A UNIFIED APPROACH TO FORMAL LOCAL COHOMOLOGY AND LOCAL TATE COHOMOLOGY

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ABSTRACT. Let R be a commutative Noetherian ring. We introduce a theory of formal local cohomology for complexes of R-modules. As an application, we establish some relations between formal local cohomology, local homology, local cohomology and local Tate cohomology through some natural isomorphisms. We investigate vanishing of formal local cohomology modules. Also, we give a characterization of Cohen-Macaulay complexes.

1. Introduction. Throughout, R is a commutative Noetherian ring with a nonzero identity. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and M a finitely generated R-module. When R is local with maximal ideal $\mathfrak{m}$, for each $i \geq 0$, Schenzel [18] has called $\mathfrak{F}_{\mathfrak{a}}^i(M) = \varprojlim_n H_{\mathfrak{m}}^i(M/\mathfrak{a}^n M)$ the ith formal local cohomology module of M with respect to $\mathfrak{a}$. Our aim in this note is to establish a theory of formal local cohomology in $\mathcal{D}(R)$, the derived category of R-modules. Let $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$ and $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$ denote the derived local cohomology and derived local homology functors with respect to $\mathfrak{a}$. (We recall their definitions in the beginning of the next section.) The compositions of these derived functors were studied extensively in [1] and [15]. For a complex $X \in \mathcal{D}(R)$, we call $\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(X) := \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{b}}(X))$ the formal local complex of X with respect to $(\mathfrak{a}, \mathfrak{b})$. Also, for each integer i, we call $\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}^i(X) := H_{-i}(\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(X))$ the ith formal local cohomology module of X with respect to $(\mathfrak{a}, \mathfrak{b})$. It is worth pointing out that in the case R is local with maximal ideal $\mathfrak{m}$, we show that $\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^i(M) = \mathfrak{F}_{\mathfrak{a}}^i(M)$ for all i, see Corollary 2.4 below.

In Section 2, we establish several general properties of formal local cohomology. In the case that R possesses a normalized dualizing

complex, we present a duality result, see Lemma 2.2 below. This duality lemma facilitates working with formal local cohomology modules. We continue this section by applying the theory of formal local cohomology for examining local Tate cohomology modules. Let $X \in \mathcal{D}(R)$. The local Tate cohomology modules $\widehat{H}_{\mathfrak{a}}^i(X)$ were introduced by Greenlees [12], and their study was continued by Alonso Tarrío, Jeremías López and Lipman [1]. We deduce a long exact sequence

$$\cdots \longrightarrow H_{\mathfrak{a}}^i(X) \longrightarrow \mathfrak{F}_{\mathfrak{a},\mathfrak{a}}^i(X) \longrightarrow \widehat{H}_{\mathfrak{a}}^i(X) \longrightarrow H_{\mathfrak{a}}^{i+1}(X) \longrightarrow \mathfrak{F}_{\mathfrak{a},\mathfrak{a}}^{i+1}(X) \longrightarrow \cdots,$$

which, in turn, yields several corollaries. In particular, from this sequence, we immediately deduce that $\widehat{H}_{\mathfrak{a}}^i(M) \cong H_{\mathfrak{a}}^{i+1}(M)$ for all R-modules M and all $i > 0$.

Section 3 is the core of this paper. In this section, we present several results on vanishing of formal local cohomology modules. Let X be a complex. We introduce the notion fdepth $(\mathfrak{a}, \mathfrak{b}, X)$ as the infimum of the integers i such that $\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}^i(X) \neq 0$. In the prime characteristic case, this notion is closely related to the notion of Frobenius depth which was defined by Hartshorne and Speiser in [14]. When X is homologically bounded and all of its homology modules are finitely generated, we establish the inequality

$$\mathrm{depth}\,(\mathfrak{b}, M) - \mathrm{cd}_{\mathfrak{a}}(R) \leq \mathrm{fdepth}\,(\mathfrak{a}, \mathfrak{b}, X) \leq \dim_R X/\mathfrak{a}X.$$

Let $(R, \mathfrak{m})$ be a local ring and $0 \not\simeq X$ a homologically bounded complex with finitely generated homology modules. We show that the supremum of the integers i such that $\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^i(X) \neq 0$ is equal to $\dim_R X/\mathfrak{a}X$. Set $H(X)^{\sharp} := \oplus_{i \in \mathbf{Z}} H_i(X)$. Then we show that

$$\mathrm{depth}\,X - \mathrm{cd}_{\mathfrak{a}}(H(X)^{\sharp}) \leq \mathrm{fdepth}\,(\mathfrak{a}, X) \leq \dim X - \mathrm{cd}_{\mathfrak{a}}(H(X)^{\sharp}).$$

This immediately provides a new characterization of Cohen-Macaulay complexes, see Corollary 3.7 below.

2. Formal local cohomology and local Tate cohomology. Throughout, the symbol $\simeq$ will denote isomorphisms in the category $\mathcal{D}(R)$, the derived category of R-modules. We denote the full subcategory of R-modules by $\mathcal{C}_0(R)$. The full subcategory of homologically

bounded complexes is denoted by $\mathcal{D}_{\square}(R)$ and that of complexes homo-
logically bounded to the right (respectively, left) is denoted by $\mathcal{D}_{\sqsupset}(R)$
(respectively, $\mathcal{D}_{\sqsubset}(R)$). Also, if $\sharp$ is one of the symbols $\square$ or $\sqsupset$, then
$\mathcal{D}_{\sharp}^f(R)$ stands for the full subcategory of complexes $X \in \mathcal{D}_{\sharp}(R)$ all of
whose homology modules are finitely generated.

