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CUSPIDALITY OF PULLBACKS OF SIEGEL-HILBERT EISENSTEIN SERIES ON HERMITIAN SYMMETRIC DOMAINS

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ABSTRACT. Let G_N be either a symplectic, unitary or Hermitian orthogonal group of rank $2N = 2m + 2n$ with $m \leq n$. We show that the restriction of a Siegel-Hilbert Eisenstein series on G_N to the diagonally embedded group $G_m \times G_n$ has a nontrivial cuspidal component in the smaller variable. As a consequence, we explicitly construct classes of Siegel-Hilbert cuspforms with rational-valued Fourier coefficients.

1. Introduction. The arithmeticity of Eisenstein series on general reductive groups is well-studied; see [**9**] for an example. The arithmetic nature of cuspforms is more mysterious. Pullbacks of certain Eisenstein series to smaller groups is a well-established tool for constructing cuspforms that inherit some of the arithmetic properties of the Eisenstein series. This has been a powerful method for proving arithmetic results about cuspforms and values of L-functions.

Let $N = m + n$ with $m \leq n$. In [**3**] it is shown that, for a certain Siegel Eisenstein series E_N on $Sp(N)$, the pullback to $Sp(m) \times Sp(n)$ has nontrivial cuspidal part in the smaller variable $g \in Sp(m)$. This was shown by modifying an argument in [**7**]. In this paper, we generalize this result to a larger class of Eisenstein series on symplectic, unitary and Hermitian orthogonal groups. We investigate the cuspidality of the pullbacks of Siegel-Hilbert Eisenstein series on these groups, give a general decomposition theorem for such pullbacks and present a unified approach for these different groups. Such results have applications to the arithmeticity of cuspforms [**1, 7**], L-functions [**5, 14**] and to the Bloch-Kato conjecture [**2**].

Let k be a totally real number field of degree ℓ over $\mathbf{Q}$, and let F be either k, a complex quadratic extension of k or a quaternionic extension of k. Let $\mathbf{A}$ denote the adeles of F, and let $\mathcal{O}_F$ denote the

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integers of F. In the following, $G_N(\mathbf{A})$ will be either $Sp(N)$, $U(N,N)$ or $O^*(4N)$. Let $N>1$. We consider only holomorphic automorphic forms. Let χ denote a character on the Siegel parabolic $P_N(\mathbf{A})$, and let $I(\chi)=\mathrm{Ind}_{P_N(\mathbf{A})}^{G_N(\mathbf{A})}\chi$ denote the induced representation space. For v a local place of F and a character χ_v on the local group $P_N(F_v)$, let $I_v(\chi_v)=\mathrm{Ind}_{P_N(F_v)}^{G_N(F_v)}\chi_v$. For $\varepsilon\in I(\chi)$ we have the factorization $\varepsilon=\prod_v\varepsilon_v$ where $\varepsilon_v\in I_v(\chi_v)$, and we can define a Siegel-Hilbert Eisenstein series on $G_N(\mathbf{A})$ by

$$E_N(g;\kappa)=\sum_{\gamma\in P_N(F)\backslash G_N(F)}\varepsilon(\gamma g).$$

We choose ε to give an Eisenstein series of a given weight κ and level $\mathfrak{n}$ where $\mathfrak{n}$ is an ideal in the integers of F. Precise definitions of these objects are given in Sections 2 and 3. Let $\iota:G_m(\mathbf{A})\times G_n(\mathbf{A})\hookrightarrow G_N(\mathbf{A})$ be the embedding from Section 3, and let $E(\iota(g_1,g_2);\kappa)$ be the pullback of the Eisenstein series to $G_m(\mathbf{A})\times G_n(\mathbf{A})$.

Theorem 1. *Let $E_N(g;\kappa)$ be a Siegel-Hilbert Eisenstein series on $G_N(\mathbf{A})$ of weight $\kappa>2N+1$ and level $\mathfrak{n}$. Then we have the following.*

i) *The pullback decomposes*

$$E_N(\iota(g_1,g_2);\kappa)=\Psi_N(g_1,g_2)+\Phi_N(g_1,g_2)$$

where $\langle\Psi_N(\cdot,g_2),f_1\rangle=\langle\Psi_N(g_1,\cdot),f_2\rangle=0$ for f_1 a cuspform on $G_m(\mathbf{A})$, f_2 a cuspform on $G_n(\mathbf{A})$ and $\Phi_N(g_1,g_2)$ is a cuspform in the variable $g_1\in G_m(\mathbf{A})$.

ii) *The Eisenstein series can be chosen so that Φ_N is nontrivial. If $m=n$, then Φ_N is nontrivial for any choice of $\varepsilon_v\in I(\chi_v)$ with $v<\infty$ and $v\nmid\mathfrak{n}$.*

iii) *For f a cuspform on $G_n(\mathbf{A})$, we have that $\langle\Phi_N(g_1,\cdot),f\rangle=0$ if and only if $m<n$.*

Cuspforms constructed in this way inherit certain arithmetic properties of the Eisenstein series. In particular, we show that their Fourier coefficients are rational-valued.

Corollary 1. *Let ξ be as in Lemma* 1*, and let χ be a character of the Siegel parabolic $P_N(\mathbf{A})$. There is some $\varepsilon \in I(\chi)$ so that*

$$\sum_{\gamma \in G_m(F)\backslash G_m(F)\times G_n(F)} \varepsilon(\xi\iota(g_1, g_2))$$

is a nontrivial cuspform in $g_1 \in G_m(\mathbf{A})$ with F-valued Fourier coefficients.

In Section 2 we define the groups and automorphic forms used here. In Sections 3 and 4 we study the restriction of a Siegel-Hilbert Eisenstein series on $G_N(\mathbf{A})$ to embedded copies of smaller groups of the same type. We prove Theorem 1 in Section 3 by first proving the decomposition in (i), then parts (ii) and (iii), and then completing the last part of (i) by showing that Φ_N is a cuspform in its smaller variable. In Section 4 we apply these results to prove Corollary 1.

2. Automorphic forms and Eisenstein series. Let 1_N denote the $N \times N$ identity matrix, and let 0_N denote the $N \times N$ zero matrix. Let $J_N = \begin{pmatrix} 0_N & \eta 1_N \\ 1_N & 0_N \end{pmatrix}$ where $\eta = \pm 1$ is fixed for a given type of group. Let k be a totally real number field of degree ℓ over $\mathbf{Q}$, and let F be a finite dimensional division algebra over k. Following [**6**], for the rational representation $\mathfrak{r} : F \mapsto M_{d\times d}(k)$, define

$$\mathfrak{r} : GL_N(F) \longrightarrow GL_N(M_{d\times d}(k)) \cong GL_{Nd}(k).$$

Here $M_{d\times d}(k)$ denotes the set of $d \times d$ matrices with entries in k. For a commutative $\mathbf{C}$-algebra A we also define $\mathfrak{r} : GL_N(F \otimes_{\mathbf{Q}} A) \to GL_M(A)$. Let $\alpha \mapsto \alpha^\sigma$ be an involution of F and, for $g \in M_{N\times N}(F)$, let g^σ denote entry-wise action by σ. Let g^T denote the usual matrix transpose, and let g^* denote the convolution $g^{\sigma T}$. Let $\mathcal{O}_F$ be an order in F so that $\mathbf{Z} \subseteq \mathcal{O}_F$ and $\mathcal{O}_F$ is stable under σ. For a commutative $\mathbf{Z}$-algebra A define

$$G_N(A) = \{g \in GL_{2N}(\mathcal{O}_F \otimes_{\mathbf{Z}} A) \mid g^* J_N g = J_N,\ \det(\mathfrak{r}(g)) = 1\}.$$

By the class number h_F of F we mean the number of isomorphism classes of locally free left (or right) $\mathcal{O}_F$-ideals in $\mathcal{O}_F \otimes \mathcal{Q}$.

Let $\mu(g,z) = \det(cz+d)^{-1}$, and define $\mu(g,z)^\kappa = \prod_{j=1}^{\ell} \mu(g_j,z_j)^\kappa$ using a multi-index notation where $g = (g_j)_j \in G_N(F)$ and $z = (z_j)_j \in \mathfrak{H}_N^\ell$. As $G_N(F)$ acts on $\mathfrak{H}_N^\ell$, the space of holomorphic Hilbert modular forms of weight the ℓ-tuple $\kappa = (\kappa,\dots,\kappa)$, level $\mathfrak{n}$, and character ρ is the space of holomorphic $\mathbf{C}$-valued functions $f(z)$ on $\mathfrak{H}_N^\ell$ so that $f(g(z))\mu(g,z)^\kappa = \rho(g)f(z)$ for all $z \in \mathfrak{H}_N^\ell$ and $g \in \Gamma_0^N(\mathfrak{n})$.

