau^2) = \sup\left\{\left|\frac{(f\tau^2)(y)-(f\tau^2)(x)}{y-x}\right| : x \neq y, |x|,|y| \geq M\right\} \tag{7}$$
$$= \sup\left\{\left|\frac{(f\tau)(y)-(f\tau)(x)}{y-x}\tau(y) + \left(f(x)\frac{\tau(y)-\tau(x)}{y-x}\right)\tau(x)\right| : x \neq y,\ |x|,|y| \geq M\right\} \longrightarrow 0 \quad (M \to \infty).$$

From (4) and (7), $\tau^2 f \in A$ follows. Thus, we obtain that $A_{\tau(1)} = A_{\tau(2)}$. Moreover, since $\tau \notin M(A)$, it follows from Theorem 5.4 (i) that

$A_{\tau(1)} \subsetneqq A$, and hence τ is a rank 1 local A-function. Thus, we conclude that $A_{\text{loc}} \cap C_b(\mathbf{R}) \setminus M(A) \subseteq A^1_{\text{loc}}$. The reverse inclusion is trivial, so the equality holds. □

8.4. Theorem. *Suppose that $\mathcal{A} = \mathcal{A}(G)$ is the Fourier algebra on a non-compact LCA group G. Then there exists a rank* 1 *local $\mathcal{A}$-function.*

8.5. Lemma. *Let $G = G_1 \times G_2$ be the direct product of* LCA *groups G_1 and G_2. Let π_1 be the natural projection of G onto G_1. Let $\mathcal{A}(G_1)$ be the Fourier algebra on G_1. For any continuous function σ on G_1, put $\widetilde{\sigma} := \sigma \circ \pi_1$. Suppose that $\sigma \in \mathcal{A}(G_1)_{\text{loc}}$. Then we have the following.*

(i) $\widetilde{\sigma} \in \mathcal{A}_{\text{loc}}$.

(ii) $\sigma \in M(\mathcal{A}(G_1))$ *if and only if* $\widetilde{\sigma} \in M(\mathcal{A})$.

Proof. (i) Let $K \in \mathcal{K}(G)$ be given arbitrarily. Put $K_1 = \pi_1(K)$. Then K_1 is compact. By Proposition 7.2 there is an $f \in \mathcal{A}(G_1)$ such that $f = \sigma$ on K_1. It is easy to see that $\widetilde{f} \in M(\mathcal{A})$ and $\widetilde{f} = \widetilde{\sigma}$ on K. For a function e_K in $\mathcal{A}_c$ such that $e_K = 1$ on K, we have $\widetilde{f}e_K \in \mathcal{A}$ with $\widetilde{f}e_K = \widetilde{\sigma}$ on K. Thus, $\widetilde{\sigma} \in \mathcal{A}_{\text{loc}}$ follows from Proposition 7.2, again.

(ii) It is easy to see that $\widetilde{\sigma} \in M(\mathcal{A}(G_1))$ for a $\sigma \in M(\mathcal{A})$. Conversely, suppose that $\widetilde{\sigma} \in M(\mathcal{A})$. Since $\widetilde{\sigma}$ is constant on each coset of G_2, $\widetilde{\sigma}$ is the Fourier-Stieltjes transform of a measure $\widetilde{\mu} \in M(\widehat{G})$ concentrated in $\widehat{G}_1 \times \{0\}$ by [**20**, Theorem 2.7.1]. Define a measure μ by $\mu(E) = \widetilde{\mu}(E \times \{0\})$ (E : Borel set of $\widehat{G}_1$). Then we have $\mu \in M(\widehat{G}_1)$ with $\widehat{\mu} = \sigma$. Hence, $\sigma \in M(\mathcal{A}(G_1))$. □

8.6. Lemma. *If G is a non-compact* LCA *group which has a compact open subgroup G_0, then there exists a function $\phi \in \mathcal{A}_{\text{loc}}$ such that*

(i) *ϕ is constant on each coset of G_0, and $\phi(x) \in \{-1, 1\}$ for all $x \in G$,*

(ii) $\phi \notin M(\mathcal{A})$,

(iii) $\phi^2 \in M(\mathcal{A})$.

Proof. G/G_0 is an infinite discrete group and hence not a Sidon set. Therefore, there exists a function φ on G/G_0 into $\{-1, 1\}$ such that $\varphi \notin M(\mathcal{A}(G/G_0))$ by [**20**, Theorem 5.7.4]. Let π be the natural projection of G onto G/G_0. Put $\phi = \varphi \circ \pi$. Then we see the following.

(i) Since ϕ is constant on each coset of G_0, we have $\phi \in \mathcal{A}_{\text{loc}}$ by Proposition 7.2.

(ii) That $\phi \notin M(\mathcal{A})$ follows from $\varphi \notin M(\mathcal{A}(G/G_0))$ is well known, and we omit the proof.

(iii) $\phi^2 = 1_G \in M(\mathcal{A})$. □

8.7. Lemma. *Let $\mathcal{A} = \mathcal{A}(\mathbf{R})$ be the Fourier algebra on $\mathbf{R}$. Suppose φ is a C^∞-function on $\mathbf{R}$ such that $\varphi(x) = 1$ if $x \in [1, \infty)$ and $\varphi(x) = -1$ if $x \in (-\infty, -1]$, then we have $\varphi \notin M(\mathcal{A})$ and $\varphi^2 \in M(\mathcal{A})$.*

Proof. It is easy to see that $\varphi \in \mathcal{A}_{\text{loc}}$ by Proposition 7.2. To show $\varphi \notin M(\mathcal{A})$, suppose on the contrary that $\varphi \in M(\mathcal{A})$. Then $1 + \varphi \in M(\mathcal{A})$. Since $1 + \varphi(x) = 0$ for all $x \le -1$, it follows from the theorem of F. and M. Riesz that $1 + \varphi \in \mathcal{A}$ and so $1 + \varphi$ vanishes at infinity. But since $1 + \varphi = 2$ for all $x \ge 1$, we arrive at a contradiction. On the other hand, since $\varphi^2 - 1$ is a C^∞-function on $\mathbf{R}$ with compact support, it follows that $\varphi^2 - 1 \in \mathcal{A}$ and so $\varphi^2 \in M(\mathcal{A})$. □

8.8. Lemma. *Suppose $G = \mathbf{R}^d \times L$, where $d \ge 1$ and L is an* LCA *group. Then there exists a $\phi \in \mathcal{A}_{\text{loc}}$ such that $\phi \notin M(\mathcal{A})$ and $\phi^2 \in M(\mathcal{A})$.*

Proof. We write $G = r \times H$, where $H = \mathbf{R}^{d-1} \times L$, and let π be the natural projection of G onto $\mathbf{R}$. Let φ be the function on $\mathbf{R}$ in Lemma 8.7, and put $\phi := \varphi \circ \pi$. Then we can conclude by Lemmas 8.5 and 8.7 that ϕ is a function in $\mathcal{A}_{\text{loc}}$ which satisfies $\phi \notin M(\mathcal{A})$ and $\phi^2 \in M(\mathcal{A})$. □

Proof of Theorem 8.4. By the structure theorem of LCA groups (cf., [**12**, Theorem 24.30]), G is isomorphic to $\mathbf{R}^d \times L$, where $0 \le d$ and L is an LCA group which has a compact open subgroup. Then, using Lemma 8.6, if $d = 0$ and Lemma 8.8 if $d \ge 1$, we can choose a $\phi \in \mathcal{A}_{\text{loc}}$ which satisfies $\phi \notin M(\mathcal{A})$ and $\phi^2 \in M(\mathcal{A})$. Then it is easy to see that ϕ is a local $\mathcal{A}$-function of rank 1. □

8.9. Theorem. (i) *If $\tau \in M(A)$ satisfies $\tau(x) \neq 0$ $(x \in G)$ and $\tau^{-1} \notin M(A)$, then we have $\tau^{-1} \in A_{\mathrm{loc}}^{\infty}$.*

(ii) *If $\mathcal{A} = \mathcal{A}(G)$ is the Fourier algebra on a non-compact* LCA *group G, then we have $\mathcal{A}_{\mathrm{loc}}^{\infty} \cap C_b(G) \neq \varnothing$.*

Proof. (i) Suppose that $\tau \in M(A)$ satisfies $\tau(x) \neq 0$ for all $x \in G$ and $\tau^{-1} \notin M(A)$. Then $\tau^{-1} \in A_{\mathrm{loc}}$ (cf., Example 5.2 (ii)). Let $n \in \mathbf{N}$ be arbitrary. Since $\tau^{-1} \notin M(A)$, there exists a $g \in A$ such that $\tau^{-1} g \notin A$. Put $f = \tau^{n-1} g$. Then $(\tau^{-1})^k f = \tau^{n-1-k} g \in A$ for $k (0 \leq k \leq n-1)$, but $(\tau^{-1})^n f = \tau^{-1} g \notin A$. Hence, $A_{\tau^{-1}(n)} \subsetneqq A_{\tau^{-1}(n-1)}$. Since n is arbitrary in $\mathbf{N}$, we have $\tau^{-1} \in A_{\mathrm{loc}}^{\infty}$.