Let $\mathfrak{a}$ be an ideal of R. The right derived functor of the $\mathfrak{a}$-section
functor

$$\Gamma_{\mathfrak{a}}(-) := \varinjlim_n \operatorname{Hom}_R(R/\mathfrak{a}^n, -) : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$$

exists in $\mathcal{D}(R)$. Let $X \in \mathcal{D}(R)$. Then the complex $\mathbf{R}\Gamma_{\mathfrak{a}}(X)$ is defined
by $\mathbf{R}\Gamma_{\mathfrak{a}}(X) := \Gamma_{\mathfrak{a}}(I)$, where I is any K-injective resolution of X. (For
more details on the theory of K-resolutions, we refer the reader to
[**19**].) For any integer i, the ith local cohomology module of X with
respect to $\mathfrak{a}$ is defined by $H_{\mathfrak{a}}^i(X) := H_{-i}(\mathbf{R}\Gamma_{\mathfrak{a}}(X))$. Let $\check{C}(\underline{\mathfrak{a}})$ denote
the Čech complex of R on a set $\underline{\mathfrak{a}} = a_1, \dots, a_n$ of generators of $\mathfrak{a}$. By
[**17**, Theorem 1.1 iv)],

$$(*) \qquad \mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq X \otimes_R^{\mathbf{L}} \check{C}(\underline{\mathfrak{a}}).$$

Let $X \in \mathcal{D}_{\sqsupset}^f(R)$ and $Y \in \mathcal{D}_{\square}(R)$. As $\check{C}(\underline{\mathfrak{a}})$ is a bounded complex of
flat R-modules, tensor evaluation property (see [**3**, A.4.23]) along with
$(*)$ yield that

$$(**) \qquad \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\operatorname{Hom}_R(X,Y)) = \mathbf{R}\operatorname{Hom}_R(X, \mathbf{R}\Gamma_{\mathfrak{a}}(Y)).$$

The left derived functor of the $\mathfrak{a}$-adic completion functor

$$\Lambda^{\mathfrak{a}}(-) = \varprojlim_n (R/\mathfrak{a}^n \otimes_R -) : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$$

exists in $\mathcal{D}(R)$, and so for a complex $X \in \mathcal{D}(R)$, the complex $\mathbf{L}\Lambda^{\mathfrak{a}}(X)$ is
defined by $\mathbf{L}\Lambda^{\mathfrak{a}}(X) := \Lambda^{\mathfrak{a}}(P)$, where P is any K-projective resolution
of X. For any integer i, the ith local homology module of a complex
$X \in \mathcal{D}(R)$ with respect to $\mathfrak{a}$ is defined by $H_i^{\mathfrak{a}}(X) := H_i(\mathbf{L}\Lambda^{\mathfrak{a}}(X))$.
By [**1**, page 4, (0.3)] (see also [**17**, Section 4] for corrections) for any
$X \in \mathcal{D}(R)$, one has

$$(\dagger) \qquad \mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \mathbf{R}\operatorname{Hom}_R(\check{C}(\underline{\mathfrak{a}}), X).$$

Using adjointness, (†) and (∗) yield the following isomorphisms
(‡)
$$\mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(X,Y)) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(X),Y) \simeq \mathbf{R}\mathrm{Hom}_R(X,\mathbf{L}\Lambda^{\mathfrak{a}}(Y))$$
in $\mathcal{D}(R)$ for all complexes X and Y, see e.g., [**11**, 2.6]. In the sequel, we will use the isomorphisms (∗), (∗∗), (†) and (‡) without any further comments.

Definition 2.1. Let $\mathfrak{a},\mathfrak{b}$ be two ideals of R and $X \in \mathcal{D}(R)$. We define the *formal local complex* of X with respect to $(\mathfrak{a},\mathfrak{b})$ by $\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(X) := \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{b}}(X)) \in \mathcal{D}(R)$. Also, for each integer i, the ith *formal local cohomology module* of X with respect to $(\mathfrak{a},\mathfrak{b})$ is defined by $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(X) := H_{-i}(\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(X))$. When R is local and $\mathfrak{b}$ is its maximal ideal, we abbreviate $\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(X)$ and $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(X)$, respectively, by $\mathfrak{F}_{\mathfrak{a}}(X)$ and $\mathfrak{F}^i_{\mathfrak{a}}(X)$.

A complex $X \in \mathcal{D}(R)$ is said to have finite injective dimension if it is isomorphic, in $\mathcal{D}(R)$, to a bounded complex of injective R-modules. A dualizing complex of R is a complex $D \in \mathcal{D}^f_{\square}(R)$ such that the homothety morphism $R \longrightarrow \mathbf{R}\mathrm{Hom}_R(D,D)$ is an isomorphism in $\mathcal{D}(R)$ and D has finite injective dimension. A dualizing complex D is said to be normalized if $\sup D = \dim R$. Assume that R possesses a dualizing complex D. Then, we know that $\dim R$ should be finite. Now, one can check easily that $\Sigma^{\dim R - \sup X}D$ is a normalized dualizing complex of R. In what follows, whenever R possesses a normalized dualizing complex D, we denote $\mathbf{R}\mathrm{Hom}_R(-,D)$ by $(-)^{\dagger}$.

Lemma 2.2 (Duality). *Let $\mathfrak{a},\mathfrak{b}$ be two ideals of R and $X \in \mathcal{D}^f_{\square}(R)$. Assume that R possesses a normalized dualizing complex D. Then $\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(X) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(X^{\dagger}),\mathbf{R}\Gamma_{\mathfrak{b}}(D))$.*

Proof. One can see easily that there is a natural isomorphism $(X^{\dagger})^{\dagger} \simeq X$. Since $X^{\dagger} \in \mathcal{D}^f_{\square}(R)$, we have

$$\begin{aligned}
\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(X) &\simeq \mathbf{R}\mathrm{Hom}_R(\check{C}(\underline{\mathfrak{a}}),\mathbf{R}\Gamma_{\mathfrak{b}}(X)) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\check{C}(\underline{\mathfrak{a}}),\mathbf{R}\Gamma_{\mathfrak{b}}(\mathbf{R}\mathrm{Hom}_R(X^{\dagger},D))) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\check{C}(\underline{\mathfrak{a}}),\mathbf{R}\mathrm{Hom}_R(X^{\dagger},\mathbf{R}\Gamma_{\mathfrak{b}}(D))) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(X^{\dagger}),\mathbf{R}\Gamma_{\mathfrak{b}}(D)). \qquad \square
\end{aligned}$$

For a local ring $(R, \mathfrak{m}, k)$, we denote the Matlis duality functor $\mathrm{Hom}_R(-, E_R(k))$ by $(-)^\vee$.