Let $\mathbf{A}$ denote the adeles of F, and let $\mathbf{A}_0$ and $\mathbf{A}_\infty$ denote the finite and infinite adeles, respectively. Let v be a local place with norm $|\cdot|_v$ normalized so that $|\varpi_v| = q_v^{-1}$, where ϖ_v is the uniformizer and q_v is the cardinality of the residue field. For $v < \infty$, set $K_v = G_n(\mathcal{O}_v)$, and let $K_{\mathbf{A}_0} = \prod_{v<\infty} K_v$. For $v \mid \infty$, define

$$K_v = \{g \in G_N(F_v) \mid \mathfrak{r}(g) \in U'(N)\},$$

where $U'(N) = U(2N) = \{g \in GL_{2N}(F_v) \mid g^*g = 1_{2N}\}$ for $F = k$ or a quaternionic extension of k. For F a complex quadratic extension of k set $U'(N) = U(N) \times U(N)$. Then K_v is a maximal compact subgroup of $G_N(F_v)$, and set $K_\infty = \prod_{v\mid\infty} K_v$ and note that K_∞ is the isotropy group in $G_N(F_\infty)$ of $i \cdot 1_N \in \mathfrak{H}_N^\ell$. This gives a diffeomorphism from $G_N(F_\infty)/K_\infty$ to $\mathfrak{H}_N^\ell$. For f an automorphic form of weight κ, level $\mathfrak{n}$ and character ρ on $\mathfrak{H}_N^\ell$ define $\widetilde{f}(g) = f(g(i \cdot 1_n))\mu(g, i \cdot 1_n)^\kappa$. Then $\widetilde{f}$ is an automorphic form on $G_n(F_\infty)$ of weight κ, level $\mathfrak{n}$ and character ρ. There is a unique compact open subgroup $\mathbf{K}_0 \subset G_N(\mathbf{A}_0)$ so that $\Gamma_0^N(\mathfrak{n}) = G_N(F) \cap G_N(F_\infty)\mathbf{K}_0$. By Strong approximation there is a diffeomorphism

$$\Gamma_0^N(\mathfrak{n}) \backslash G_N(F_\infty) \longrightarrow G_N(F) \backslash G_N(\mathbf{A}) / \mathbf{K}_0.$$

If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_\infty = \prod_{v\mid\infty} K_v$, then let $\rho_\kappa(g) = \det(a+ib)^\kappa$ and let $K_{\mathbf{A}} = K_\infty \mathbf{K}_0$. For $g \in G_N(\mathbf{A})$ and $z \in \mathfrak{H}_N^\ell$ define $\mu(g,z) = (\mu(g_v,z_v))_{v \in \mathbf{A}_\infty}$. For $\widetilde{f}$, an automorphic form on $G_N(F_\infty)$ and $\gamma \in G_N(F)$, $g \in G_N(F_\infty)$ and $k \in K_{\mathbf{A}}$ we define a function, which we also label f, on $G_N(\mathbf{A})$ by $f(\gamma g k) = \widetilde{f}(g)\rho(k)$. Then f is a left $G_N(F)$-invariant, right (K_∞, ρ_κ)-equivariant, right $\mathbf{K}_0$-invariant function on $G_N(\mathbf{A})$. Thus, f is a holomorphic automorphic form on $G_N(\mathbf{A})$ of weight κ, level $\mathfrak{n}$ and character ρ. In the sequel, by automorphic forms we mean adelic automorphic forms in this sense, and we denote the space of such automorphic forms by $\mathcal{M}_N(\kappa,\mathfrak{n})$. Let $\mathcal{C}_N(\kappa,\mathfrak{n})$ denote the subspace of cuspforms. Let $Z_N(\mathbf{A})$ be the center of $G_N(\mathbf{A})$. The Petersson inner product on $\mathcal{C}_N(\kappa, \mathfrak{n})$ is defined to be

$$\langle f_1, f_2\rangle = \int_{Z_N(\mathbf{A})G_N(F)\backslash G_N(\mathbf{A})} f_1(g)\overline{f_2(g)}\, dg.$$

The measure chosen is induced from a Haar measure on $G_N(\mathbf{A})$. By the Iwasawa decomposition, we have $G_N(F) = P_N(F)K(F)$ where $P_N(F) = L_N(F)U_N(F)$ with $L_N(F) = \left\{\begin{pmatrix} A & 0 \\ 0 & A^{*-1}\end{pmatrix} \mid A \in GL_N(F)\right\}$ the Levi component and $U_N(F) = \left\{\begin{pmatrix} 1 & B \\ 0 & 1\end{pmatrix} \mid B \in S_N(F)\right\}$ the unipotent radical, where $S_N(F) = \{B \in M_{N\times N}(F) \mid B^* = \eta B\}$. Thus, $P_N(F)$ is the Siegel parabolic of $G_N(F)$. More generally, for $0 \leq j \leq N$, we have the unipotent radicals

$$U_j(F) = \left\{\begin{pmatrix} 1_{N-j} & * & * & * \\ 0 & 1_j & * & 0 \\ 0 & 0 & 1_{N-j} & 0 \\ 0 & 0 & * & 1_j \end{pmatrix} \in G_N(F)\right\}$$

and Levi components

$$L_j(F) = \left\{\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & A^{*-1} & 0 \\ 0 & c & 0 & d \end{pmatrix} \in G_N(F) \;\middle|\; A \in GL_{N-j}(F),\ \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in G_j(F)\right\}$$

and the respective parabolics $P_j(F) = L_j(F)U_j(F)$ of $G_N(F)$. The context will be clear as to the group $G_N(F)$ which contains a given parabolic. Note that, for a cuspform f on $G_N(\mathbf{A})$, we have $\int_{U_j(F)\backslash U_j(\mathbf{A})} f(ug)\, du = 0$ for any $j \leq N$.

In the sequel, we write $e(x)$ for e^x. Also, let $\det(A)$ denote the usual determinant if $F = k$, the usual determinant over F followed by the norm for F over k if F is a complex quadratic extension of k and the determinant of the image of the rational representation followed by the reduced norm if F is a quaternionic extension of k. For $w_n = \begin{pmatrix} 0_n & 1_n \\ 1_n & 0_n\end{pmatrix}$ and $g \in G_N$ let $g^\natural = w_n g w_n$ and note that $g \to g^\natural$ stabilizes the maximal compact subgroup of G_N. For an automorphic form f on $G_N(\mathbf{A})$, let $f^\natural(g) = \overline{f(g^\natural)}$ where $\overline{f}$ is complex conjugation. Note that $f^\natural$ is a holomorphic automorphic form on $G_N(\mathbf{A})$.

3. Pullbacks of Eisenstein series and cuspforms. Let $N = m+n$ and consider the embedding $\iota : G_m(\mathbf{A}) \times G_n(\mathbf{A}) \hookrightarrow G_N(\mathbf{A})$ given by

$$\iota : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}.$$

For the case $F = k = \mathbf{Q}$, the restriction of Siegel Eisenstein series on $G_N(\mathbf{A}) = Sp_N(\mathbf{A})$ to $Sp_m(\mathbf{A}) \times Sp_n(\mathbf{A})$ is well studied; see [**4, 8**] for a detailed discussion of the doubling method.

By the Iwasawa decomposition, for $g \in G_N(\mathbf{A})$, we can write $g = pk$ with $p = \left(\begin{smallmatrix} A & B \\ 0_N & A^{*-1} \end{smallmatrix}\right) \in P_N(\mathbf{A})$ and $k \in K$. For an integer $\kappa > 2N+1$, define a character χ on $P_N(\mathbf{A})$ by $\chi(p) = |\det(A)|_{\mathbf{J}}^{\kappa}$, where $\mathbf{J}$ denotes the ideles of F. Recall that the induced representation space $I(\chi) = \mathrm{Ind}_{P_N(\mathbf{A})}^{G_N(\mathbf{A})}\chi$ consists of smooth complex-valued functions α on $G_N(\mathbf{A})$ so that $\alpha(pg) = \chi(p)\alpha(g)$ for all $p \in P_N(\mathbf{A})$ and $g \in G_N(\mathbf{A})$. For a section $\varepsilon \in I(\chi)$, we define the following Siegel-Hilbert Eisenstein series on $G_N(\mathbf{A})$,

$$E_N(g;\kappa) = \sum_{\gamma \in P_N(F)\backslash G_N(F)} \varepsilon(g;\kappa). \tag{3.1}$$

This series converges uniformly in compacta for $\kappa > 2N+1$ and is a holomorphic modular form on $G_N(\mathbf{A})$ of weight κ. The level $\mathfrak{n}$ depends on the choice of section ε. Let $E_N(\iota(g_1, g_2);\kappa)$ denote the restriction of $E_N(g;\kappa)$ to $G_m(\mathbf{A}) \times G_n(\mathbf{A})$. The following lemma is proved in [**6**]. Recall that $\eta = \pm 1$ and is fixed for $G_N(F)$.