(ii) By [**20**, Theorem 5.3.4], there exists a $\tau \in M(\mathcal{A})$ such that $1 \leq \tau(x)$ for all $x \in G$ and $\tau^{-1} \notin M(A)$. Then, by (i), we have $\tau^{-1} \in \mathcal{A}_{\mathrm{loc}}^{\infty} \cap C_b(G)$. $\square$

8.10. Proposition. *Let $\tau \in A_{\mathrm{loc}}^{1}$. Then $A_{\tau(1)}$ is the largest one in the family of Segal algebras $\mathcal{S}$ in A satisfying $\tau \in M(\mathcal{S})$.*

Proof. Suppose that $\mathcal{S}$ is a Segal algebra in A which satisfies $\tau \in M(\mathcal{S})$. From Proposition 8.2 (i), we have $\tau \in M(A_{\tau(1)})$. Since $f\tau \in \mathcal{S}$ $(f \in \mathcal{S})$, $A_{\tau(1)} = \{f \in A : \tau f \in A\} \supseteq \mathcal{S}$ follows as required. $\square$

8.11. *Remarks.* (i) Let X be a locally compact, non-compact Hausdorff space, and let $C_0(X)$ be the algebra of all continuous functions on X which vanishes at infinity provided with the norm $\| \ \|_{\infty}$. Suppose that $A = C_0(X)$. Then we can identify Φ_A with X. Furthermore, we have $A_{\mathrm{loc}} = C(X) (:= \{\phi : \text{a continuous function on } X\})$ and $M(A) = C_b(X)$. Therefore, $A_{\mathrm{loc}} \cap C_b(X) \setminus M(A) = \varnothing$. This means that there are no rank n $(1 \leq n < \infty)$ local A-functions.

(ii) Let $A = \mathrm{Lip}_1^0(\mathbf{R})$. For any positive integer $n \geq 2$, there are no rank n local A-functions in A_{loc} by Theorem 8.3 (ii). Furthermore, there are no bounded rank ∞ local A-functions by the same theorem.

(iii) If $\mathcal{A}$ is the Fourier algebra on a non-compact LCA, Theorems 8.4 and 8.9 say that the set $\mathcal{A} \cap C_b(G) \setminus M(\mathcal{A})$ is non-empty and contains rank 1 local $\mathcal{A}$-functions and rank ∞-functions.

8.12. Problem. Let $\mathcal{A}$ be the Fourier algebra on a non-compact LCA group G. Let n be a positive integer grater than 1. Are there any rank n local $\mathcal{A}$-functions?

8.13. Definition. For $\sigma, \tau \in A_{\mathrm{loc}}$, we denote $\sigma \preceq \tau$ if and only if $\cap_{k=1}^{\infty} A_{\sigma(k)} \subseteq \cap_{k=1}^{\infty} A_{\tau(k)}$. This relation $\preceq$ is a partial semi-order in A_{loc}.

The next theorem characterizes the multiplier algebra of $A_{\tau(n)}, \tau \in A_{\mathrm{loc}}^n$.

8.14. Theorem. *$M(A_{\tau(n)}) = \{\sigma \in \mathcal{A}_{\mathrm{loc}} : \tau \preceq \sigma\}$ holds for every $n \in \mathbf{N}$ and $\tau \in A_{\mathrm{loc}}^n$.*

Proof. ($\subseteq$). If $\sigma \in M(A_{\tau(n)})$ and $f \in A_{\tau(n)}$, we have $\sigma^k f \in A_{\tau(n)} \subseteq A$ $(k = 1, 2, 3, \dots)$. This implies $f \in \cap_{k=1}^{\infty} A_{\sigma(k)}$, that is, $\cap_{k=1}^{\infty} A_{\tau(k)} = A_{\tau(n)} \subseteq \cap_{k=1}^{\infty} A_{\sigma(k)}$, and hence $\tau \preceq \sigma$ follows.

($\supseteq$). Let $\sigma \in A_{\mathrm{loc}}$ with $\tau \preceq \sigma$. Then $A_{\tau(n)} = \cap_{k=1}^{\infty} A_{\tau(k)} \subseteq \cap_{k=1}^{\infty} A_{\sigma(k)}$. In this case, if $f \in A_{\tau(n)}$, then $\sigma f \in A$ and $\tau^k f \in A_{\tau(n)}$. Hence, $\tau^k(\sigma f) = \sigma(\tau^k f) \in A$ for all $k = 1, 2, 3, \dots, n$. So $\sigma \in M(A_{\tau(n)})$. ◻

8.15. Corollary. *Suppose $\tau_i \in A_{\mathrm{loc}}^{n_i}$ $(i = 1, 2)$. Then $A_{\tau_1(n_1)} \subseteq A_{\tau_2(n_2)}$ if and only if $M(A_{\tau_2(n_2)}) \subseteq M(A_{\tau_1(n_1)})$.*

Proof. Suppose $A_{\tau_1(n_1)} \subseteq A_{\tau_2(n_2)}$. By Theorem 8.14, we have

$$\begin{aligned} M(A_{\tau_2(n_2)}) &= \{\sigma \in A_{\mathrm{loc}} : \tau_2 \preceq \sigma\} \\ &= \{\sigma \in A_{\mathrm{loc}} : A_{\tau_2(n_2)} = \cap_{k=1}^{\infty} A_{\tau_2(k)} \subseteq \cap_{k=1}^{\infty} A_{\sigma(k)}\} \\ &\subseteq \{\sigma \in A_{\mathrm{loc}} : A_{\tau_1(n_1)} = \cap_{k=1}^{\infty} A_{\tau_1(k)} \subseteq \cap_{k=1}^{\infty} A_{\sigma(k)}\} \\ &= M(A_{\tau_1(n_1)}). \end{aligned}$$

Conversely, suppose that $M(A_{\tau_2(n_2)}) \subseteq M(A_{\tau_1(n_1)})$. Then $\tau_2 \in M(A_{\tau_2(n_2)}) \subseteq M(A_{\tau_1(n_1)})$ by Proposition 8.2 (i), and hence $\tau_1 \preceq \tau_2$ by Theorem 8.14, that is, $A_{\tau_1(n_1)} = \cap_{k=1}^{\infty} A_{\tau_1(k)} \subseteq \cap_{k=1}^{\infty} A_{\tau_2(k)} = A_{\tau_2(n_2)}$. ◻

9. BED- and BSE- properties of $A_{\tau(n)}(1 \le n \le \infty)$**.** We suppose in this section the following condition (γ_A) on A.

(γ_A) A is BSE.

For example, Fourier algebras on non-compact LCA groups G and $\mathrm{Lip}_1(\mathbf{R})$ satisfy the condition (γ_A). Since A has a bounded approximate identity composed of elements in A_c, A is also BED (cf., [**14**, Theorem 4.7]).

A bounded weak approximate identity of $\mathcal{S}$ in the sense of Jones-Lahr is, by definition, a bounded net $\{u_\omega\}_{\omega\in\Omega}$ in $\mathcal{S}$ such that $\lim_{\omega\in\Omega} u_\omega(x)f(x) = f(x)$ $(f \in \mathcal{S}, x \in X)$ (cf., [**15, 21**]).

9.1. Definition. (i) Let $\tau \in A_{\mathrm{loc}}$ and n be a non-negative integer. We put

$$M(A)_{\tau(n)} := \{\sigma \in M(A) : \sigma\tau^k \in M(A) \quad (0 \le k \le n)\},$$
$$\|\sigma\|_{\tau(n)} := \sum_{k=0}^{n} \|\sigma\tau^k\|_{M(A)} \quad (\sigma \in M(A)_{\tau(n)}).$$

Note that $(M(A)_{\tau(0)}, \|\ \|_{\tau(0)})$ is nothing but $(M(A), \|\ \|_{M(A)})$.

9.2. Proposition. *For* $\tau \in A_{\mathrm{loc}}$ *and* $n \in \mathbf{N}$, $(M(A)_{\tau(n)}, \|\ \|_{\tau(n)})$ *is a Banach ideal of* $M(A)$.