Corollary 2.3. *Let $(R, \mathfrak{m})$ be a local ring possessing a normalized dualizing complex D, $\mathfrak{a}$ an ideal of R and $X \in \mathcal{D}_{\square}^f(R)$. Then $\mathfrak{F}_\mathfrak{a}(X) \simeq \mathbf{R}\Gamma_\mathfrak{a}(X^\dagger)^\vee$.*

Proof. By [**13**, Proposition 6.1], $\mathbf{R}\Gamma_\mathfrak{m}(D) = E_R(k)$. Applying Lemma 2.2 to $\mathfrak{b} := \mathfrak{m}$ yields that

$$\mathfrak{F}_\mathfrak{a}(X) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_\mathfrak{a}(X^\dagger), \mathbf{R}\Gamma_\mathfrak{m}(D)) \simeq \mathbf{R}\Gamma_\mathfrak{a}(X^\dagger)^\vee. \qquad \square$$

Next, we present the following corollary. It shows that Definition 2.1 extends Schenzel's definition.

Corollary 2.4. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and $X \in \mathcal{D}(R)$. If either X is a bounded complex of flat R-modules whose all homology modules are finitely generated or X is a finitely generated R-module, then $\mathfrak{F}_\mathfrak{a}^i(X) \cong \varprojlim_n H_\mathfrak{m}^i(X/\mathfrak{a}^n X)$ for all $i \in \mathbf{Z}$.*

Proof. First assume that X is a bounded complex of flat R-modules all of whose homology modules are finitely generated. Then $X \otimes_R^\mathbf{L} \check{C}(\underline{\mathfrak{m}}) \simeq X \otimes_R \check{C}(\underline{\mathfrak{m}})$ is a bounded complex of flat R-modules and $\mathbf{R}\Gamma_\mathfrak{m}(X) \simeq X \otimes_R \check{C}(\underline{\mathfrak{m}})$. Hence $X \otimes_R \check{C}(\underline{\mathfrak{m}})$ is a K-flat resolution of $\mathbf{R}\Gamma_\mathfrak{m}(X)$. It is known that, for a complex Z and any K-flat resolution F of Z, one has $\mathbf{L}\Lambda^\mathfrak{a}(Z) \simeq \Lambda^\mathfrak{a}(F)$. Thus,

$$\mathfrak{F}_\mathfrak{a}(X) = \mathbf{L}\Lambda^\mathfrak{a}(\mathbf{R}\Gamma_\mathfrak{m}(X)) \simeq \Lambda^\mathfrak{a}(X \otimes_R \check{C}(\underline{\mathfrak{m}})).$$

Now, we have

$$\Lambda^\mathfrak{a}(X \otimes_R \check{C}(\underline{\mathfrak{m}})) = \varprojlim_n (R/\mathfrak{a}^n \otimes_R (X \otimes_R \check{C}(\underline{\mathfrak{m}}))) \simeq \varprojlim_n (X/\mathfrak{a}^n X \otimes_R \check{C}(\underline{\mathfrak{m}})).$$

For each nonnegative integer n, all homology modules of the complex $X/\mathfrak{a}^n X \otimes_R \check{C}(\underline{\mathfrak{m}})(\simeq \mathbf{R}\Gamma_\mathfrak{m}(X/\mathfrak{a}^n X))$ are Artinian. The Mittag-Leffler condition [**20**, Proposition 3.5.7] implies that $\varprojlim_n^1 H_{-i}(X/\mathfrak{a}^n X \otimes_R$

$\check{C}(\underline{\mathfrak{m}})) = 0$ for all integers i. From [**20**, Theorem 3.5.8], we deduce the exact sequence

$$0 \longrightarrow \varprojlim_n {}^1 H_{-i+1}(X/\mathfrak{a}^n X \otimes_R \check{C}(\underline{\mathfrak{m}})) \longrightarrow H_{-i}(\varprojlim_n (X/\mathfrak{a}^n X \otimes_R \check{C}(\underline{\mathfrak{m}})))$$
$$\longrightarrow \varprojlim_n H_{-i}(X/\mathfrak{a}^n X \otimes_R \check{C}(\underline{\mathfrak{m}})) \longrightarrow 0,$$

for all i. Thus, for any integer i, we have

$$\mathfrak{F}_\mathfrak{a}^i(X) \cong H_{-i}(\varprojlim_n (X/\mathfrak{a}^n X \otimes_R \check{C}(\underline{\mathfrak{m}}))) \cong \varprojlim_n H_{-i}(X/\mathfrak{a}^n X \otimes_R \check{C}(\underline{\mathfrak{m}}))$$
$$\cong \varprojlim_n H_{-i}(\mathbf{R}\Gamma_\mathfrak{m}(X/\mathfrak{a}^n X)) = \varprojlim_n H_\mathfrak{m}^i(X/\mathfrak{a}^n X).$$

Next, assume that X is a finitely generated R-module. Without loss of generality, we may and do assume that R is complete. Hence, R possesses a normalized dualizing complex, and so Corollary 2.3 implies that $H_{-i}(\mathfrak{F}_\mathfrak{a}(X)) \cong H_\mathfrak{a}^{-i}(X^\dagger)^\vee$. Now, [**18**, Theorem 3.5] finishes the proof in this case.   □

Next, we bring two more results concerning the computation of formal local cohomology modules. The first one indicates that the theory of formal local cohomology can be considered as a unified generalization of the two theories of local cohomology and local homology.