Lemma 1 [**6**, Proposition 3.1]. *Let $N = m+n$ with $m \le n$. The double coset space*

$$P_N(F)\backslash G_N(F)/\iota(G_m(F) \times G_n(F))$$

has irredundant representatives $\xi_0, \dots, \xi_m$ *where*

$$\xi_j = \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & 1_m & 0 & 0 \\ 0 & \eta I_j & 1_n & 0 \\ I_j^* & 0 & 0 & 1_m \end{pmatrix} \quad \text{and} \quad I_j = \begin{pmatrix} 0_{m-j} & 0 \\ 0 & 1_j \end{pmatrix}.$$

The respective isotropy groups of ξ_j *in* $G_m(F) \times G_n(F)$ *are*

$$H_j(F) \cong \left\{ J_{m,j}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \times J_{n,j}\left(\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right) \in G_m(F) \times G_n(F) \,\middle|\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_j(F) \right\}$$

where

$$J_{m,j}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left\{ \begin{pmatrix} \alpha & * & * & * \\ 0 & a & * & b \\ 0 & 0 & \alpha^{*-1} & 0 \\ 0 & c & * & d \end{pmatrix} \,\middle|\, \alpha \in GL_{m-j}(F) \right\}.$$

Note that $J_{m,j}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ contains the unipotent radical $U_j(F)$ of $G_m(F)$ for $0 \le j < m$.

For $G_N(\mathbf{A}) = Sp(N)$, an Eisenstein kernel ε was chosen in [**3, 7**] so that the contribution of certain cells to the pullback of a given Eisenstein series vanished. In fact, in [**3, 7**] all of the terms in the pullback of the Eisenstein series vanish except the term contributed by the orbit that is Zariski-open in $G_N(F)$ (the "big-cell").

For symplectic groups, for $F = k$, a version of the following decomposition result was studied by Böcherer [**1**], Brown [**2**] and Garrett [**6, 7**]. In [**14**] Shimura studied this decomposition for unitary groups, so for F a complex quadratic extension of k.

Lemma 2. *Let* $E_N(g;\kappa)$ *be as in* (3.1). *Then*

$$E_N(\iota(g_1, g_2);\kappa) = \Psi_N(g_1, g_2) + \Phi_N(g_1, g_2)$$

where $\langle \Psi_N(\cdot, g_2), f_1 \rangle = \langle \Psi_N(g_1, \cdot), f_2 \rangle = 0$ *for cuspforms* f_1 *on* $G_m(\mathbf{A})$ *and* f_2 *on* $G_n(\mathbf{A})$, *respectively.*

Proof. Let $H_i(F)$ be the isotropy group of the representative ξ_i, and let $\xi = \xi_m$ be the representative of the open orbit. Consider

$$\gamma \in P_N(F)\backslash G_N(F) = \bigsqcup_{i=1}^{m} P_N(F)\backslash P_N(F)\xi_i\iota(G_m(F) \times G_n(F)).$$

For $\gamma \in P_N(F)\backslash P_N(F)\xi_i\iota(G_m(F) \times G_n(F))$ we have $\xi_i\gamma \in H_i(F)\backslash\xi_i \iota(G_m(F) \times G_n(F))$, and as $H_i(F)\xi_i = \xi_i H_i(F)$, we have $\gamma \in H_i(F)\backslash \iota(G_m(F)\times G_n(F))$. In the following, to avoid clutter, we abuse notation and leave out the ι in the range of the summation indices. Thus, we have

(3.2)
$$\begin{aligned}
E_N(\iota(g_1, g_2); \kappa) &= \sum_{\gamma\in P_N(F)\backslash G_N(F)} \varepsilon(\gamma\iota(g_1, g_2); \kappa) \\
&= \sum_{i=1}^{m} \sum_{\gamma\in H_i(F)\backslash(G_m(F)\times G_n(F))} \varepsilon(\xi_i\gamma\iota(g_1, g_2); \kappa) \\
&= \sum_{i=1}^{m-1} \sum_{\gamma\in H_i(F)\backslash(G_m(F)\times G_n(F))} \varepsilon(\xi_i\gamma\iota(g_1, g_2); \kappa) \\
&\quad + \sum_{\gamma\in H_m(F)\backslash(G_m(F)\times G_n(F))} \varepsilon(\xi\gamma\iota(g_1, g_2); \kappa) \\
&= \Psi_N(g_1, g_2) + \Phi_N(g_1, g_2)
\end{aligned}$$

where $\Phi_N(g_1, g_2)$ is the contribution of the open orbit to the pullback.

For $\gamma \in H_j(F)\backslash(G_m(F) \times G_n(F))$ we have $\gamma = (\gamma_1, \gamma_2)$ where $\gamma_1 \in J_{m,j}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\backslash G_m(F)$ and $\gamma_2 \in J_{n,j}\begin{pmatrix} d & c \\ b & a \end{pmatrix}\backslash G_n(F)$. For $0 \le j \le m-1$ and f a generic cuspform in $\mathcal{C}_m(\kappa, \mathfrak{n})$ we can unwind the following integral and get

$$\begin{aligned}
\langle\Psi_N(*, g_2), f\rangle &= \int_{Z_m(\mathbf{A})G_m(F)\backslash G_m(\mathbf{A})} \sum_{\gamma\in H_j(F)\backslash(G_m(F)\times G_n(F))} \\
&\quad \times \varepsilon(\xi_j\gamma\iota(g_1, g_2); \kappa)\overline{f(g_1)}\, dg_1 \\
&= \int_{Z_m(\mathbf{A})J_{m,j}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\backslash G_m(\mathbf{A})} \sum_{\gamma_2\in J_{n,j}\begin{pmatrix} d & c \\ b & a \end{pmatrix}\backslash G_n(F)}
\end{aligned}$$

$$\times\, \varepsilon(\xi_j \iota(g_1, \gamma_2 g_2); \kappa)\overline{f(g_1)}\; dg_1.$$

Writing $J_{m,j}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = L_j(F)U_j(F)$ where $U_j(F)$ is the unipotent radical of the parabolic $P_j(F)$ in $G_m(F)$, the above can be rewritten

$$\int_{Z_m(\mathbf{A})L_j(\mathbf{A})U_j(\mathbf{A})\backslash G_m(\mathbf{A})} \int_{L_j(F)\backslash L_j(\mathbf{A})} \int_{U_j(F)\backslash U_j(\mathbf{A})} \times \sum_{\gamma_2 \in J_{n,j}\left(\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right)\backslash G_n(F)} \varepsilon(\xi_j \iota(ulg, \gamma_2 g_2); \kappa)\overline{f(ulg)}\, du\, dl\, dg.$$

A direct calculation shows that $\varepsilon(\xi_j \iota(ulg, \gamma_2 g_2); \kappa) = \varepsilon(\xi_j \iota(lg, \gamma_2 g_2); \kappa)$ and so, by the left $U_j(\mathbf{A})$-invariance of ε, this becomes

$$\int_{Z_m(\mathbf{A})L_j(\mathbf{A})U_j(\mathbf{A})\backslash G_m(\mathbf{A})} \int_{L_j(F)\backslash L_j(\mathbf{A})} \sum_{\gamma_2 \in J_{n,j}\left(\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right)\backslash G_n(F)} \times\, \varepsilon(\xi_j \iota(lg, \gamma_2 g_2); \kappa)\left(\int_{U_j(F)\backslash U_j(\mathbf{A})} \overline{f(ulg)}\, du\right) dl\, dg.$$

For $j < m$, $U_j(\mathbf{A})$ is the unipotent radical of the parabolic $P_j(\mathbf{A})$ in $G_m(\mathbf{A})$. Thus, as $f(g)$ is a cuspform, the inner integral is 0. Similarly we get $\langle \Psi_N(g_1, *), f\rangle = 0$. It follows that the first of the two terms in (3.2) is orthogonal in each variable to the space of cuspforms. □

In [**2, 7**] a specific Siegel Eisenstein series is chosen on $Sp(n)$ so that Ψ_N as above is identically zero. It immediately follows that Φ_N is nontrivial in that case. We show that an Eisenstein series can be chosen on G_N so that Φ_N is nontrivial, with no condition placed on Ψ_N. Further, we show that there is somewhat more flexibility in choosing such an Eisenstein series.