Proof. It is easy to see that $M(A)_{\tau(n)}$ is a linear subspace and $\|\ \|_{\tau(n)}$ is a norm on $M(A)_{\tau(n)}$, and we will show that $\|\ \|_{\tau(n)}$ is complete. To see this, let $\{\sigma_i\}_{i=1}^{\infty}$ be a Cauchy sequence in $M(A)_{\tau(n)}$. Then $\lim_{i,j\to\infty} \sum_{k=0}^{n} \|\sigma_i\tau^k - \sigma_j\tau^k\|_{M(A)} = 0$, and there exist $\rho_k \in M(A)$ $(k = 0, \ldots, n)$ such that $\lim_{i\to\infty} \|\sigma_i\tau^k - \rho_k\|_{M(A)} = 0$. Then $\rho_0(x) = \lim_{i\to\infty} \sigma_i(x)$ $(x \in X)$. Since

$$\rho_k(x) = \lim_{i\to\infty} (\sigma_i\tau^k)(x) = \lim_{i\to\infty} \sigma_i(x)\tau^k(x) = \rho_0(x)\tau^k(x)$$
$$(x \in X, k = 1, \ldots, n),$$

$\rho_0\tau^k = \rho_k \in M(A)$ $(k = 1, \ldots, n)$ follows. Hence, $\rho_0 \in M(A)_{\tau(n)}$.

Therefore,

$$\begin{aligned}\lim_{i\to\infty}\|\sigma_i-\rho_0\|_{\tau(n)} &= \lim_{i\to\infty}\sum_{k=0}^{n}\|\sigma_i\tau^k-\rho_0\tau^k\|_{M(A)}\\ &= \lim_{i\to\infty}\sum_{k=0}^{n}\|\sigma_i\tau^k-\rho_k\|_{M(A)}=0.\end{aligned}$$

Finally, we see that $\rho\sigma\in M(A)_{\tau(n)}$ and $\|\rho\sigma\|_{\tau(n)}\le\|\rho\|_{M(A)}\|\sigma\|_{\tau(n)}$ for all $\rho\in M(A)$ and $\sigma\in M(A)_{\tau(n)}$. In fact, since $(\rho\sigma)\tau^k=\rho(\sigma\tau^k)\in M(A)$ $(1\le k\le n)$, we have $\rho\sigma\in M(A)_{\tau(n)}$, and it follows that

$$\begin{aligned}\|\rho\sigma\|_{\tau(n)} &= \sum_{k=0}^{n}\|(\rho\sigma)\tau^k\|_{M(A)}\le\|\rho\|_{M(A)}\sum_{k=0}^{n}\|\sigma\tau^k\|_{M(A)}\\ &= \|\rho\|_{M(A)}\|\sigma\|_{\tau(n)}. \qquad \square\end{aligned}$$

9.3. Theorem. *The equalities* $C_{\mathrm{BSE}\,(A_{\tau(n)})}(X)=M(A)_{\tau(n)}=M(A,A_{\tau(n)})$ *hold for* $\tau\in A_{\mathrm{loc}}$ *and* $n\in\mathbf{N}$.

Proof. We divide the proof into three parts: (i) $C_{\mathrm{BSE}\,(A_{\tau(n)})}(X)\subseteq M(A)_{\tau(n)}$, (ii) $M(A)_{\tau(n)}\subseteq M(A,A_{\tau(n)})$, and (iii) $M(A,A_{\tau(n)})\subseteq C_{BSE(A_{\tau(n)})}(X)$.

(i) If $\sigma\in C_{\mathrm{BSE}\,(A_{\tau(n)})}(X)$, there is a bounded net $\{f_\omega\}_{\omega\in\Omega}$ in $A_{\tau(n)}$ such that

$$\lim_{\omega\in\Omega}f_\omega(x)=\sigma(x)\quad(x\in X) \tag{8}$$

by [**21**, Theorem 4 (i)]. Then we have, by (8),

$$\begin{gathered}\sigma(x)\tau^k(x)=\lim_{\omega\in\Omega}f_\omega(x)\tau^k(x)=\lim_{\omega\in\Omega}(f_\omega\tau^k)(x)\\ (k=1,\dots,n,\ x\in X).\end{gathered} \tag{9}$$

Since $\sup_{\omega\in\Omega}\|f_\omega\tau^k\|_A\le\sup_{\omega\in\Omega}\|f_\omega\|_{\tau(n)}<\infty$ $(k=0,\dots,n)$, we have by (9) and [**21**, Theorem 4 (i)] that $\sigma\tau^k\in C_{BSE}(X)=M(A)$ $(k=0,\dots,n)$, that is, $\sigma\in M(A)_{\tau(n)}$.

(ii) If $\sigma \in M(A)_{\tau(n)}$, we have $\sigma\tau^k \in M(A)$ $(1 \le k \le n)$. Thus, for each $f \in A$, we have $(f\sigma)\tau^k = f(\sigma\tau^k) \in A$ $(1 \le k \le n)$. This implies $f\sigma \in A_{\tau(n)}$ $(f \in A)$, and hence $\sigma \in M(A, A_{\tau(n)})$.

(iii) Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be a bounded approximate identity of A. If $\sigma \in M(A, A_{\tau(n)})$, then $\{\sigma e_\lambda\}_{\lambda\in\Lambda}$ is a bounded net in $A_{\tau(n)}$ such that $\lim_{\lambda\in\Lambda} \sigma e_\lambda(x) = \sigma(x)$ $(x \in X)$, and hence $\sigma \in C_{\mathrm{BSE}\,(A_{\tau(n)})}(X)$ by [**21**, Theorem 4 (i)]. $\square$

9.4. Theorem. *The equality* $C^0_{\mathrm{BSE}\,(A_{\tau(n)})}(X) = A_{\tau(n)}$ *holds for* $\tau \in A_{\mathrm{loc}}$ *and* $n \in \mathbf{N}$. *Therefore,* $A_{\tau(n)}$ *is* BED.

Proof. Since ${A_{\tau(n)}}_c$ is dense in $A_{\tau(n)}$ by Theorem A$'$, we have $A_{\tau(n)} \subseteq C^0_{\mathrm{BSE}\,(A_{\tau(n)})}(X)$ by [**14**, Proposition 4.1]. We must show the reverse inclusion. Let $\sigma \in C^0_{\mathrm{BSE}\,(A_{\tau(n)})}(X)$ be arbitrary. By Theorem 9.3, we have $\sigma\tau^k \in M(A)$ $(k = 0, \dots, n)$. Since $\|f\|_{\tau(n)} \ge \|f\tau^k\|_A$ $(f \in A_{\tau(n)}, k = 0, \dots, n)$, we have

(10)
$$\begin{aligned}\|\tau^k p\|_{A^*_{\tau(n)}} &= \sup\left\{\left|\sum_{x\in X} \tau^k(x)\widehat{p}(x)f(x)\right| : f \in A_{\tau(n)}, \|f\|_{\tau(n)} \le 1\right\}\\ &= \sup\left\{\left|\sum_{x\in X} \widehat{p}(x)(f\tau^k)(x)\right| : f \in A_{\tau(n)}, \|f\|_{\tau(n)} \le 1\right\}\\ &\le \sup\left\{\left|\sum_{x\in X} \widehat{p}(x)g(x)\right| : g \in A, \|g\|_A \le 1\right\}\\ &= \|p\|_{A^*} \quad (p \in \mathrm{span}\,(X),\ k = 0, \dots, n).\end{aligned}$$

For each $K \in \mathcal{K}(X)$, we have by (10),

(11)
$$\begin{aligned}&\|\sigma\tau^k\|_{\mathrm{BSE},K}\\ &= \sup\left\{\left|\sum_{x\in X} \widehat{p}(x)\sigma(x)\tau^k(x)\right| : p \in \mathrm{span}\,(X \setminus K), \|p\|_{A^*} \le 1\right\}\\ &\le \sup\left\{\left|\sum_{x\in X} \widehat{q}(x)\sigma(x)\right| : q \in \mathrm{span}\,(X \setminus K), \|q\|_{A^*_{\tau(n)}} \le 1\right\}\\ &= \|\sigma\|_{\mathrm{BSE}\,(A_{\tau(n)}),K} \quad (k = 0, \dots, n).\end{aligned}$$

Since $\sigma \in C^0_{\mathrm{BSE}\,(A_{\tau(n)})}(X)$, we have $\lim_{K\in\mathcal{K}(X)} \|\sigma\|_{\mathrm{BSE}\,(A_{\tau(n)}),K} = 0$. So we have, by (11),

$$\lim_{K\in\mathcal{K}(X)} \|\tau^k\sigma\|_{\mathrm{BSE},K} = 0 \quad (k = 0, \dots, n),$$

that is, $\sigma\tau^k \in C^0_{\mathrm{BSE}}(X) = A$ $(k = 0, \dots, n)$. This proves $\sigma \in A_{\tau(n)}$. □

9.5. Definition. Let $\tau \in A_{\mathrm{loc}}$. We define $M(A)_{\tau(\infty)}$ and $\|\ \|_{\tau(\infty)}$ by

$$M(A)_{\tau(\infty)} := \Big\{\sigma \in M(A) : \sigma\tau^k \in M(A)\ (k = 1, 2, 3, \dots), \\ \sum_{k=0}^{\infty} \|\sigma\tau^k\|_{M(A)} < \infty\Big\},$$

$$\|\sigma\|_{\tau(\infty)} := \sum_{k=0}^{\infty} \|\sigma\tau^k\|_{M(A)} \quad (\sigma \in M(A)_{\tau(\infty)}).$$