Proposition 2.5. *Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R and $X \in \mathcal{D}(R)$. The following assertions hold.*

  i) $\mathfrak{F}_{\mathfrak{a},\mathfrak{a}}(X) \simeq \mathbf{L}\Lambda^\mathfrak{a}(X)$.

  ii) *Assume that* $\mathrm{Supp}_R X \subseteq \mathrm{V}(\mathfrak{b})$. *Then* $\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(X) \simeq \mathbf{L}\Lambda^\mathfrak{a}(X)$.

  iii) *Assume that R possesses a normalized dualizing complex D, $X \in \mathcal{D}_\square^f(R)$ and* $\mathrm{Supp}_R X \subseteq \mathrm{V}(\mathfrak{a})$. *Then* $\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(X) \simeq \mathbf{R}\Gamma_\mathfrak{b}(X)$.

  Proof. i) holds by [**1**, (0.3)*, Corollary].

  ii) holds by [**15**, Corollary 3.2.1].

  iii) For any prime ideal $\mathfrak{p}$ of R, we have $(X^\dagger)_\mathfrak{p} \simeq \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(X_\mathfrak{p}, D_\mathfrak{p})$, and so $\mathrm{Supp}_R X^\dagger \subseteq \mathrm{Supp}_R X$. Thus, $X^\dagger$ is homologically bounded and

$\mathrm{Supp}_R X^\dagger \subseteq \mathrm{V}(\mathfrak{a})$. Now, [**15**, Corollary 3.2.1] yields that $\mathbf{R}\Gamma_{\mathfrak{a}}(X^\dagger) \simeq X^\dagger$. Set $(-)^* := \mathbf{R}\mathrm{Hom}_R(-, \mathbf{R}\Gamma_{\mathfrak{b}}(D))$. Then Lemma 2.2 yields that

$$\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(X) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(X^\dagger)^* \simeq (X^\dagger)^* \simeq \mathbf{R}\Gamma_{\mathfrak{b}}(\mathbf{R}\mathrm{Hom}_R(X^\dagger, D)) \simeq \mathbf{R}\Gamma_{\mathfrak{b}}(X). \quad \square$$

Proposition 2.6. *Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R, $X \in \mathcal{D}_\sqsupseteq^f(R)$ and $Y \in \mathcal{D}_\square(R)$. Then*

i) $\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(\mathbf{R}\mathrm{Hom}_R(X, Y)) \simeq \mathbf{R}\mathrm{Hom}_R(X, \mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(Y))$.

ii) $\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(X \otimes_R^{\mathbf{L}} Y) \simeq X \otimes_R^{\mathbf{L}} \mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(Y)$.

Proof. One has

$$\begin{aligned}
\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(\mathbf{R}\mathrm{Hom}_R(X, Y)) &\simeq \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{b}}(\mathbf{R}\mathrm{Hom}_R(X, Y))) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\check{C}(\underline{\mathfrak{a}}), \mathbf{R}\Gamma_{\mathfrak{b}}(\mathbf{R}\mathrm{Hom}_R(X, Y))) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\check{C}(\underline{\mathfrak{a}}), \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\Gamma_{\mathfrak{b}}(Y))) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\mathrm{Hom}_R(\check{C}(\underline{\mathfrak{a}}), \mathbf{R}\Gamma_{\mathfrak{b}}(Y))) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(X, \mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(Y)).
\end{aligned}$$

ii) One has

$$\begin{aligned}
\mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(X \otimes_R^{\mathbf{L}} Y) &\simeq \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{b}}(X \otimes_R^{\mathbf{L}} Y)) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\check{C}(\underline{\mathfrak{a}}), \mathbf{R}\Gamma_{\mathfrak{b}}(X \otimes_R^{\mathbf{L}} Y)) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\check{C}(\underline{\mathfrak{a}}), \mathbf{R}\Gamma_{\mathfrak{b}}(Y) \otimes_R^{\mathbf{L}} X) \\
&\overset{\sharp}{\simeq} \mathbf{R}\mathrm{Hom}_R(\check{C}(\underline{\mathfrak{a}}), \mathbf{R}\Gamma_{\mathfrak{b}}(Y)) \otimes_R^{\mathbf{L}} X \\
&\simeq X \otimes_R^{\mathbf{L}} \mathfrak{F}_{\mathfrak{a},\mathfrak{b}}(Y).
\end{aligned}$$

Regarding the isomorphism $\sharp$, we have to give some explanations. By [**4**, 5.8], the projective dimension of $\check{C}(\underline{\mathfrak{a}})$ is finite. On the other hand, as $Y \in \mathcal{D}_\square(R)$ and $\check{C}(\underline{\mathfrak{a}})$ is a bounded complex of flat R-modules, we deduce that $\mathbf{R}\Gamma_{\mathfrak{a}}(Y) \simeq \check{C}(\underline{\mathfrak{a}}) \otimes_R^{\mathbf{L}} Y$ is homologically bounded. Thus, [**5**, Proposition 2.2 vi)] implies the isomorphism $\sharp$. $\quad \square$

The theory of local Tate cohomology was introduced by Greenlees [**12**]. Let $\{\phi^t : X^t \to X^{t+1}\}_{t \in \mathbf{N}}$ be a family of morphisms of complexes.

It induces a morphism of complexes $\varphi = (\varphi_i) : \bigoplus_{t \in \mathbf{N}} X^t \to \bigoplus_{t \in \mathbf{N}} X^t$ given by $\varphi_i((x_i^t)) = (x_i^t) - (\phi_i^t(x_i^t))$ in spot i. The telescope of $\{\phi^t : X^t \to X^{t+1}\}_{t \in \mathbf{N}}$ is defined by $\mathrm{Tel}\,(X^t) := \mathrm{Cone}\,(\varphi)$. Let $a \in R$. For each natural integer t, let $K(a^t)$ denote the Koszul complex of R with respect to a^t. Clearly, multiplication by a induces a family of morphisms of complexes $K(a^t) \to K(a^{t+1})$. The projective stabilized Koszul complex with respect to a is defined by $K(a^\infty) := \mathrm{Tel}\,(K(a^t))$. Let $\underline{\mathfrak{a}} = a_1, \dots, a_n$ be a sequence of elements of R. The projective stabilized Koszul complex with respect to $\underline{\mathfrak{a}}$ is defined by $K(\underline{\mathfrak{a}}^\infty) := \mathrm{Tel}\,(K(a_1^t)) \otimes_R \dots \otimes_R \mathrm{Tel}\,(K(a_n^t))$. The stabilized Čech complex with respect to $\underline{\mathfrak{a}}$ is defined by $\check{C}(\underline{\mathfrak{a}}^\infty) := \mathrm{Cone}\,(K(\underline{\mathfrak{a}}^\infty) \to R)$.