Lemma 3. *Let $\Phi_N(g_1, g_2)$ be as in* (3.2). *We can choose $\varepsilon \in I(\chi)$ so that $\Phi_N(g_1, g_2)$ is a nontrivial automorphic form on $G_m(\mathbf{A}) \times G_n(\mathbf{A})$.*

Proof. It is straightforward that $\Phi_N(g_1, g_2)$ is an automorphic form on $G_m(\mathbf{A}) \times G_n(\mathbf{A})$ in the sense of Section 2. For $a \in \mathbf{Z}_{>0}$ consider the compact subgroup

$$K_{\mathbf{A}}(a) = \{\iota(g_{1v}, g_{2v}) \in K_{\mathbf{A}} \mid \iota(g_{1v}, g_{2v}) \equiv 1_{2N} \pmod{\varpi_v^a} \text{ for } v < \infty\},$$

and note that for $a > a'$, we have $K_{\mathbf{A}}(a) \subset K_{\mathbf{A}}(a')$. We similarly define $K_F(a)$ and $K_v(a)$ and note that $\lim_{a\to\infty} K_F(a) = \{1_{2N}\}$.

Recall that $H_m(F)$ is a subgroup of $G_m(F) \times G_n(F)$. So, for $h = \iota(g_1, g_2) \in H_m(F_v)$, we have $\xi h \xi^{-1} = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right) \in P_N(F_v)$ where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right)$. Note that $\det(C) = \det(A)$. As $\xi h \xi^{-1} \in P_N(F_v)$ we have $C = A^{*-1}$, and this implies $\det(A) = \det(A)^{-1}$. It follows that $|\det(A)|_v = 1$ and as the local character is $\chi_v\left(\begin{smallmatrix} A & B \\ 0 & A^{*-1} \end{smallmatrix}\right) = |\det(A)|_v^{\kappa}$ we get $\chi_v(\xi h \xi^{-1}) = 1$. So for $f(*) \in \mathrm{Ind}_{P_N(F_v)}^{P_N(F_v)\xi\iota(G_m(F_v)\times G_n(F_v))}\chi_v$ we have $f(\xi * \xi^{-1}) \in \mathrm{Ind}_{H_m(F_v)}^{G_m(F_v)\times G_n(F_v)}\mathbf{1}$.

For any $f \in \mathrm{Ind}_{P_N(F_v)}^{P_N(F_v)\xi\iota(G_m(F_v)\times G_n(F_v))}\chi_v$ we can associate some $\widetilde{f} \in \mathrm{Ind}_{P_N(F_v)}^{G_N(F_v)}\chi_v$. Set $\widetilde{f}(g) = f(g)$ for $g \in P_N(F_v)\xi\iota(G_m(F_v) \times G_n(F_v))$. As $P_N(F_v)\xi\iota(G_m(F_v) \times G_n(F_v))$ is Zariski-open in $G_N(F_v)$, then for $g \in G_N(F_v) - P_N(F_v)\xi\iota(G_m(F_v) \times G_n(F_v))$ there is a sequence $g_i \to g$ with $g_i \in P_N(F_v)\xi\iota(G_m(F_v) \times G_n(F_v))$. As f is smooth, we have that $\lim f(g_i)$ exists and we set $\widetilde{f}(g)$ equal to this limit. Therefore, there is a sequence of injections $\mathrm{Ind}_{H_m(F_v)}^{G_m(F_v)\times G_n(F_v)}\mathbf{1} \hookrightarrow \mathrm{Ind}_{P_N(F_v)}^{P_N(F_v)\xi\iota(G_m(F_v)\times G_n(F_v))}\chi_v \hookrightarrow I_v(\chi_v)$.

Let $\mathrm{ch}_{H_m(F_v)K_v(a)}(*) \in \mathrm{Ind}_{H_m(F_v)}^{G_m(F_v)\times G_n(F_v)}\mathbf{1}$ denote the characteristic function of $H_m(F_v)K_v(a)$. By the injection above, there exists some $\varepsilon_v \in I_v(\chi_v)$ so that

$$\varepsilon_v(\xi * \xi^{-1})|_{G_m(F_v)\times G_n(F_v)} = \mathrm{ch}_{H_m(F_v)K_v(a)}(*).$$

Set $\varepsilon_{a,v} = \mathrm{ch}_{H_m(F_v)K_v(a)}$, and let $\varepsilon_a = \prod_v \varepsilon_{a,v}$. We can choose ε_v so that it is supported on $P_N(F_v)K_v'$ for some compact open subgroup K_v' of $G_N(F_v)$ that contains a principal congruence subgroup of level r, denoted $K_v(r)$. Thus, we can choose $\varepsilon \in I(\chi)$ so that the local component ε_v is supported on $P_N(F_v)K_v(r)$ and then $\varepsilon_{a,v}$ is supported on $H_m(F_v)K_v(a)$ for $r \geq a$. Note that as $K_v(r) \subseteq K_v(a)$ we have $\varepsilon_{r,v}(g) = \varepsilon_{a,v}(g)$ for $g \in H_m(F_v)K_v(r)$.

We have $\varepsilon_{av}(\gamma\iota(g_1, g_2)) = \mathrm{ch}_{H_m(F_v)K_v(a)}(\gamma\iota(g_1, g_2)) = \varepsilon_v(\xi\gamma\iota(g_1, g_2)\xi^{-1}) = \varepsilon_v(\xi\gamma\iota(g_1, g_2))$ by the right K_v-invariance of ε_v. Thus, $\varepsilon_a(\gamma\iota(g_1, g_2)) = \varepsilon(\xi\gamma\iota(g_1, g_2))$, and let

$$\begin{aligned}\Phi_{N,a}(g_1, g_2) &= \sum_{\gamma\in H_m(F)\backslash(G_m(F)\times G_n(F))} \varepsilon(\gamma\iota(g_1, g_2)) \\ &= \sum_{\gamma\in H_m(F)\backslash(G_m(F)\times G_n(F))} \varepsilon(\xi\gamma\iota(g_1, g_2)).\end{aligned}$$

As ε_a is supported on $H_m(\mathbf{A})K_{\mathbf{A}}(r)$,

$$\begin{aligned}\lim_{a\to\infty} \Phi_{N,a}(1_{2m}, 1_{2n}) &= \lim_{a\to\infty} \sum_{\gamma\in H_m(F)\backslash(G_m(F)\times G_n(F))} \varepsilon_a(\gamma) \\ &= \lim_{a\to\infty} \sum_{\gamma\in H_m(F)\backslash H_m(F)K_F(a)} \varepsilon_a(\gamma).\end{aligned}$$

Now, $\varepsilon_a(1_{2N}) = \varepsilon(\xi \cdot 1_{2N}) = \mathrm{ch}_{H_m(F)K_F(a)}(1_{2N}) = 1$, and since $\lim_{a\to\infty} H_m(F)\backslash H_m(F)K_F(a) = \{1_{2N}\}$, it follows that

$$\lim_{a\to\infty} \sum_{\gamma\in H_m(F)\backslash H_m(F)K_F(a)} \varepsilon_a(\gamma) = 1.$$

Thus, $\Phi_{N,a}$ is nontrivial for sufficiently large a. It follows that we can choose a and r sufficiently large and $\varepsilon \in I(\chi)$ so that the cuspidal part of the restriction of the corresponding Eisenstein series is nontrivial. That is, there exists some $\varepsilon \in I(\chi)$ so that

$$\Phi_N(g_1, g_2) = \sum_{\gamma\in H_m(F)\backslash(G_m(F)\times G_n(F))} \varepsilon(\xi\gamma\iota(g_1, g_2))$$

is a nontrivial automorphic form on $G_m(\mathbf{A}) \times G_n(\mathbf{A})$. $\square$

We now show that, for $m = n$, we can choose any $\varepsilon_v \in I_v(\chi_v)$ where $v < \infty$ and $v \nmid \mathfrak{n}$ so that Φ_N is nontrivial. Let $m = n$, and let f be a cuspform on $G_n(\mathbf{A})$ that is a Hecke eigenfunction. Unwinding the following integral, we get