9.6. Proposition. *For $\tau \in A_{\mathrm{loc}}$, $(M(A)_{\tau(\infty)}, \|\ \|_{\tau(\infty)})$ is a Banach ideal of $M(A)$.*

Proof. It is easy to see that $(M(A)_{\tau(\infty)}, \|\ \|_{\tau(\infty)})$ is a normed linear space, and we verify that $\|\ \|_{\tau(\infty)}$ is complete. Let $\{\sigma_n\}$ be a Cauchy sequence in $M(A)_{\tau(\infty)}$. Then $\lim_{i,j\to\infty} \sum_{k=0}^{\infty} \|\sigma_i\tau^k - \sigma_j\tau^k\|_{M(A)} = 0$, and there exist $\rho_k \in M(A)$ $(k = 0, 1, 2, \dots)$ such that $\lim_{n\to\infty} \|\sigma_n\tau^k - \rho_k\|_{M(A)} = 0$. Then $\rho_0(x) = \lim_{n\to\infty} \sigma_i(x)$ $(x \in X)$, and since $\rho_k(x) = \lim_{n\to\infty}(\sigma_n\tau^k)(x) = \lim_{n\to\infty} \sigma_n(x)\tau^k(x) = \rho_0(x)\tau^k(x)$ $(x \in X)$, we get $\rho_0\tau^k = \rho_k \in M(A)$ for $k = 1, 2, 3, \dots$.

Now we will show that $\rho_0 \in M(A)_{\tau(\infty)}$ and $\|\sigma_n - \rho_0\|_{\tau(\infty)} \to 0$ $(n \to \infty)$. Since $\{\sigma_n\}$ is a Cauchy sequence in $M(A)_{\tau(\infty)}$, it forms a bounded set in $M(A)_{\tau(\infty)}$, that is, there is a $C > 0$ such that $\sum_{k=0}^{\infty} \|\sigma_n\tau^k\|_{M(A)} \le C$ $(n = 1, 2, 3, \dots)$. For each $k_0 \in \mathbf{N}$, if we choose an $n_0 \in \mathbf{N}$ so that $\sum_{k=0}^{k_0} \|\rho_0\tau^k - \sigma_{n_0}\tau^k\|_{M(A)} < 1$, we have

$$\sum_{k=0}^{k_0} \|\rho_0\tau^k\|_{M(A)} \le \sum_{k=0}^{k_0} \|\rho_0\tau^k - \sigma_{n_0}\tau^k\|_{M(A)} + \sum_{k=0}^{k_0} \|\sigma_{n_0}\tau^k\|_{M(A)} < 1 + C,$$

which implies $\sum_{k=1}^{\infty} \|\rho_0\tau^k\|_{M(A)} \le 1 + C$, and hence $\rho_0 \in M(A)_{\tau(\infty)}$.

Our next aim is to prove $\lim_{n\to\infty}\|\sigma_n-\rho_0\|_{\tau(\infty)}=0$. Given $\varepsilon>0$, let $n_1\in\mathbf{N}$ be such that $\|\sigma_n-\sigma_{n_1}\|_{\tau(\infty)}<\varepsilon/3$ $(n\geq n_1)$. Choose a $k_1\in\mathbf{N}$ so that $\sum_{k=k_1+1}^{\infty}\|\rho_0\tau^k\|_{M(A)}<\varepsilon/6$ and $\sum_{k=k_1+1}^{\infty}\|\sigma_{n_1}\tau^k\|_{M(A)}<\varepsilon/6$. Choose an $n_2\in\mathbf{N}$ $(n_2\geq n_1)$ so that $\sum_{k=0}^{k_1}\|\rho_0\tau^k-\sigma_n\tau^k\|_{M(A)}<\varepsilon/3$ $(n\geq n_2)$. Then we have

$$\begin{aligned}
\|\rho_0-\sigma_n\|_{\tau(\infty)} &= \sum_{k=0}^{k_1}\|\rho_0\tau^k-\sigma_n\tau^k\|_{M(A)} \\
&\quad + \sum_{k=k_1+1}^{\infty}\|\rho_0\tau^k-\sigma_n\tau^k\|_{M(A)} \\
&\leq \varepsilon/3+\sum_{k=k_1+1}^{\infty}\Big(\|\sigma_n\tau^k-\sigma_{n_1}\tau^k\|_{M(A)} \\
&\qquad\qquad + \|\rho_0\tau^k\|_{M(A)}+\|\sigma_{n_1}\tau^k\|_{M(A)}\Big) \\
&\leq \frac{1}{3}\varepsilon+\frac{1}{3}\varepsilon+\frac{1}{6}\varepsilon+\frac{1}{6}\varepsilon \\
&= \varepsilon \quad (n\geq n_2).
\end{aligned}$$

Therefore $\lim_{n\to\infty}\|\rho_0-\sigma_n\|_{\tau(\infty)}=0$, and hence $\|\ \|_{\tau(\infty)}$ is a complete norm on $M(A)_{\tau(\infty)}$.

Finally, for any $\rho\in M(A)$ and $\sigma\in M(A)_{\tau(\infty)}$, we get $\rho\sigma\in M(A)_{\tau(\infty)}$ and $\|\rho\sigma\|_{\tau(\infty)}\leq\|\rho\|_{M(A)}\|\sigma\|_{\tau(\infty)}$ by quite a similar way as in the proof of Proposition 9.2. □

9.7. Theorem. *For $\tau\in A_{\mathrm{loc}}$ with $\|\tau\|_\infty<1/C_0$, the equalities $C_{\mathrm{BSE}\,(A_{\tau(\infty)})}(X)=M(A)_{\tau(\infty)}=M(A,A_{\tau(\infty)})$ hold.*

Proof. To begin with a proof we note that $A_{\tau(\infty)}$ is a Segal algebra in A by Theorem 6.2 (i). We divide the proof into three parts: (i) $C_{\mathrm{BSE}\,(A_{\tau(\infty)})}(X)\subseteq M(A)_{\tau(\infty)}$, (ii) $M(A)_{\tau(\infty)}\subseteq M(A,A_{\tau(\infty)})$ and (iii) $M(A,A_{\tau(\infty)})\subseteq C_{\mathrm{BSE}\,(A_{\tau(\infty)})}(X)$.

(i) Let $\sigma\in C_{\mathrm{BSE}\,(A_{\tau(\infty)})}(X)$. There is a bounded net $\{f_\omega\}_{\omega\in\Omega}$ in $A_{\tau(\infty)}$ and a $C_1>0$ such that

$$\lim_{\omega\in\Omega}f_\omega(x)=\sigma(x)\quad(x\in X),\qquad\|f_\omega\|_{\tau(\infty)}\leq C_1\quad(\omega\in\Omega). \tag{12}$$

Hence,

$$\sigma(x)\tau^k(x) = \lim_{\omega\in\Omega} f_\omega(x)\tau^k(x) \quad (x \in X, k = 1,2,3,\dots). \tag{13}$$

By (12) and (13), we have $\sigma\tau^k \in C_{\mathrm{BSE}}(X) = M(A)$ $(k = 0,1,2,\dots)$. Let $n_0 \in \mathbf{N}$ and $p_0,\dots,p_{n_0} \in \mathrm{span}\,(X)$ be chosen such that $\|p_k\|_{A^*} \le 1$ $(k = 0,\dots,n_0)$. Choose $\omega_0 \in \Omega$ such that

$$\sum_{k=0}^{n_0}\sum_{x\in X} |\widehat{p_k}(x)||\sigma(x)\tau^k(x) - f_{\omega_0}(x)\tau^k(x)| \le 1. \tag{14}$$

Since $\|f\|_{\mathrm{BSE}} \le \|f\|_A$ $(f \in A)$, it follows that

$$\sum_{k=0}^{n_0} \|f_\omega\tau^k\|_{\mathrm{BSE}} \le \sum_{k=0}^{n_0} \|f_\omega\tau^k\|_A \le C_1 \quad (\omega \in \Omega). \tag{15}$$

By (14) and (15), we have

$$\begin{aligned}\sum_{k=0}^{n_0}\left|\sum_{x\in X}\widehat{p_k}(x)\sigma(x)\tau^k(x)\right| &\le \sum_{k=0}^{n_0}\left|\sum_{x\in X}\widehat{p_k}(x)(f_{\omega_0}\tau^k)(x)\right| \\ &\quad + \sum_{k=0}^{n_0}\sum_{x\in X}|\widehat{p_k}(x)||\sigma(x)\tau^k(x) - f_{\omega_0}(x)\tau^k(x)| \\ &\le C_1 + 1.\end{aligned}$$

This implies that $\sum_{k=0}^{n_0}\|\sigma\tau^k\|_{\mathrm{BSE}} \le C_1 + 1$. Taking $n_0 \to \infty$, we have $\sum_{k=0}^{\infty}\|\sigma\tau^k\|_{\mathrm{BSE}} \le C_1 + 1$. Since A is BSE by the condition (γ_A) there is a C_2 such that $\|\sigma\|_{M(A)} \le C_2\|\sigma\|_{\mathrm{BSE}}$ $(\sigma \in M(A))$ by [**21**, Corollary 6]. Therefore, we have $\sum_{k=0}^{\infty}\|\sigma\tau^k\|_{M(A)} \le \sum_{k=0}^{\infty} C_2\|\sigma\tau^k\|_{\mathrm{BSE}} \le C_2(1 + C_1)$, that is, $\sigma \in M(A)_{\tau(\infty)}$.