Definition 2.7. Let $\mathfrak{a}$ be an ideal of R and $X \in \mathcal{D}(R)$. Let $\underline{\mathfrak{a}} = a_1, \dots, a_n$ be a generating set of $\mathfrak{a}$. The *local Tate complex* of X with respect to $\mathfrak{a}$ is defined by $T(X) := \mathrm{Hom}_R(K(\underline{\mathfrak{a}}^\infty), X) \otimes_R \check{C}(\underline{\mathfrak{a}}^\infty)$. Also, for each integer i, the ith *local Tate cohomology module* of X with respect to $\mathfrak{a}$ is defined by $\widehat{H}_{\mathfrak{a}}^i(X) := H_{-i}(T(X))$.

Lemma 2.8 (The algebraic Warwick duality). *Let $\underline{\mathfrak{a}} = a_1, \dots, a_n$ be a sequence of elements of R and $X \in \mathcal{D}(R)$. Let $TT(X) := \mathrm{Hom}_R(\check{C}(\underline{\mathfrak{a}}^\infty), X \otimes_R \Sigma K(\underline{\mathfrak{a}}^\infty))$. Then there is a natural isomorphism $T(X) \simeq TT(X)$ in $\mathcal{D}(R)$.*

Proof. For modules, this is proved in [**12**, Theorem 4.1]. In view of [**12**, Corollary 4.6], we can check easily that the conclusion is true also for complexes. □

The next result is contained in [**1**, Proposition 5.1.3]. Here we prove it by applying a simpler argument.

Proposition 2.9. *Let $\mathfrak{a}$ be an ideal of R and $X \in \mathcal{D}_\square(R)$. We have the following long exact sequence*

$$\cdots \longrightarrow H_{\mathfrak{a}}^i(X) \longrightarrow H_{-i}^{\mathfrak{a}}(X) \longrightarrow \widehat{H}_{\mathfrak{a}}^i(X) \longrightarrow H_{\mathfrak{a}}^{i+1}(X) \longrightarrow H_{-i-1}^{\mathfrak{a}}(X) \longrightarrow \cdots.$$

Proof. Let $\underline{\mathfrak{a}} = a_1, \dots, a_n$ be a generating set of $\mathfrak{a}$. In view of Lemma 2.8, we have the following natural isomorphisms in $\mathcal{D}(R)$:

$$T(X) \simeq TT(X)$$
$$\simeq \Sigma\,\mathrm{Hom}_R(\check{C}(\underline{\mathfrak{a}}^\infty), X \otimes_R K(\underline{\mathfrak{a}}^\infty))$$

$$= \Sigma \operatorname{Hom}_R(\operatorname{Cone}(K(\mathfrak{a}^\infty) \longrightarrow R), X \otimes_R K(\mathfrak{a}^\infty))$$
$$\simeq \Sigma\Sigma^{-1}\operatorname{Cone}(\operatorname{Hom}_R(R, X \otimes_R K(\mathfrak{a}^\infty))$$
$$\longrightarrow \operatorname{Hom}_R(K(\mathfrak{a}^\infty), X \otimes_R K(\mathfrak{a}^\infty)))$$
$$\simeq \Sigma\Sigma^{-1}\operatorname{Cone}(X \otimes_R K(\mathfrak{a}^\infty) \longrightarrow \operatorname{Hom}_R(K(\mathfrak{a}^\infty), X \otimes_R K(\mathfrak{a}^\infty)))$$
$$= \operatorname{Cone}(X \otimes_R K(\mathfrak{a}^\infty) \longrightarrow \operatorname{Hom}_R(K(\mathfrak{a}^\infty), X \otimes_R K(\mathfrak{a}^\infty))).$$

Note that the third quism above is obtained by [**9**, 3.48]. In view of [**12**, Remark 1.1], $K(\mathfrak{a}^\infty)$ is a projective resolution of $\check{C}(\mathfrak{a})$. So, we deduce the following exact sequence in $\mathcal{D}(R)$

$$0 \longrightarrow \mathbf{R}\operatorname{Hom}_R(\check{C}(\mathfrak{a}), X \otimes_R^{\mathbf{L}} \check{C}(\mathfrak{a})) \longrightarrow T(X) \longrightarrow \Sigma(X \otimes_R^{\mathbf{L}} \check{C}(\mathfrak{a})) \longrightarrow 0.$$

Now, by Proposition 2.5 i), the induced long exact sequence of homologies of this exact sequence is our desired long exact sequence. $\qquad\square$

Let $\mathfrak{a}$ be an ideal of R and $X \in \mathcal{D}(R)$. We denote

$$\sup\{i \in \mathbf{Z} \mid H_{\mathfrak{a}}^i(X) \neq 0\}(= -\inf \mathbf{R}\Gamma_{\mathfrak{a}}(X))$$

by $\operatorname{cd}_{\mathfrak{a}}(X)$. Also, recall that $\operatorname{depth}(\mathfrak{a}, X)$ is defined by $\operatorname{depth}(\mathfrak{a}, X) := -\sup \mathbf{R}\operatorname{Hom}_R(R/\mathfrak{a}, X)$. The following corollary is immediate.