(3.3)

$$\langle \Phi_N(\cdot, g_2), f\rangle = \int_{Z_n(\mathbf{A})G_n(F)\backslash G_n(\mathbf{A})}$$

$$\times \sum_{\gamma \in H_n(F)\backslash(G_n(F)\times G_n(F))} \varepsilon(\xi\gamma\iota(g_1,g_2);\kappa)\overline{f(g_1)}\,dg_1$$
$$= \int_{Z_n(\mathbf{A})\backslash G_n(\mathbf{A})} \varepsilon(\xi\iota(g_1,g_2);\kappa)\overline{f(g_1)}\,dg_1.$$

Note that, for $g_1 \in G_n(\mathbf{A})$, we have $\xi\iota(g_1, g_1^\natural)\xi^{-1} \in P_n(\mathbf{A})$, and so $\varepsilon(\xi\iota(g_1, g_1^\natural)\xi^{-1}g;\kappa) = \varepsilon(g;\kappa)$ from the equivariance properties of ε. Thus,

$$\varepsilon(\xi\iota(g_1,g_2);\kappa) = \varepsilon(\xi\iota(g_2^{\natural-1}, g_2^{-1})\xi^{-1}\xi\iota(g_1,g_2);\kappa) = \varepsilon(\xi\iota(g_2^{\natural-1}g_1, 1_{2n});\kappa)$$

and so (3.3) is

$$\begin{aligned}
&\int_{Z_n(\mathbf{A})\backslash G_n(\mathbf{A})} \varepsilon(\xi\iota(g_1,g_2);\kappa)\overline{f(g_1)}\,dg_1 \\
&= \int_{Z_n(\mathbf{A})\backslash G_n(\mathbf{A})} \varepsilon(\xi\iota(g_2^{\natural-1}g_1, 1_{2n});\kappa)\overline{f(g_1)}\,dg_1 \\
&= \int_{Z_n(\mathbf{A})\backslash G_n(\mathbf{A})} \varepsilon(\xi\iota(g_1, 1_{2n});\kappa)\overline{f(g_2^\natural g_1)}\,dg_1 \\
&= \int_{Z_n(\mathbf{A})\backslash G_n(\mathbf{A})} \varepsilon(\xi\iota(g_1^{-1}, 1_{2n});\kappa)\overline{f(g_2^\natural g_1^{-1})}\,dg_1.
\end{aligned}$$

Let T_v be the convolution operator

$$(T_v f_v)(g) = \int_{G_n(F_v)} \varepsilon_v(\xi\iota(g_1^{-1}, 1_{2n});\kappa)\overline{f_v(gg_1^{-1})}\,dg_1, \tag{3.4}$$

and note that the T_v's commute for varying v. These integrals are absolutely convergent since integral (3.3) is absolutely convergent. As $\varepsilon(g)$ factors over v, we can factor the global integral in (3.3) as a product of the operators T_v,

$$\langle \Phi_N(\cdot, g), f\rangle = \left(\prod_v T_v\right) f^\natural(g) = \prod_v \left(T_v f_v^\natural\right)(g).$$

Let $\widetilde{\varepsilon}_v(g) = \varepsilon_v(\xi\iota(g, 1_{2n});\kappa)$, so we write $T_v f_v^\natural = f_v^\natural * \widetilde{\varepsilon}_v$ where $f_v * \eta_v(g) = \int_{G_n(F_v)} \eta_v(g_1) f_v(gg_1^{-1})\,dg_1$. For $k_1, k_2 \in K_v$,

$$\widetilde{\varepsilon}_v(k_1 g k_2) = \varepsilon_v\left(\xi\iota(k_1 g k_2, 1_{2n});\kappa\right) = \varepsilon_v\left(\xi\iota(gk_2, k_1^{\natural-1});\kappa\right).$$

This last term is $\varepsilon_v(\xi\iota(g, 1_{2n}); \kappa)$ by the right K_v-invariance of ε_v. Following [**7**], let $\{K\}$ be a nested collection of compact open sets in $G_n(F_v)$ so that $\bigcup K = G_n(F_v)$, and we can take each set K to be stable under the involution $g \to g^\natural$. Let $\widetilde{\varepsilon}_K = \widetilde{\varepsilon}_v \mathrm{ch}_K \in H_v(G, K)$. Then $\widetilde{\varepsilon}_K$ is $\mathbf{Q}$-valued as $\widetilde{\varepsilon}_v$ is, and $\lim_K \widetilde{\varepsilon}_K = \widetilde{\varepsilon}_v$. As f is an eigenfunction of the spherical Hecke algebra at v, then $f_v * \widetilde{\varepsilon}_K = c_K f_v$ for some number c_K, and the absolute convergence of $(T_v f_v)(g)$ implies the absolute convergence of $f_v * \widetilde{\varepsilon}_v$. Therefore, $(\lim_K c_K) f_v = \lim_K (c_K f_v) = \lim_K (f_v * \widetilde{\varepsilon}_K) = f_v * \widetilde{\varepsilon}_v$, and so $\lim_K c_K$ exists and we denote it by $\overline{c}_v(f)$. Since $\widetilde{\varepsilon}_v$ and $\widetilde{\varepsilon}_K$ are $\mathbf{Q}$-valued we have

$$\begin{aligned}
\left(T_v f_v^\natural\right)(g) &= \left(f_v^\natural * \widetilde{\varepsilon}_v\right)(g) = \lim_K \left(f_v^\natural * \widetilde{\varepsilon}_K\right)(g) \\
&= \left(\lim_K (f_v * \widetilde{\varepsilon}_K)(g)\right)^\natural = \left((\lim_K c_K) f_v(g)\right)^\natural \\
&= \left(\overline{c}_v(f) f_v(g)\right)^\natural = c_v(f) f_v^\natural(g).
\end{aligned}$$

This implies that $c_v(f)$ depends only on the local data of f and the homomorphism T_v of the Hecke algebra at v. So we have

$$\langle \Phi(\cdot, g_2), f\rangle = \prod_v \left(T_v f_v^\natural\right)(g) = \prod_v c_v(f) f_v^\natural(g) = c(f) f^\natural(g)$$

where $c(f) = \prod_v c_v(f)$. As (3.3) is absolutely convergent, then the infinite product is absolutely convergent.

It follows that the integral $\langle\langle \Phi_N, f_1\rangle, f_2\rangle$ factors into a product of local integrals of the form

$$\int_{G_n(F_v)\times G_n(F_v)} \varepsilon_v(\xi\iota(g_1, g_2); \kappa)\overline{f_{1v}(g_1) f_{2v}(g_2)}\, dg_1\, dg_2. \tag{3.5}$$

We have $\varepsilon_v(\xi * \xi^{-1}; \kappa) = \mathrm{Ind}_{H_n(F_v)}^{G_n(F_v)\times G_n(F_v)} \chi_v^\xi$ because ε_v is right K_v-invariant. Consider automorphic representations $\pi_{j,\mathbf{A}}$ on $G_n(\mathbf{A})$ so that $f_j \in \pi_{j,\mathbf{A}}$ and $\pi_{j,\mathbf{A}} \cong \otimes_v' \pi_{j,v}$ where the product is the restricted tensor product. For $v < \infty$ and $v \nmid \mathfrak{n}$, the automorphic representations $\pi_{j,v}$ are spherical. For $f_j \in \pi_{j,\mathbf{A}}$, as above, the local integral (3.5) defines an element in the space of intertwining operators on local representation spaces,

$$\mathrm{Hom}_{G_n(F_v)\times G_n(F_v)}(\mathrm{Ind}_{H_n(F_v)}^{G_n(F_v)\times G_n(F_v)} \chi_v^\xi \otimes (\pi_{1,v} \otimes \pi_{2,v}), \mathbf{1}). \tag{3.6}$$

By dualization (see [**8**]) this is isomorphic to

$$\mathrm{Hom}_{G_n(F_v)\times G_n(F_v)}(\pi_{1,v}\otimes\pi_{2,v},\mathrm{Ind}_{H_n(F_v)}^{G_n(F_v)\times G_n(F_v)}\widehat{\chi}_v^{\xi}\delta_H) \tag{3.7}$$

where $\widehat{\chi}_v$ is the smooth dual of χ_v and δ_H is the modular function of H_n. By definition, the isomorphism from (3.6) to (3.7) is given by $\phi\mapsto I_\phi$ where $I_\phi(v_1,v_2)(w)=\phi(w\otimes(v_1,v_2))$ with $v_i\in\pi_{i,v}$ and $w\in\mathrm{Ind}_{H_n(F_v)}^{G_n(F_v)\times G_n(F_v)}\widehat{\chi}^{\xi}\delta_\Theta$. It follows that, if $\phi(w\otimes(v_1,v_2))\neq 0$ for some vectors v_1,v_2 and w as above, then I_ϕ is also a nontrivial intertwining operator.