(ii) If $\sigma \in M(A)_{\tau(\infty)}$, we have $\sigma\tau^k \in M(A)$ $(k = 1,2,3,\dots)$ and $\sum_{k=0}^{\infty}\|\sigma\tau^k\|_{M(A)} = \|\sigma\|_{\tau(\infty)} < \infty$. Then we have $(\sigma f)\tau^k = f(\sigma\tau^k) \in A$ $(k = 1,2,3,\dots)$ for an $f \in A$ and

$$\begin{aligned}\sum_{k=0}^{\infty}\|(\sigma f)\tau^k\|_A = \sum_{k=0}^{\infty}\|f(\sigma\tau^k)\|_A &\le \sum_{k=0}^{\infty}\|\sigma\tau^k\|_{M(A)}\|f\|_A \\ &= \|\sigma\|_{\tau(\infty)}\|f\|_A < \infty.\end{aligned}$$

Thus, $\sigma f \in A_{\tau(\infty)}$. We have proved that $\sigma f \in A_{\tau(\infty)}$ $(f \in A)$. Hence, $\sigma \in M(A, A_{\tau(\infty)})$ is observed.

(iii) The proof is the same as that of (iii) in Theorem 9.3, and we omit the proof. $\square$

9.8. Theorem. *The equality* $C^0_{\mathrm{BSE}\,(A_{\tau(\infty)})}(X) = A_{\tau(\infty)}$ *holds for* $\tau \in A_{\mathrm{loc}}$ *with* $\|\tau\|_\infty < 1/C_0$*, that is,* $A_{\tau(\infty)}$ *is* BED.

Proof. Since $A_{\tau(\infty)_c}$ is dense in $A_{\tau(\infty)}$ from Theorem A$'$, the relation $C^0_{\mathrm{BSE}\,(A_{\tau(\infty)})}(X) \supseteq A_{\tau(\infty)}$ follows from [**14**, Proposition 4.1]. To show the reverse inclusion, let $\sigma \in C^0_{\mathrm{BSE}\,(A_{\tau(\infty)})}(X)$ be arbitrary. Then $\sigma \in M(A)_{\tau(\infty)}$ holds by Theorem 9.7, that is, $\sigma\tau^k \in M(A)$ $(k = 0, 1, 2, \dots)$ and $\sum_{k=0}^{\infty} \|\sigma\tau^k\|_{M(A)} < \infty$.

In the same way as in the proof of Theorem 9.4 we get

$$\|\sigma\tau^k\|_{\mathrm{BSE},K} \le \|\sigma\|_{\mathrm{BSE}\,(A_{\tau(\infty)}),K} \quad (k = 0, 1, 2, ...)$$

for every $K \in \mathcal{K}(X)$. It follows that

$$\lim_{K \in \mathcal{K}(X)} \|\sigma\tau^k\|_{\mathrm{BSE},K} \le \lim_{K \in \mathcal{K}(X)} \|\sigma\|_{\mathrm{BSE}\,(A_{\tau(\infty)}),K} = 0$$
$$(k = 0, 1, 2, \dots).$$

Hence, $\sigma\tau^k \in C^0_{BSE}(X) = A$ $(k = 0, 1, 2, \dots)$. Since A is BSE and BED, the identity maps of A to $C^0_{BSE}(X)$ and $M(A)$ to $C_{BSE}(X)$ are Banach algebra isomorphisms, and so there exist $C_1, C_2 > 0$ such that

$$\|f\|_A \le C_1 \|f\|_{\mathrm{BSE}} \quad (f \in A)$$

and

$$\|\sigma\|_{\mathrm{BSE}} \le C_2 \|\sigma\|_{M(A)} \quad (\sigma \in M(A)).$$

Therefore, we obtain

$$\sum_{k=0}^{\infty} \|\sigma\tau^k\|_A \le C_1 \sum_{k=0}^{\infty} \|\sigma\tau^k\|_{\mathrm{BSE}} \le C_1 C_2 \sum_{k=0}^{\infty} \|\sigma\tau^k\|_{M(A)} < \infty.$$

This implies that $\sigma \in A_{\tau(\infty)}$. $\square$

9.9. Proposition. (i) *Suppose that $\mathcal{S}$ is a Segal algebra in A. Then $C_{\mathrm{BSE}\,(\mathcal{S})}(X) \subseteq C_{\mathrm{BSE}}\,(X) = M(A)$ hold.*

(ii) *Suppose that a Segal algebra $\mathcal{S}$ in A is* BSE *and* BED. *Then $\mathcal{S}$ coincides with A.*

Proof. (i) Let $f \in C_{\mathrm{BSE}\,(\mathcal{S})}(X)$. There exists a bounded net $\{f_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{S}$ such that $f(x) = \lim_{\lambda\in\Lambda} f_\lambda(x)$ $(x \in X)$ by [**21**, Theorem 4 (i)]). Since a bounded net in $\mathcal{S}$ is also a bounded net in A, we have $f \in C_{\mathrm{BSE}}(X)$. Since $C_{\mathrm{BSE}}(X) = M(A)$ by the condition (γ_A) posed in the beginning of this section, we get the desired inclusion.

(ii) If a Segal algebra $\mathcal{S}$ in A is BSE, then $\|\ \|_{\mathrm{BSE}\,(\mathcal{S})}$ and $\|\ \|_{M(\mathcal{S})}$ are equivalent norms in $M(\mathcal{S})$ (cf., [**21**, page 151, Remark]). If $\mathcal{S}$ is BSE and BED, $\|\ \|_{\mathrm{BSE}\,(\mathcal{S})}$ and $\|\ \|_{\mathcal{S}}$ are equivalent norms in $\mathcal{S}$. Consequently, two norms $\|\ \|_{\mathcal{S}}$ and $\|\ \|_{\mathrm{op}}$ (the multiplication operator norm in $\mathcal{S}$) are equivalent. This implies that $\mathcal{S}$ contains a bounded approximate identity by Theorem C′ (i). Hence $\mathcal{S} = A$ follows from Theorem C′ (ii). ◻

9.10. Theorem. *Let $\mathcal{S}$ be a Segal algebra in A. Then the following are equivalent*:

(a) *$\mathcal{S}$ has a bounded weak approximate identity in the sense of Jones-Lahr.*

(b) *$\mathcal{S}$ is* BSE, *that is, $M(\mathcal{S}) = C_{\mathrm{BSE}\,(\mathcal{S})}(X)$.*

Moreover, if $\mathcal{S}$ satisfies (a) *or* (b), *then $M(\mathcal{S}) = M(A)$ holds.*

Proof. (a) ⇒ (b). By (a) and [**21**, Corollary 5], we have

$$M(\mathcal{S}) \subseteq C_{\mathrm{BSE}\,(\mathcal{S})}(X). \tag{16}$$

On the other hand, by Propositions 4.5 and 9.9, we have

$$C_{\mathrm{BSE}\,(\mathcal{S})}(X) \subseteq C_{\mathrm{BSE}}(X) = M(A) \subseteq M(\mathcal{S}). \tag{17}$$

By (16) and (17), we get $C_{\mathrm{BSE}\,(\mathcal{S})}(X) = M(\mathcal{S})$, which implies (b).

(b) ⇒ (a). This implication follows easily from [**21**, Corollary 5]. Moreover, if $\mathcal{S}$ satisfies (a) (or equivalently (b)), $M(A) = M(\mathcal{S})$ holds by (16) and (17). ◻

9.11. *Remarks.* (a) $S_p(G)$ and $A_p(G)$ of Example 2.2 have bounded weak approximate identities in the sense of Jones-Lahr [**13**].

(b) If $\tau \in A_{\rm loc} \setminus M(A)$ and $n \in \mathbf{N}$, then $A_{\tau(n)}$ is a proper Segal algebra which is BED by Theorems 5.4 and 9.4. Therefore $A_{\tau(n)}$ is not BSE by Proposition 9.9 (ii).