Corollary 2.10. *Let $\mathfrak{a}$ be an ideal of R and $X \in \mathcal{D}_\square(R)$. If either $i < \operatorname{depth}(\mathfrak{a}, X) - 1$ or $i > \operatorname{cd}_{\mathfrak{a}}(X)$, then $\widehat{H}_{\mathfrak{a}}^i(X) \cong H_{-i}^{\mathfrak{a}}(X)$.*

Corollary 2.11. *Let $\mathfrak{a}$ be an ideal of R and M an R-module. Then $\widehat{H}_{\mathfrak{a}}^i(M) \cong H_{\mathfrak{a}}^{i+1}(M)$ for all $i > 0$. Moreover, if M is $\mathfrak{a}$-adic complete, then $\widehat{H}_{\mathfrak{a}}^i(M) \cong H_{\mathfrak{a}}^{i+1}(M)$ for all $i \neq 0, -1$.*

Proof. Since $H_{-i}^{\mathfrak{a}}(M) = 0$ for all $i > 0$, from Proposition 2.9, it turns out that $\widehat{H}_{\mathfrak{a}}^i(M) \cong H_{\mathfrak{a}}^{i+1}(M)$ for all $i > 0$. Now, assume that M is $\mathfrak{a}$-adic complete. Then $H_i^{\mathfrak{a}}(M) = 0$ for all $i \neq 0$. Hence, from Proposition 2.9, we deduce that $\widehat{H}_{\mathfrak{a}}^i(M) \cong H_{\mathfrak{a}}^{i+1}(M)$ for all $i \neq 0, -1$. $\qquad\square$

3. Vanishing results. We start this section with the following definition.

Definition 3.1. Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R and $X \in \mathcal{D}(R)$. We define *formal depth* of X with respect to $(\mathfrak{a}, \mathfrak{b})$ by fdepth $(\mathfrak{a}, \mathfrak{b}, X) :=$ inf $\{i \in \mathbf{Z} : \mathfrak{F}^i_{\mathfrak{a}, \mathfrak{b}}(X) \neq 0\}$. When R is local with maximal ideal $\mathfrak{m}$, we abbreviate fdepth $(\mathfrak{a}, \mathfrak{m}, X)$ by fdepth $(\mathfrak{a}, X)$.

Let $\mathfrak{a}$ be an ideal of R and $X \in \mathcal{D}(R)$. Recall that $\dim_R X$ is defined by $\dim_R X := \sup\{\dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}$. It is known that, for any complex $X \in \mathcal{D}^f_\square(R)$, we have $- \sup \mathbf{R}\Gamma_\mathfrak{a}(X) = \operatorname{depth}(\mathfrak{a}, X)$ and $\operatorname{cd}_\mathfrak{a}(X) \leq \dim_R X$ with equality if R is local and $\mathfrak{a}$ is its maximal ideal, see [**8**, Theorem 7.8 and Proposition 7.10]. From Corollary 2.3, we can record the following immediate corollary.

Corollary 3.2. *Let $(R, \mathfrak{m})$ be a local ring possessing a normalized dualizing complex D, $\mathfrak{a}$ an ideal of R and $X \in \mathcal{D}^f_\square(R)$. Then*

$$\operatorname{fdepth}(\mathfrak{a}, X) = -\sup\{i \in \mathbf{Z} : H^\mathfrak{a}_i(\mathbf{R}\Gamma_\mathfrak{m}(X)) \neq 0\}$$
$$= -\operatorname{cd}_\mathfrak{a}(X^\dagger) \geq -\dim_R X^\dagger.$$

Next is our first main result.

Theorem 3.3. *Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R and $X \in \mathcal{D}^f_\square(R)$. Let $K(\underline{\mathfrak{a}})$ denote the Koszul complex of R with respect to a generating set $\underline{\mathfrak{a}} = a_1, \dots, a_n$ of $\mathfrak{a}$. The following assertions hold.*

i) fdepth $(\mathfrak{a}, \mathfrak{a}, X) = -\sup \mathbf{L}\Lambda^\mathfrak{a}(X)$.

ii) fdepth $(\mathfrak{a}, \mathfrak{b}, X) \geq \operatorname{depth}(\mathfrak{b}, X) - \operatorname{cd}_\mathfrak{a}(R)$.

iii) $\sup\{i \in \mathbf{Z} : \mathfrak{F}^i_{\mathfrak{a}, \mathfrak{b}}(X) \neq 0\} = \operatorname{cd}_\mathfrak{b}(K(\underline{\mathfrak{a}}) \otimes^{\mathbf{L}}_R X) \leq \dim_R X/\mathfrak{a}X$. *In particular, if R is local, then* $\sup\{i \in \mathbf{Z} : \mathfrak{F}^i_\mathfrak{a}(X) \neq 0\} = \dim_R X/\mathfrak{a}X$.

Proof. i) is clear by Proposition 2.5 i).

ii) For any two complexes $V \in \mathcal{D}_\sqsupset(R)$ and $W \in \mathcal{D}_\sqsubset(R)$, [**3**, Proposition A.4.6] yields that

$$\sup \mathbf{R}\operatorname{Hom}_R(V, W) \leq \sup W - \inf V.$$

Hence, one has

$$\operatorname{fdepth}(\mathfrak{a}, \mathfrak{b}, X) = \inf\{i \in \mathbf{Z} : H_{-i}(\mathbf{L}\Lambda^\mathfrak{a}(\mathbf{R}\Gamma_\mathfrak{b}(X))) \neq 0\}$$

$$= -\sup\{i \in \mathbf{Z} : H_i(\mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{b}}(X))) \neq 0\}$$
$$= -\sup \mathbf{R}\mathrm{Hom}_R(\check{C}(\mathfrak{a}), \mathbf{R}\Gamma_{\mathfrak{b}}(X))$$
$$\geq -\sup \mathbf{R}\Gamma_{\mathfrak{b}}(X) + \inf \check{C}(\mathfrak{a})$$
$$= \mathrm{depth}\,(\mathfrak{b}, X) - \mathrm{cd}_{\mathfrak{a}}(R).$$

iii) For any complex $Y \in \mathcal{D}_{\sqsupset}(R)$, [**11**, Theorem 2.11] asserts that $\inf \mathbf{L}\Lambda^{\mathfrak{a}}(Y) = \inf (K(\mathfrak{a}) \otimes_R^{\mathbf{L}} Y)$. Also, by Grothendieck's vanishing theorem (see [**9**, Theorem 7.8]), for any complex $Z \in \mathcal{D}_{\square}^f(R)$, we know that $\mathrm{cd}_{\mathfrak{b}}(Z) \leq \dim_R Z$ with equality if R is local and $\mathfrak{b}$ is its maximal ideal. One has