Recall from the proof of Lemma 3 that $\chi_v^\xi(h)=1$ for $h\in H_n(F_v)$. As $H_n(F_v)$ is unimodular, we have that $\delta_H=1$ and therefore $\widehat{\chi}_v^{\xi}\delta_H\equiv\mathbf{1}$. From this, and as we are in the category of smooth representations, we can apply Frobenius reciprocity to (3.7). Therefore the space of intertwinings (3.7) is isomorphic to

$$\mathrm{Hom}_{H_n(F_v)}(\mathrm{Res}_{H_n(F_v)}^{G_n(F_v)\times G_n(F_v)}(\pi_{1,v}\otimes\pi_{2,v}),\mathbf{1}), \tag{3.8}$$

and thus we have that the local intertwining operator defined by integral (3.3) is in a space of intertwinings isomorphic to

$$\mathrm{Hom}_{H_n(F_v)}(\pi_{1,v}\otimes\pi_{2,v},\mathbf{1}). \tag{3.9}$$

Note that the Frobenius reciprocity mapping from (3.7) to (3.8) is $\psi\mapsto\phi_\psi$, where $\phi_\psi(v_1,v_2)=\psi(v_1,v_2)(1_{4n})$ with $v_i\in\pi_{i,v}$. Thus, if $\psi(v_1,v_2)(1_{4n})\neq 0$, it follows that ϕ_ψ is a nontrivial intertwining operator. By the irreducibility of $\pi_{1,v}$ and $\pi_{2,v}$, we have that (3.9) has dimension 1 if $\pi_{2,v}\cong\widehat{\pi}_{1,v}$ and the dimension is 0 otherwise, where $\widehat{\pi}_{1,v}$ is the contragredient of $\pi_{1,v}$. See [**8**] for a detailed discussion of this.

More generally, let $\alpha_v\in\mathrm{Hom}_{H_n(F_v)}(\pi_v\otimes\widehat{\pi}_v,\mathbf{1})$ for π_v a spherical automorphic representation of $H_n(F_v)$. From [**8**] this space has dimension 1, and the nonvanishing of the nontrivial intertwining operator in this space is independent of the exact value of the Satake parameters of π_v. That is, if $\alpha_v(e_v,\widehat{e}_v)\neq 0$ for some spherical vector $e_v\in\pi_v$, then α_v is (a multiple of) a nontrivial normalized intertwining in that space. It follows that, for σ_v, any spherical automorphic representation of $H_n(F_v)$ with spherical vector e'_v we must also have $\alpha_v(e'_v,\widehat{e}'_v)\neq 0$.

If α_v is a nontrivial intertwining operator in (3.9), then the preimage of α_v of the isomorphism from (3.6) to (3.9) is also a nontrivial intertwining operator. Now, the nonvanishing of an intertwining in (3.9) is independent of the vector $\varepsilon_v \in I_v(\chi_v)$. Therefore, the nonvanishing of an intertwining in (3.6) is also independent of the particular vector $\varepsilon_v \in I_v(\chi_v)$, and it follows that the nonvanishing of integral (3.5) is independent of the particular ε_v for $v < \infty$ and $v \nmid \mathfrak{n}$. This proves (ii) of Theorem 1. Part (iii) follows from an argument similar to that of Lemma 2.

To show that Φ_N is a cuspform in the smaller variable we need a preliminary lemma. For $v|\infty$, let $\Omega_m(F_v)$ denote the cone of positive definite $m \times m$ matrices with entries in F_v.

Lemma 4. *Let $v \mid \infty$ and $\alpha \in \Omega_m(F_v)$. Then*

$$\int_{S_m(F_v)} e^{-i\mathrm{Tr}(\beta y)}(\det(\alpha - i\beta))^{-s}\, d\beta = 0$$

if y is not positive definite.

Proof. Let r denote the dimension of F_v over $\mathbf{R}$ and h the dimension of $S_m(F_v)$. Thus $h = m + (1/2)rm(m-1)$. The gamma function attached to $\Omega_m(F_v)$ is

$$\begin{aligned}\Gamma_{\Omega_m}(s) &= \int_{\Omega_m(F_v)} e^{-\mathrm{Tr}\,(x)}(\det(x))^s \frac{dx}{(\det(x))^{h/m}} \\ &= \pi^{\frac{rm(m-1)}{4}} \prod_{j=0}^{m-1} \Gamma\left(s - \frac{rj}{2}\right),\end{aligned} \tag{3.10}$$

where $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\, dt$ is the usual gamma function and $\mathrm{Tr}\,(\alpha)$ is the usual matrix trace. Let $\alpha \in \Omega_m(F_v)$, and then α has a unique square root in $\Omega_m(F_v)$ which we denote $\sqrt{\alpha}$. As $\mathrm{Tr}\,(\alpha x) = \mathrm{Tr}\,(\sqrt{\alpha}x\sqrt{\alpha})$ we can replace x with $\sqrt{\alpha}x\sqrt{\alpha}$ in (3.10) and

$$\Gamma_{\Omega_m}(s) = (\det(\alpha))^s \int_{\Omega_m(F_v)} e^{-\mathrm{Tr}\,(\alpha x)}(\det(x))^s \frac{dx}{(\det(x))^{h/m}}.$$

Using the analytic continuation of $\Gamma_{\Omega_m(F_v)}$ we have, for $\beta \in S_m(F_v)$,

(3.11)
$$\begin{aligned}\Gamma_{\Omega_m}(s) &= (\det(\alpha - i\beta))^s \\ &\quad\times \int_{\Omega_m(F_v)} e^{-\mathrm{Tr}\,((\alpha - i\beta)x)} (\det(x))^s \,\frac{dx}{(\det(x))^{h/m}} \\ &= (\det(\alpha - i\beta))^s \\ &\quad\times \int_{\Omega_m(F_v)} e^{-\mathrm{Tr}\,(\alpha x)} e^{i\mathrm{Tr}\,(\beta x)} (\det(x))^{s-h/m}\, dx.\end{aligned}$$

Let $f(\alpha) = e^{-\mathrm{Tr}\,(\alpha x)}(\det(x))^{s-h/n}$ for $\alpha \in \Omega_m(F_v)$ and 0 otherwise. The Fourier transform and the inverse Fourier transform on $S_m(F_v)$ are defined by

$$f^{\wedge}(y) = \int_{S_m(F_v)} e^{-i\mathrm{Tr}\,(xy)} f(x)\, dx$$

and

$$f^{\vee}(y) = \int_{S_m(F_v)} e^{i\mathrm{Tr}\,(xy)} f(x)\, dx,$$

respectively. The Fourier inversion formula gives $(f^{\wedge})^{\vee}(y) = (f^{\vee})^{\wedge}(y) = (2\pi)^{-m}\pi^{-rm(m-1)/2} f(y)$. Now, (3.11) can be rewritten $\Gamma_{\Omega_m}(s) = (\det(\alpha - i\beta))^s f^{\vee}(\beta)$ and so Fourier inversion gives

$$\begin{aligned}&\Gamma_{\Omega_m}(s) \int_{S_m(F_v)} e^{-i\mathrm{Tr}\,(\beta y)} (\det(\alpha - i\beta))^{-s}\, d\beta \\ &= \left(\frac{\Gamma_{\Omega_m}(s)}{(\det(\alpha - \beta))^s}\right)^{\wedge}(y) = \left(f^{\vee}\right)^{\wedge}(y) = (2\pi)^{-m}\pi^{-rm(m-1)/2} f(y).\end{aligned}$$

It follows that the integral

$$\int_{S_m(F_v)} e^{-i\mathrm{Tr}\,(\beta y)} (\det(\alpha - i\beta))^{-s}\, d\beta = 0$$

if y is not positive definite. $\square$

The following result completes the proof of (i) of Theorem 1.