(c) If $\tau \in A_{\rm loc}$ with $\|\tau\|_\infty < 1/C_0$ and $A_{\tau(\infty)} \subsetneqq A$, then $A_{\tau(\infty)}$ is not BSE. This follows from Theorem 9.8 and Proposition 9.9 (ii).

10. Applications of local A-functions. In this section $\mathcal{A}$ stands for the Fourier algebra on a non-compact LCA group G. For $f \in \mathcal{A}$ and $y \in G$ the translation of f by y is denoted by $f_y : f_y(x) = f(x-y)$ $(x \in G)$.

As an application of $A_{\rm loc}$ we show a representation theorem for the multiplier algebra of the smallest isometrically translation invariant Segal algebra in $\mathcal{A}$.

10.1. Definition. Let V be a non-empty open subset of G with compact closure $\overline{V}$. We define two subsets $\Lambda_{\bar V}(G)$ and $\widetilde{\Lambda}_{\bar V}(G)$ of $\mathcal{A}$ by

$$\begin{aligned}\Lambda_{\bar V}(G) &:= \{f \in \mathcal{A} : \ \operatorname{supp} f \subset \overline{V}\},\\ \widetilde{\Lambda}_{\bar V}(G) &:= \{f \in \mathcal{A} : \ \text{there exists } y \in G \text{ s.t. } \operatorname{supp} f \subset \overline{V} + y\}.\end{aligned}$$

Thus, we have $\widetilde{\Lambda}_{\bar V}(G) = \cup_{y \in G} \Lambda_{\bar V + y}(G)$. One can easily see that $\widetilde{\Lambda}_{\bar V}(G)$ is contained in $\mathcal{A}_c$, and hence contained in every Segal algebra in $\mathcal{A}$.

Suppose that $f_n \in \widetilde{\Lambda}_{\bar V}(G)$, $n = 1, 2, \dots$, with $\sum_{n=1}^\infty \|f_n\|_{\mathcal{A}} < \infty$. Then there is an $f \in \mathcal{A}$ such that $\|f - \sum_{n=1}^N f_n\|_{\mathcal{A}} \to 0$ $(N \to \infty)$. Here we write $f = \sum_{n=1}^\infty f_n$ and call it a $\overline{V}$-representation of f.

We put $\mathcal{S}_{\bar V}(G) := \{f \in \mathcal{A} : f$ has at least one $\overline{V}$ $-$ representation$\}$, and for $f \in \mathcal{S}_{\bar V}(G)$, we define $\|f\|_{\bar V} := \inf\{\sum_{n=1}^\infty \|f_n\|_{\mathcal{A}} : f = \sum_{n=1}^\infty f_n (\overline{V} - \text{representation})\}$.

Under the above definitions of $\mathcal{S}_{\bar V}(G)$ and $\|\ \|_{\bar V}$, $(\mathcal{S}_{\bar V}(G), \|\ \|_{\bar V})$ is a Segal algebra in $\mathcal{A}$, which is isometrically translation invariant in the sense that, if $f \in \mathcal{S}_{\bar V}(G)$ then $f_y \in \mathcal{S}_{\bar V}(G)$ and $\|f\|_{\bar V} = \|f_y\|_{\bar V}$ hold for all $y \in G$. Moreover, it is minimal in the sense that $\mathcal{S}_{\bar V}(G)$ is contained in any isometrically translation invariant Segal algebra in $\mathcal{A}$.

Actually, it can be easily seen from the definitions of $\mathcal{S}_{\bar{V}}(G)$ that the smallest isometrically character invariant Segal algebra $\mathcal{S}^1_{\bar{V}}(\widehat{G})$ (constructed in [**18**, Chapter 6]) is isometrically isomorphic to $\mathcal{S}_{\bar{V}}(G)$ through the Fourier transform.

10.2. *Remark.* The smallest isometrically character invariant Segal algebra in $L^1(\widehat{G})$ was found and constructed in 1981 by Feichtinger [**9**], which is also called the Feichtinger Segal algebra.

10.3. Theorem. *Put* $C^{\bar{V}}_{\mathrm{BSE}}(G) := \{\tau \in C(G) : \|\tau\|^{\bar{V}} := \sup_{x\in G} \|\tau\|_{\mathrm{BSE},G\setminus(\bar{V}+x)} < \infty\}$. *Then the following hold.*

(i) $(C^{\bar{V}}_{\mathrm{BSE}}(G), \|\ \|^{\bar{V}})$ *is a Banach algebra.*

(ii) $C^{\bar{V}}_{\mathrm{BSE}}(G) = M(\mathcal{S}_{\bar{V}}(G))$, *and* $(C^{\bar{V}}_{\mathrm{BSE}}(G), \|\ \|^{\bar{V}})$ *is isomorphic to* $(M(\mathcal{S}_{\bar{V}}(G)), \|\ \|_{M(\mathcal{S}_{\bar{V}}(G))})$.

10.4. *Remark.* It is known that there are some Wiener amalgam spaces $W(\mathcal{A}, \ell(I)^1)$ and $W(\mathcal{A}, \ell(I)^\infty)$ such that $W(\mathcal{A}, \ell(I)^1)$ is isomorphic to $\mathcal{S}_{\overline{V}}(G)$ and its multiplier algebra is given by $W(\mathcal{A}, \ell(I)^\infty)$ (cf., [**8, 10, 11**]).

10.5. Lemma. $\|f\|_{\mathcal{A}} \le \|f\|_{\bar{V}}$ $(f \in \mathcal{S}_{\bar{V}}(G))$.

Proof. Let $f \in \mathcal{S}_{\bar{V}}(G)$, and let $f = \sum_{n=1}^{\infty} f_n$ be a $\overline{V}$-representation. Then $\|f\|_{\mathcal{A}} \le \sum_{n=1}^{\infty} \|f_n\|_{\mathcal{A}}$. Taking the infimum over all the $\overline{V}$-representations of f in this inequality, we get $\|f\|_{\mathcal{A}} \le \|f\|_{\bar{V}}$. □

10.6. Lemma. *There exists a constant* $M > 0$ *which satisfies the conditions that, for each* $\tau \in \mathcal{A}_{\mathrm{loc}}$ *and* $x_0 \in G$, *there is an* $e_{\tau,x_0} \in \mathcal{A}$ *satisfying*

$$e_{\tau,x_0} = \tau \quad \text{on } \overline{V} + x_0 \text{ and } \|e_{\tau,x_0}\|_{\mathcal{A}} \le M\|\tau\|_{\mathrm{BSE},G\setminus(\overline{V}+x_0)}. \tag{18}$$

Proof. Let $\tau \in \mathcal{A}_{\mathrm{loc}}$ and $x_0 \in G$ be fixed arbitrarily. Since $\mathcal{A}/I(\overline{V})$ is BED by [**14**, Theorem 5.2], there exists an $M_1 > 0$ such that

$$\|g + I(\overline{V})\|_{\mathcal{A}/I(\bar{V})} \le M_1 \|g|_{\bar{V}}\|_{\mathrm{BSE}\,(\mathcal{A}/I(\bar{V}))} \quad (g \in \mathcal{A}). \tag{19}$$

For each $g \in \mathcal{A}$, we have

$$\begin{aligned} \|g + I(\overline{V})\|_{\mathcal{A}/I(\bar{V})} &= \inf\left\{\|g + f\|_{\mathcal{A}} : f \in I(\overline{V})\right\} \\ &= \inf\left\{\|g_{x_0} + f_{x_0}\|_{\mathcal{A}} : f \in I(\overline{V})\right\} \\ &= \inf\left\{\|g_{x_0} + f\|_{\mathcal{A}} : f \in I(\overline{V} + x_0)\right\} \\ &= \|g_{x_0} + I(\overline{V} + x_0)\|_{\mathcal{A}/I(\bar{V}+x_0)}. \end{aligned} \tag{20}$$

Since $\|p\|_{(\mathcal{A}/I(\bar{V}))^*} = \|p\|_{\mathcal{A}^*}$ $(p \in \mathrm{span}\,(\overline{V}))$, by Lemma 7.1 (i), we have

$$\begin{aligned} &\|g|_{\bar{V}}\|_{\mathrm{BSE}\,(\mathcal{A}/I(\bar{V}))} \\ &= \sup\left\{\left|\sum_{x\in\bar{V}} \widehat{p}(x)g(x)\right| : p \in \mathrm{span}\,(\overline{V}), \|p\|_{\mathcal{A}^*} \le 1\right\} \\ &= \sup\left\{\left|\sum_{x\in\bar{V}+x_0} \widehat{p}(x-x_0)g(x-x_0)\right| : p \in \mathrm{span}\,(\overline{V}), \|p\|_{\mathcal{A}^*} \le 1\right\} \\ &= \sup\left\{\left|\sum_{x\in\bar{V}+x_0} \widehat{p}(x)g_{x_0}(x)\right| : p \in \mathrm{span}\,(\overline{V}+x_0), \|p\|_{\mathcal{A}^*} \le 1\right\} \\ &= \|g_{x_0}|_{\bar{V}+x_0}\|_{\mathrm{BSE}\,(\mathcal{A}/I(\bar{V}+x_0))}. \end{aligned} \tag{21}$$