$$\sup\{i \in \mathbf{Z} : \mathfrak{F}_{\mathfrak{a},\mathfrak{b}}^i(X) \neq 0\} = \sup\{i \in \mathbf{Z} : H_{-i}(\mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{b}}(X))) \neq 0\}$$
$$= -\inf \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{b}}(X))$$
$$= -\inf (K(\mathfrak{a}) \otimes_R^{\mathbf{L}} \mathbf{R}\Gamma_{\mathfrak{b}}(X))$$
$$= -\inf \mathbf{R}\Gamma_{\mathfrak{b}}(K(\mathfrak{a}) \otimes_R^{\mathbf{L}} X)$$
$$= \mathrm{cd}_{\mathfrak{b}}(K(\mathfrak{a}) \otimes_R^{\mathbf{L}} X)$$
$$\leq \dim_R(K(\mathfrak{a}) \otimes_R^{\mathbf{L}} X)$$
$$= \dim_R X/\mathfrak{a}X.$$

It is easy to see that $\mathrm{Supp}_R(K(\mathfrak{a}) \otimes_R^{\mathbf{L}} X) = \mathrm{Supp}_R X/\mathfrak{a}X$ and, for any prime ideal $\mathfrak{p}$ of R, we have $\inf (K(\mathfrak{a}) \otimes_R^{\mathbf{L}} X)_{\mathfrak{p}} = \inf X_{\mathfrak{p}}$. This yields the last equality above.   □

Let $\mathfrak{a}$ be an ideal of R and $X \in \mathcal{D}_{\square}^f(R)$. Finding a good upper bound for $\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X)$ is of some interest (see, e.g., [**17**, page 179]). Theorem 3.3 immediately implies the following corollary.

Corollary 3.4. *Let $\mathfrak{a}$ be an ideal of R and $X \in \mathcal{D}_{\square}^f(R)$. Then* $\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X) \leq \mathrm{cd}_{\mathfrak{a}}(R) - \mathrm{depth}\,(\mathfrak{a}, X).$

For proving our second main result, we need to prove the following lemma.

Lemma 3.5. *Let $\mathfrak{a}$ be an ideal of R and M a finitely generated R-module. Let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a family of finitely generated R-modules such*

that $\cup_{\lambda \in \Lambda} \mathrm{Supp}_R N_\lambda = \mathrm{Supp}_R M$. *Then*

$$\mathrm{cd}_{\mathfrak{a}}(M) = \sup\{\mathrm{cd}_{\mathfrak{a}}(N_\lambda)|\lambda \in \Lambda\}.$$

Proof. By [**7**, Theorem 2.2], for any finitely generated R-module N, we conclude that $\mathrm{cd}_{\mathfrak{a}}(N) = \mathrm{cd}_{\mathfrak{a}}(\oplus_{\mathfrak{p} \in \mathrm{Supp}_R N} R/\mathfrak{p})$. Hence,

$$\mathrm{cd}_{\mathfrak{a}}(N) = \sup\{\mathrm{cd}_{\mathfrak{a}}(R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R N\},$$

which easily yields the claim. $\quad\square$

The following result extends [**2**, Theorem 4.6] to complexes. It should be noted that its proof is completely different from our proof for [**2**, Theorem 4.6].

Theorem 3.6. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* $X \in \mathcal{D}_\square^f(R)$ *a non-homologically trivial complex. Set* $H(X)^\sharp := \oplus_{i \in \mathbf{Z}} H_i(X)$. *Then*

$$\mathrm{depth}\, X - \mathrm{cd}_{\mathfrak{a}}(H(X)^\sharp) \leq \mathrm{fdepth}\,(\mathfrak{a}, X) \leq \dim X - \mathrm{cd}_{\mathfrak{a}}(H(X)^\sharp).$$

Proof. [**15**, Corollary 3.4.4] yields an isomorphism

$$\mathbf{R}\Gamma_{\mathfrak{m}}(X) \otimes_R \widehat{R} \simeq \mathbf{R}\Gamma_{\mathfrak{m}\widehat{R}}(X \otimes_R \widehat{R})$$

in $\mathcal{D}(R)$. Let P be a K-projective resolution of $\mathbf{R}\Gamma_{\mathfrak{m}}(X)$. Then $P \otimes_R \widehat{R}$ is a K-projective resolution of the $\widehat{R}$-complex $\mathbf{R}\Gamma_{\mathfrak{m}\widehat{R}}(X \otimes_R \widehat{R})$. Now, consider the following natural isomorphisms of R-complexes

$$\begin{aligned}
\Lambda^{\mathfrak{a}\widehat{R}}(P \otimes_R \widehat{R}) &= \varprojlim_n (\widehat{R}/(\mathfrak{a}\widehat{R})^n \otimes_{\widehat{R}} (P \otimes_R \widehat{R})) \\
&\simeq \varprojlim_n (R/\mathfrak{a}^n \otimes_R P) \\
&= \Lambda^{\mathfrak{a}}(P),
\end{aligned}$$

which implies that the two complexes $\mathfrak{F}_{\mathfrak{a}}(X)$ and $\mathfrak{F}_{\mathfrak{a}\widehat{R}}(X \otimes_R \widehat{R})$ are isomorphic in $\mathcal{D}(R)$. Thus, $\mathrm{fdepth}\,(\mathfrak{a}, X) = \mathrm{fdepth}\,(\mathfrak{a}\widehat{R}, X \otimes_R \widehat{R})$.