Lemma 5. *Let $\Phi_N(g_1, g_2)$ be as in Lemma 2. Then $\Phi_N(g_1, g_2)$ is a cuspform in the variable $g_1 \in G_m(\mathbf{A})$.*

Proof. Since

$$H_m(F)\backslash(G_m(F) \times G_n(F)) \cong G_m(F) \times P_j(F)\backslash G_n(F),$$

then any $\gamma \in H_m(F)\backslash(G_m(F) \times G_n(F))$ can be written $\gamma = (\gamma_1, \gamma_2)$ for $\gamma_1 \in G_m(F)$ and $\gamma_2 \in P_j(F)\backslash G_n(F)$. Thus,

$$\begin{aligned}
&\sum_{\gamma \in H_m(F)\backslash(G_m(F)\times G_n(F))} \varepsilon(\xi\gamma\iota(g_1, g_2); \kappa) \\
&= \sum_{(\gamma_1,\gamma_2)\in G_m(F)\times P_j(F)\backslash G_n(F)} \varepsilon(\xi\iota(\gamma_1 g_1, \gamma_2 g_2); \kappa) \\
&= \sum_{\gamma_2 \in P_j(F)\backslash G_n(F)} \sum_{\gamma_1 \in G_m(F)/U_0(F)} \sum_{u \in U_0(F)} \varepsilon(\xi\iota(\gamma_1 u g_1, \gamma_2 g_2); \kappa) \\
&= \sum_{\gamma_2 \in P_j(F)\backslash G_n(F)} \sum_{\gamma_1 \in G_m(F)/U_0(F)} \sum_{u \in U_0(F)} \varepsilon(\xi\iota(u g_1, \gamma_1^{\natural-1}\gamma_2 g_2); \kappa).
\end{aligned}$$

For ψ_0 the standard additive character on $\mathbf{A}_{\mathcal{Q}}$, let $\psi = \psi_0 \circ \mathrm{Tr}_{F/\mathcal{Q}}$ be the corresponding additive character on $\mathbf{A}_F$. The αth-Fourier coefficient of the inner sum is, for $\left(\begin{smallmatrix} 1_m & B \\ 0_m & 1_m \end{smallmatrix}\right) \in U_0(\mathbf{A})$,

$$\begin{aligned}
&\int_{U_0(F)\backslash U_0(\mathbf{A})} \sum_{u \in U_0(F)} \varepsilon\left(\xi\iota\left(u \begin{pmatrix} 1_m & B \\ 0_m & 1_m \end{pmatrix} g_1, g_2\right); \kappa\right)\overline{\psi(\alpha B)}\, dB \\
&\qquad = \int_{U_0(\mathbf{A})} \varepsilon\left(\xi\iota\left(\begin{pmatrix} 1_m & B \\ 0_m & 1_m \end{pmatrix} g_1, g_2\right); \kappa\right)\overline{\psi(\alpha B)}\, dB.
\end{aligned}$$

For $v \,|\, \infty$, the vth factor of the above integral is $\int_{U_0(F_v)} \varepsilon_v(\xi\iota(\left(\begin{smallmatrix} 1_m & B \\ 0_m & 1_m \end{smallmatrix}\right) g_1, g_2); \kappa)\overline{\psi_v(\alpha B)}\, dB$. By the Iwasawa decomposition and change of variables we set $g_1 = \left(\begin{smallmatrix} A & 0_m \\ 0_m & A^{*-1} \end{smallmatrix}\right)$ and $g_2 = \left(\begin{smallmatrix} 1_m & B' \\ 0_m & 1_m \end{smallmatrix}\right)\left(\begin{smallmatrix} A' & 0_m \\ 0_m & A'^{*-1} \end{smallmatrix}\right)$. Therefore,

$$\xi\left(\iota\begin{pmatrix}1_m & B\\ 0_m & 1_m\end{pmatrix}g_1,g_2\right)=\begin{pmatrix}1_{2m} & 0_{2m}\\ w_m & 1_{2m}\end{pmatrix}\begin{pmatrix}1_{2m} & C\\ 0_{2m} & 1_{2m}\end{pmatrix}\begin{pmatrix}D & 0_{2m}\\ 0_{2m} & D^{*-1}\end{pmatrix}$$
$$=\begin{pmatrix}D & CD^{*-1}\\ w_mD & w_mCD^{*-1}+D^{*-1}\end{pmatrix}$$

where $C=\left(\begin{smallmatrix}B & 0_m\\ 0_m & B'\end{smallmatrix}\right)$ and $D=\left(\begin{smallmatrix}A & 0_m\\ 0_m & A'\end{smallmatrix}\right)$. By the Iwasawa decomposition of G_v, we can write $\xi\iota\left(\left(\begin{smallmatrix}1_m & B\\ 0_m & 1_m\end{smallmatrix}\right)g_1,g_2\right)=ulk$ for $u\in U_v$, $l\in L_v$, $k\in K_v$ and

$$ulk=\begin{pmatrix}1_{2m} & C'\\ 0_{2m} & 1_{2m}\end{pmatrix}\begin{pmatrix}D' & 0_{2m}\\ 0_{2m} & D'^{*-1}\end{pmatrix}\begin{pmatrix}X & Y\\ -Y & X\end{pmatrix}.$$

Equating the terms gives $D'(X+iY)=D-C'w_mD+i(C-C'w_mC-C')D^{*-1}$. By the left U_v-invariance of ε_v for $v|\infty$, we have $\varepsilon_v(ulk;\kappa,\mathfrak{n},\omega)=|\det(D'(X+iY))|^\kappa=|\det(D+iCD^{*-1})|^\kappa$. Therefore, we get

$$(3.12)\quad \int_{S_m(F_v)}\varepsilon_v\left(\xi\iota\left(\begin{pmatrix}1_m & B\\ 0_m & 1_m\end{pmatrix}g_1,g_2\right);\kappa,\mathfrak{n},\omega\right)\overline{\psi_v(\alpha B)}\,dB$$
$$=\int_{S_m(F_v)}\left|\det\left(\begin{pmatrix}A & 0_m\\ 0_m & A'\end{pmatrix}+i\begin{pmatrix}B & 0_m\\ 0_m & B'\end{pmatrix}\begin{pmatrix}A^{*-1} & 0_m\\ 0_m & A'^{*-1}\end{pmatrix}\right)\right|^\kappa \overline{\psi_v(\alpha B)}\,dB$$
$$=|\det(A'+iB'A'^{*-1})|^\kappa \times\int_{S_m(F_v)}|\det(A+iBA^{*-1})|^\kappa\overline{\psi_v(\alpha B)}\,dB.$$

Note that $\psi_v(\alpha iABA^*)=\psi_v(Ai\alpha A^*B)$ and $Ai\alpha A^*$ is positive definite if and only if α is positive definite.

We have $A\in\Omega_m(F_v)$ and $\psi_v(x)=e^{-i\mathrm{Tr}(x)}$ so for α not positive definite we have from Lemma 4 that the integral of (3.12) becomes

$$\int_{S_m(F_v)}|\det(A+iBA^{*-1})|^\kappa\overline{\psi_v(\alpha B)}\,dB =\int_{S_m(F_v)}e^{-i\mathrm{Tr}\,(\alpha B)}|\det(A-iB)|^\kappa\,dB=0.$$

Therefore, for such an α we have that the αth-Fourier coefficient of $\Phi_N(g_1, g_2)$ in the first variable is 0, and so this is a cuspform in the variable g_1. ◻

Lemmas 2, 3 and 5 and the discussion following Lemma 3 give Theorem 1.

4. Cuspforms with rational Fourier coefficients For $N = n_1+\cdots+n_r$, we have an embedding $\iota : G_{n_1}(F)\times\cdots\times G_{n_r}(F) \hookrightarrow G_N(F)$ given by

(4.1)

$$\iota\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots, \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix}\right) = \begin{pmatrix} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_r & & & b_r \\ c_1 & & & d_1 & & \\ & \ddots & & & \ddots & \\ & & c_r & & & d_r \end{pmatrix}.$$

Let $f(\iota(g_1, \dots, g_n))$ denote the restriction of f on $G_N(F)$ to $\iota(G_{n_1}(F)\times \cdots \times G_{n_r}(F))$.