By (19), (20) and (21), we have

$$\begin{aligned} \|g_{x_0} + I(\overline{V} + x_0)\|_{\mathcal{A}/I(\bar{V}+x_0)} \le M_1 \|g_{x_0}|_{\bar{V}+x_0}\|_{\mathrm{BSE}\,(\mathcal{A}/I(\bar{V}+x_0))} \\ (g \in \mathcal{A}). \end{aligned} \tag{22}$$

By Proposition 7.2, there is an $f_0 \in \mathcal{A}$ such that $f_0 = \tau$ on $\overline{V} + x_0$. Moreover, we can choose an $e_{\tau,x_0} \in \mathcal{A}$ such that $e_{\tau,x_0} = f_0$ on $\overline{V} + x_0$ and

$$\|e_{\tau,x_0}\|_{\mathcal{A}} \le 2\|f_0 + I(\overline{V} + x_0)\|_{\mathcal{A}/I(\bar{V}+x_0)}. \tag{23}$$

By (22) and Lemma 7.1 (ii), we have

$$\begin{aligned} \|f_0 + I(\overline{V} + x_0)\|_{\mathcal{A}/I(\bar{V}+x_0)} &\le M_1 \|f_0|_{\overline{V}+x_0}\|_{\mathrm{BSE}\,(\mathcal{A}/I(\bar{V}+x_0))} \\ &= M_1 \|f_0\|_{\mathrm{BSE}, G\setminus(\bar{V}+x_0)} \\ &= M_1 \|\tau\|_{\mathrm{BSE}, G\setminus(\bar{V}+x_0)}. \end{aligned} \tag{24}$$

From the choices of f_0 and e_{τ,x_0}, the equality $e_{\tau,x_0} = \tau$ on $\overline{V} + x_0$ is observed. Hence, (18) follows from (23) and (24) with $M = 2M_1$. $\square$

Proof of Theorem 10.3. (i) It is easy to see that $C_{\mathrm{BSE}}^{\bar{V}}(G)$ is a normed linear space with respect to $\| \ \|^{\bar{V}}$, as well as $\| \ \|^{\bar{V}}$ is an algebra norm by Lemma 7.3.

To prove completeness, let $\{\tau_n\}$ be a Cauchy sequence in $C_{\mathrm{BSE}}^{\bar{V}}(G)$. For any $y \in G$, we can choose an $x \in G$ so that $y \in \overline{V} + x$. Then we have

$$|\tau_n(y) - \tau_m(y)| \le \|\tau_n - \tau_m\|_{\mathrm{BSE}, G\setminus(\bar{V}+x)} \le \|\tau_n - \tau_m\|^{\bar{V}}.$$

Hence, $\{\tau_n\}$ is a uniformly convergent sequence in $C_b(G)$. Let $\tau \in C_b(G)$ be the uniform limit of $\{\tau_n\}$. Let $\varepsilon > 0$. Choose an $n_0 \in \mathbf{N}$ such that $\|\tau_n - \tau_m\|^{\bar{V}} < \varepsilon$ $(n, m \ge n_0)$. Let $x \in G$ and $p \in \mathrm{span}\,(\overline{V} + x)$ with $\|p\|_{\mathcal{A}^*} \le 1$. Then we have $|\sum_{y\in\bar{V}+x} \widehat{p}(y)(\tau_n(y) - \tau_m(y))| \le \varepsilon$ $(n, m \ge n_0)$. Fixing n and letting m go to infinity we have $|\sum_{y\in\bar{V}+x} \widehat{p}(y)(\tau_n(y) - \tau(y))| \le \varepsilon$. Since $x \in G$, $p \in \mathrm{span}\,(\overline{V} + x)$ with $\|p\|_{\mathcal{A}^*} \le 1$ and $n (n \ge n_0)$ are arbitrarily chosen, we have

$$\|\tau_n - \tau\|^{\bar{V}} = \sup_{x\in G} \sup_{\substack{p\in \mathrm{span}\,(\bar{V}+x) \\ \|p\|_{\mathcal{A}^*} \le 1}} \left| \sum_{y\in\bar{V}+x} \widehat{p}(y)(\tau_n(y) - \tau(y)) \right| \le \varepsilon \quad (n \ge n_0).$$

Thus, we get $\tau \in C_{\mathrm{BSE}}^{\bar{V}}(G)$ and $\|\tau_n - \tau\|^{\bar{V}} \to 0$ $(n \to \infty)$. Therefore, $\| \ \|^{\bar{V}}$ is complete.

(ii) $(\subseteq)$. Let τ be in $C(G)$ such that $\sup_{x\in G} \|\tau\|_{\mathrm{BSE}, G\setminus(\bar{V}+x)} < \infty$. Let $f \in \mathcal{S}_{\bar{V}}(G)$, and let $f = \sum_{n=1}^{\infty} f_n$ be a $\overline{V}$-representation such that $\mathrm{supp}\, f_n \subseteq \overline{V} + x_n$ for $x_n \in G$ $(n = 1, 2, 3, \ldots)$. By Lemma 10.6, there exists a positive constant M such that, for each $n \in \mathbf{N}$, we can choose an $e_{\tau,x_n} \in \mathcal{A}$ satisfying $e_{\tau,x_n} = \tau$ on $\overline{V} + x_n$ and $\|e_{\tau,x_n}\|_{\mathcal{A}} \le M\|\tau\|_{\mathrm{BSE}, G\setminus(\bar{V}+x_n)}$, and hence

$$\|\tau f_n\|_{\mathcal{A}} = \|e_{\tau,x_n} f_n\|_{\mathcal{A}} \le \|e_{\tau,x_n}\|_{\mathcal{A}} \|f_n\|_{\mathcal{A}} \le M\|\tau\|_{\mathrm{BSE}, G\setminus(\bar{V}+x_n)} \|f_n\|_{\mathcal{A}}.$$

Thus, we have

$$\sum_{n=1}^{\infty} \|\tau f_n\|_{\mathcal{A}} \le M \left(\sup_{x\in G} \|\tau\|_{\mathrm{BSE}, G\setminus(\bar{V}+x)} \right) \sum_{n=1}^{\infty} \|f_n\|_{\mathcal{A}} < \infty. \tag{25}$$

By (25), it is easy to see that there exists a $g \in \mathcal{A}$ such that $\|g - \sum_{k=1}^{n} f_k\tau\|_{\mathcal{A}} \to 0$ $(n \to \infty)$. Since $\operatorname{supp} f_n\tau \subseteq \operatorname{supp} f_n \subseteq \overline{V} + x_n$ for each $n \in \mathbf{N}$, g is an element in $\mathcal{S}_{\bar{V}}(G)$ with a $\overline{V}$-representation $g = \sum_{n=1}^{\infty} f_n\tau$. Since $g = \sum_{n=1}^{\infty} f_n\tau = f\tau$, we have $\tau f \in \mathcal{S}_{\bar{V}}(G)$, and hence we have $\tau \in M(\mathcal{S}_{\bar{V}}(G))$.

($\supseteq$). Choose an element $e \in \mathcal{A}_c$ such that $e = 1$ on $\overline{V}$. Let $\tau \in M(\mathcal{S}_{\bar{V}}(G))$ and $x_0 \in G$ be arbitrary. Since $e_{x_0} \in \mathcal{A}_c \subseteq \mathcal{A}_{\tau(1)}$, we have $(e_{x_0}\tau)(x) = e(x - x_0)\tau(x) = \tau(x)$ for all $x \in \overline{V} + x_0$. Hence,

$$\begin{aligned}(26)\quad & \|\tau\|_{\mathrm{BSE},G\setminus(\bar{V}+x_0)} \\ &= \sup\left\{\left|\sum_{x\in\bar{V}+x_0} \widehat{p}(x)\tau(x)\right| : p \in \operatorname{span}(\overline{V}+x_0), \|p\|_{\mathcal{A}^*} \le 1\right\} \\ &= \sup\left\{\left|\sum_{x\in\bar{V}+x_0} \widehat{p}(x)(e_{x_0}\tau)(x)\right| : p \in \operatorname{span}(\overline{V}+x_0), \|p\|_{\mathcal{A}^*} \le 1\right\} \\ &\le \|e_{x_0}\tau\|_{\mathrm{BSE}} \le \|e_{x_0}\tau\|_{\mathcal{A}}.\end{aligned}$$

Since $e_{x_0}\tau \in \mathcal{A}_c \subseteq \mathcal{S}_{\bar{V}}(G)$, we get $\|e_{x_0}\tau\|_{\mathcal{A}} \le \|e_{x_0}\tau\|_{\overline{V}}$ from Lemma 10.5.