Lemma 6. *Let f be an automorphic form on $G_N(\mathbf{A})$ with Fourier coefficients in a field $\widetilde{F}$, and let ι be the embedding* (4.1). *Then $f(\iota(g_1, \dots, g_r))$ has Fourier coefficients in $\widetilde{F}$.*

Proof. It suffices to show the result holds for $r = 2$ as induction then gives the result for general r. The Fourier expansion of f is given by

$$f(g) = \sum_{\substack{T\in\Lambda \\ T\geq 0}} W_{f,T}(g),$$

where $\Lambda \subseteq S_n(F)$ and

$$W_{f,T}(g) = \int_{U_0(F)\backslash U_0(\mathbf{A})} \overline{\psi(TB)} f(ug)\, dB$$

for $u = \begin{pmatrix} 1_N & B \\ 0_N & 1_N \end{pmatrix}$. Restricting the Fourier expansion to $(g_1, g_2) \in G_{n_1}(\mathbf{A}) \times G_{n_2}(\mathbf{A})$ we get

$$f(\iota(g_1, g_2)) = \sum_{\substack{T \in \Lambda \\ T \geq 0}} W_{f,T}(\iota(g_1, g_2)).$$

Let $\Lambda(t_1, t_2) = \left\{ T \in \Lambda \mid T = \begin{pmatrix} t_1 & * \\ * & t_2 \end{pmatrix} \geq 0 \right\}$, and note that $|\Lambda(t_1, t_2)| < \infty$ for fixed t_1, t_2. For $T \in \Lambda(t_1, t_2)$, we have

$$\psi(T\iota(g_1, g_2)) = \psi\left(\begin{pmatrix} t_1 & * \\ * & t_2 \end{pmatrix} \iota(g_1, g_2) \right) = \psi(t_1 g_1)\psi(t_2 g_2),$$

and therefore the Fourier indices of $f(\iota(g_1, g_2))$ are t_1 and t_2. We have

$$f(\iota(g_1, g_2)) = \sum_{\substack{t_1, t_2 \in \Lambda \\ T \geq 0}} \left(\sum_{T \in \Lambda(t_1, t_2)} W_{f,T}(\iota(g_1, g_2)) \right).$$

As the inner sum is finite and $W_{f,T}(\iota(g_1, g_2)) \in \widetilde{F}$, we have $\sum_{T \in \Lambda(t_1, t_2)} W_{f,T}(\iota(g_1, g_2)) \in \widetilde{F}$. The result follows. $\square$

From the arithmeticity of Eisenstein series from [**9**], we have that the series $E_N(g; \kappa)$, normalized by a constant, has F-valued Fourier coefficients. It follows from Lemma 6 that $E_N(\iota(g_1, g_2); \kappa)$ has F-valued Fourier coefficients. Let S_m denote the vector space over $\mathbf{Q}$ of modular forms in $\mathcal{M}_m(\kappa, \mathfrak{n})$ generated by modular forms with Fourier coefficients in F and modular forms orthogonal to the space of cuspforms. Then, for fixed g_2, we have $E_N(\iota(g_1, g_2); \kappa) \in S_m$ from the above discussion, and $\Psi_N(g_1, g_2) \in S_m$ from Lemma 2. Thus, from the decomposition of Lemma 2, we have $\Phi_N(g_1, g_2) \in S_m$. As $\Phi_N(g_1, g_2)$ is a cuspform in the smaller variable g_1 by Theorem 1, then it must have F-valued Fourier coefficients. This proves Corollary 1.

We illustrate these results with some examples. Let $P_n(F)$ be the Siegel parabolic of $Sp_n(F)$. By [**13**], the double coset space

$$\frac{P_n(F) \backslash Sp_n(F)}{\iota(Sp_{n_1}(F) \times Sp_{n_2}(F) \times Sp_{n_1+n_2-1}(F))}$$

is finite, where $n = 2n_1 + 2n_2 - 1$. Further, there is a unique nonnegligible, nondegenerate orbit, and all of the other orbits are negligible in the sense of [**8**]. It follows that one of the orbits is Zariski-open, and let ξ denote a representative of this open orbit. The isotropy group of ξ is

$$H_n(F) = \xi P_n(F)\xi^{-1} \cap \iota(Sp_{n_1}(F) \times Sp_{n_2}(F) \times Sp_{n_3}(F)).$$

The representatives ξ and $H_n(F)$ are given explicitly in [**13**]. Thus, we have the following.

Proposition 1. *Let χ denote a character on $P_n(\mathbf{A})$, and let $\varepsilon \in I(\chi)$. Then*

$$\Phi_n(g_1, g_2, g_3) = \sum_{\gamma \in H_n(F)\backslash Sp_{n_1}(F)\times Sp_{n_2}(F)\times Sp_{n_1+n_2-1}(F)} \varepsilon(\xi\gamma\iota(g_1, g_2, g_3)) \tag{4.2}$$

is a cuspform with F-valued Fourier coefficients in each variable.

Proof. From Lemma 6, we have that $\Phi_n(g_1, g_2, g_3)$ is an automorphic form with F-valued Fourier coefficients in each variable, where $n = 2n_1 + 2n_2 - 1$. We can restrict a Siegel Eisenstein series on $Sp_n(\mathbf{A})$ to $Sp_{n_1}(\mathbf{A}) \times Sp_{n_1+2n_2-1}(\mathbf{A})$ and, by Theorem 1, we have that the restriction is a cuspform in the smaller variable $g_1 \in Sp_{n_1}(\mathbf{A})$. Further restricting to $Sp_{n_1}(\mathbf{A}) \times Sp_{n_2}(\mathbf{A}) \times Sp_{n_1+n_2-1}(\mathbf{A})$, we have that (4.2) is a cuspform in $g_1 \in Sp_{n_1}(\mathbf{A})$. We can apply a symmetric argument to show that (4.2) is a cuspform in each variable. More precisely, we can restrict a Siegel Eisenstein series on $Sp_n(\mathbf{A})$ to $Sp_{n_2}(\mathbf{A}) \times Sp_{2n_1+n_2-1}(\mathbf{A})$ or to $Sp_{n_1+n_2-1}(\mathbf{A}) \times Sp_{n_1+n_2}(\mathbf{A})$. The smaller variable is $g_2 \in Sp_{n_2}(\mathbf{A})$ in the former case and $g_3 \in Sp_{n_1+n_2-1}(\mathbf{A})$ in the latter case, and, by Theorem 1, the restriction is a cuspform in the smaller variable. In both cases, we further restrict to $Sp_{n_1}(\mathbf{A}) \times Sp_{n_2}(\mathbf{A}) \times Sp_{n_1+n_2-1}(\mathbf{A})$ to obtain $\Phi_n(g_1, g_2, g_3)$, which gives the result. $\square$

Consider the case $n_1 = n_2 = 1$ and $F = \mathbf{Q}$. From Lemma 3 of [**12**], the double coset $P_3(\mathbf{Q})\backslash Sp_3(\mathbf{Q})/SL_2(\mathbf{Q})^3$ has five irredundant representatives, and the representative of the big-cell is given by

$$\xi = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

The isotropy group is

$$H_3(\mathcal{Q}) = \left\{ \begin{pmatrix} a & b_1 \\ 0 & a^{-1} \end{pmatrix} \times \begin{pmatrix} a & b_2 \\ 0 & a^{-1} \end{pmatrix} \times \begin{pmatrix} a & b_2 \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \mathbf{Q}^\times, \quad b_1, b_2, b_3 \in \mathbf{Q},\ b_1 + b_2 + b_3 = 0 \right\}.$$

For χ a character on $P_3(\mathbf{A})$, $\varepsilon \in I(\chi)$ and ξ the representative (4.3), we have from Proposition 1 that

$$\Phi_3(g_1, g_2, g_3) = \sum_{\gamma \in H_3(\mathbf{Q}) \backslash SL_2(\mathbf{Q})^3} \varepsilon(\xi \gamma \iota(g_1, g_2, g_3))$$

is a holomorphic cuspform with rational-valued Fourier coefficients in each variable $g_i \in SL_2(\mathbf{A})$. Note that this is the cuspidal part of the well-known decomposition of the pullback of a Siegel Eisenstein series on Sp_3. From Proposition 1, for some cuspforms f_1, f_2, f_3 on $SL_2(\mathbf{A})$, we have $\langle\langle\langle \Phi_3, f_1\rangle, f_2\rangle, f_3\rangle \neq 0$. This integral gives the well-known triple product L-function from [**5**].

The cases $n_1 = 1$, $n_2 = 2$ and $F = \mathbf{Q}$ were studied in [**10**]. Proposition 1 gives $\langle\langle\langle \Phi_5, f_1\rangle, F_2\rangle, F_3\rangle \neq 0$ for some cuspforms f_1 on $SL_2(\mathbf{A})$ and F_2, F_3 on $Sp_2(\mathbf{A})$. For F_3 in the Maass Spezialschar [**10**], we have that F_3 is in the image of the Saito-Kurakawa lift of some elliptic cuspform f_3. In that case the integral gives the degree-8 spinor L-function of F_2 twisted by f_3 [**10**].

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