Moreover, $\|e_{x_0}\|_{\bar{V}} = \|e\|_{\bar{V}}$ since $\mathcal{S}_{\bar{V}}(G)$ is isometrically translation-invariant. Thus, we have from (26) that

$$(27)\qquad \|\tau\|_{\mathrm{BSE},G\setminus(\overline{V}+x_0)} \le \|e_{x_0}\tau\|_{\mathcal{A}} \le \|e_{x_0}\tau\|_{\bar{V}} \le \|\tau\|_{M(\mathcal{S}_{\bar{V}}(G))}\|e\|_{\bar{V}}.$$

Since x_0 in (27) is arbitrary in G we have $\|\tau\|^{\bar{V}} \le \|\tau\|_{M(\mathcal{S}_{\bar{V}}(G))}\|e\|_{\bar{V}} < \infty$. $\square$

10.7. Corollary. $M(\mathcal{A}) \subsetneqq M(\mathcal{S}_{\bar{V}}(G))$.

Proof. By the structure theorem of LCA groups [**12**, Theorem 24.30], G is topologically isomorphic to $\mathbf{R}^d \times L$, where d is a non-negative integer and L is an LCA group which contains an open compact subgroup. We divide the proof into two parts: (i) the case $d \neq 0$, where we have $G = \mathbf{R} \times H$ with $H = \mathbf{R}^{d-1} \times L$; (ii) the case $d = 0$, where G contains an open compact subgroup G_0.

(i) Let π_1 be the natural projection of G onto $\mathbf{R}$. For any function φ on $\mathbf{R}$, set $\widetilde{\varphi} = \varphi \circ \pi_1$. In this case, we can choose a V, which appears in the definition of $\mathcal{S}_{\bar{V}}(G)$, so that $\pi_1(V) \subseteq [-1, 1]$. Let φ be a C^∞-function on $\mathbf{R}$ such that $\varphi = 1$ on $[1, \infty)$ and $\varphi = -1$ on $(-\infty, -1]$.

Take a C^∞-function e in $\mathcal{A}(\mathbf{R})_c$ such that $e = 1$ on $[-3, 3]$. Then $e\varphi$ is a C^∞-function on $\mathbf{R}$ with compact support, and hence $e\varphi \in \mathcal{A}(\mathbf{R})$ and so $\widetilde{e\varphi} \in M(\mathcal{A})$ by Lemma 8.5 (ii).

Let $(x_0, y_0) \in G = \mathbf{R}\times H$ be given arbitrarily. By a simple calculation we have

$$\widetilde{\varphi}|_{\bar{V}+(x_0,y_0)} = \begin{cases} \widetilde{e\varphi}|_{\bar{V}+(x_0,y_0)} & |x_0| \leq 2, \\ 1 & x_0 > 2, \\ -1 & x_0 < -2. \end{cases}$$

Hence,

$$\sup_{(x_0,y_0)} \|\widetilde{\varphi}\|_{\mathrm{BSE},G\setminus(\bar{V}+(x_0,y_0))} \leq \max\{\|\widetilde{e\varphi}\|_{\mathrm{BSE}}, \|1\|\} < \infty. \tag{28}$$

If follows that $\widetilde{\varphi} \in M(\mathcal{S}_{\bar{V}}(G))$ from (28) and Theorem 10.3.

On the other hand, $\varphi \notin M(\mathcal{A}(\mathbf{R}))$ follows from Lemma 8.7. Hence, $\widetilde{\varphi} \notin M(\mathcal{A})$ follows from Lemma 8.5 (ii). Consequently, the corollary holds in this case.

(ii) In this case we set $V = G_0$. By Lemma 8.6, there exists a function $\phi \in \mathcal{A}_{\mathrm{loc}}$ which satisfies (i) and (ii) of that lemma. Then it is easy to see that $\sup_{x\in G} \|\phi\|_{\mathrm{BSE},G\setminus(\overline{V}+x)} < \infty$, and the corollary also holds in this case. □

10.8. Corollary. $\mathcal{S}_{\bar{V}}(G)$ *is not* BSE.

Proof. By Corollary 10.7 and Theorem 9.10, $\mathcal{S}_{\bar{V}}(G)$ is not BSE. □

10.9. Corollary. *Suppose that $\mathcal{A} = \mathcal{A}(G)$ is the Fourier algebra on an infinite discrete abelian group G. Let us consider G as a subset of $\Phi_{M(\mathcal{S}_{\bar{V}}(G))}$ in the natural way. Then $\Phi_{M(\mathcal{S}_{\bar{V}}(G))}$ is homeomorphic to βG, the Stone-Čech compactification of G. In particular, G is dense in $\Phi_{M(\mathcal{S}_{\bar{V}}(G))}$.*

Proof. We take $V = \{e\}$, where e is the identity of G. Then by Theorem 10.3 $(M(\mathcal{S}_{\bar{V}}(G), \|\ \|_{M(\mathcal{S}_{\bar{V}}(G))})$ is isomorphic to $(C^{\bar{V}}_{\mathrm{BSE}}(G), \|\ \|^{\bar{V}})$. For any $f \in C_b(G)$ and $x \in G$ we have

$$\begin{aligned}\|f\|_{\mathrm{BSE},G\setminus\{x\}} &= \sup\left\{\left|\sum_{y\in\{x\}} \widehat{p}(y)f(y)\right| : p \in \mathrm{span}\,(\{x\}), \|p\|_{\mathcal{A}^*} \leq 1\right\} \\ &= |f(x)|.\end{aligned}$$

Therefore, $(C_{\mathrm{BSE}}^{\bar{V}}(G), \| \ \|^{\bar{V}}) = (C_b(G), \| \ \|_\infty)$. Since the Gelfand space of $(C_b(G), \| \ \|_\infty)$ is βG, and the Gelfand transform is the natural isomorphism of $C_b(G)$ onto $C(\beta G)$, the assertion of this corollary is observed. $\square$

10.10. Problems. Let G be an LCA group which is neither discrete nor compact.

(i) Are there any effective representations of $\Phi_{M(\mathcal{S}_{\bar{V}}(G))}$?

(ii) Is G dense in $\Phi_{M(\mathcal{S}_{\bar{V}}(G))}$?

10.11. Proposition. *If $\mathcal{S}$ is an isometrically translation invariant Segal algebra in $\mathcal{A}$, we have $M(\mathcal{S}) \subseteq M(\mathcal{S}_{\bar{V}}(G))$.*

Proof. We observe that there is a constant $C_{\bar{V}} > 0$ such that $\|f\|_{\mathcal{S}} \leq C_{\bar{V}} \|f\|_{\mathcal{A}}$ $(f \in \widetilde{\Lambda}_{\bar{V}}(G))$. Choose an $e \in \mathcal{A}_c$ such that $e = 1$ on $\overline{V}$. Let $f \in \widetilde{\Lambda}_{\bar{V}}(G)$ and $x \in G$ be such that $f_x \in \Lambda_{\bar{V}}(G)$. Then we have $\|f\|_{\mathcal{S}} = \|f_x\|_{\mathcal{S}} = \|ef_x\|_{\mathcal{S}} \leq \|e\|_{\mathcal{S}} \|f_x\|_{\mathcal{A}} = \|e\|_{\mathcal{S}} \|f\|_{\mathcal{A}}$. Thus, the inequality holds with a constant $C_{\bar{V}} = \|e\|_{\mathcal{S}}$. Let $\tau \in M(\mathcal{S})$ be arbitrary, and let $f \in \mathcal{S}_{\bar{V}}(G)$ with a $\bar{V}$-representation $f = \sum_{n=1}^\infty f_n$ be such that $\operatorname{supp} f_n \subseteq \overline{V} + x_n$ for $x_n \in G$ $(n = 1, 2, 3, \ldots)$. Then we have from above

$$\|\tau f_n\|_{\mathcal{A}} \leq \|\tau f_n\|_{\mathcal{S}} \leq \|\tau\|_{M(\mathcal{S})} \|f_n\|_{\mathcal{S}} \leq C_{\bar{V}} \|\tau\|_{M(\mathcal{S})} \|f_n\|_{\mathcal{A}} \quad (n = 1, 2, 3, \ldots).$$

Hence, there is a $g \in \mathcal{S}_{\bar{V}}(G)$ with a $\overline{V}$-representation $g = \sum_{n=1}^\infty \tau f_n$. Since $g = \sum_{n=1}^\infty \tau f_n = \tau f$, we have $\tau f \in \mathcal{S}_{\bar{V}}(G)$. Hence $\tau \in M(\mathcal{S}_{\bar{V}}(G))$. Thus, $M(\mathcal{S}) \subseteq M(\mathcal{S}_{\bar{V}}(G))$ follows. $\square$

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ENDNOTES

1. In [**14**] "BSE" is denoted by "BE." But this is not preferable and will cause confusion. In this paper, we use "BSE" as it is originally defined in [**21**].

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