Next, it is straightforward to check that $\operatorname{depth} X = \operatorname{depth}(X \otimes_R \widehat{R})$ and $\operatorname{cd}_{\mathfrak{a}}(H(X)^\sharp) = \operatorname{cd}_{\mathfrak{a}\widehat{R}}(H(X \otimes_R \widehat{R})^\sharp)$. On the other hand, by [**10**, Proposition 3.5], one has

$$\begin{aligned}
\dim X &= \sup\{\dim H_i(X) - i \mid i \in \mathbf{Z}\} \\
&= \sup\{\dim H_i(X \otimes_R \widehat{R}) - i \mid i \in \mathbf{Z}\} \\
&= \dim(X \otimes_R \widehat{R}).
\end{aligned}$$

Therefore, without loss of generality, we may and do assume that R is complete. So, R possesses a normalized dualizing complex D. By Corollary 3.2, $\operatorname{fdepth}(\mathfrak{a}, X) = -\operatorname{cd}_{\mathfrak{a}}(X^\dagger)$. By [**9**, 16.20], it turns out that $\inf X^\dagger = \operatorname{depth} X$ and $\sup X^\dagger = \dim X$. The natural quism $(X^\dagger)^\dagger \simeq X$ yields that $\operatorname{Supp}_R X^\dagger = \operatorname{Supp}_R X$, and so

$$\bigcup_{i \in \mathbf{Z}} \operatorname{Supp}_R H_i(X^\dagger) = \operatorname{Supp}_R H(X)^\sharp.$$

Hence, Lemma 3.5 implies that

$$\operatorname{cd}_{\mathfrak{a}}(H(X)^\sharp) = \sup\{\operatorname{cd}_{\mathfrak{a}}(H_i(X^\dagger)) \mid i \in \mathbf{Z}\}.$$

In particular, there exists $\operatorname{depth} X \leq j \leq \dim X$ such that $\operatorname{cd}_{\mathfrak{a}}(H(X)^\sharp) = \operatorname{cd}_{\mathfrak{a}}(H_j(X^\dagger))$. Since $X^\dagger \in \mathcal{D}_\square^f(R)$, by [**6**, Theorem 3.2], one has

$$\operatorname{cd}_{\mathfrak{a}}(X^\dagger) = \sup\{\operatorname{cd}_{\mathfrak{a}}(H_i(X^\dagger)) - i \mid \operatorname{depth} X \leq i \leq \dim X\}.$$

Thus,

$$\begin{aligned}
\operatorname{cd}_{\mathfrak{a}}(H(X)^\sharp) - \dim X &\leq \operatorname{cd}_{\mathfrak{a}}(H_j(X^\dagger)) - j \leq \operatorname{cd}_{\mathfrak{a}}(X^\dagger) \\
&\leq \operatorname{cd}_{\mathfrak{a}}(H(X)^\sharp) - \operatorname{depth} X,
\end{aligned}$$

which is what we wish to show for the proof. $\square$

The next result provides a characterization for Cohen-Macaulay complexes.

Corollary 3.7. *Let $(R, \mathfrak{m})$ be a local ring and $X \in \mathcal{D}_\square^f(R)$ a non-homologically trivial complex. Set $H(X)^\sharp := \oplus_{i \in \mathbf{Z}} H_i(X)$. The following are equivalent*:

i) X *is Cohen-Macaulay.*

ii) $\operatorname{fdepth}(\mathfrak{a}, X) = \dim X - \operatorname{cd}_{\mathfrak{a}}(H(X)^{\sharp})$ *for all ideals* $\mathfrak{a}$ *of* R.

iii) $\operatorname{fdepth}(0, X) = \dim X - \operatorname{cd}_0(H(X)^{\sharp})$.

Proof. By Theorem 3.6, the implication i) $\Rightarrow$ ii) is obvious. Also clearly, ii) implies iii).

We have

$$\mathfrak{F}_0^i(X) = H_{-i}(\mathbf{L}\Lambda^0(\mathbf{R}\Gamma_{\mathfrak{m}}(X))) \cong H_{\mathfrak{m}}^i(X).$$

Hence, $\operatorname{fdepth}(0, X) = \operatorname{depth} X$. Also, we can see easily that $\operatorname{cd}_0(H(X)^{\sharp}) = 0$. Thus, iii) yields i). □

Remark 3.8. Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$.

i) Suppose that $\mathfrak{a}$ is generated by a regular sequence $\underline{x} := x_1, \ldots, x_r$ and $K(\underline{x})$ denotes the Koszul complex of R with respect to $\underline{x}$. Then $\operatorname{fdepth}(\mathfrak{a}, K(\underline{x})) = \operatorname{depth} R - \operatorname{ht} \mathfrak{a}$. To this end, note that $R/\mathfrak{a} \simeq K(\underline{x})$. Hence,

$$\mathfrak{F}_{\mathfrak{a}}^i(K(\underline{x})) \cong \mathfrak{F}_{\mathfrak{a}}^i(R/\mathfrak{a}) \cong H_{\mathfrak{m}}^i(R/\mathfrak{a})$$

for all $i \geq 0$, and so

$$\operatorname{fdepth}(\mathfrak{a}, K(\underline{x})) = \operatorname{depth} R/\mathfrak{a} = \operatorname{depth} R - \operatorname{ht} \mathfrak{a}.$$

ii) Let $X \in \mathcal{D}_{\square}^f(R)$ be a d-dimensional Cohen-Macaulay complex. Then

$$\mathfrak{F}_{\mathfrak{a}}^i(X) \cong H_{\mathfrak{a}}^{d-i}(H_{\mathfrak{m}}^d(X)^{\vee})^{\vee} \cong H_{d-i}^{\mathfrak{a}}(H_{\mathfrak{m}}^d(X)).$$

To see this, first of all note that, without loss of generality, we may and do assume that R is complete. So, R possesses a normalized dualizing complex. Since X is Cohen-Macaulay, one has $H_{\mathfrak{m}}^i(X) = 0$ for all $i \neq d$. Hence, $\mathbf{R}\Gamma_{\mathfrak{m}}(X) \simeq \Sigma^{-d} H_{\mathfrak{m}}^d(X)$. By local duality theorem (see [**13**, Chapter V, Theorem 6.2]), one has $(X^{\dagger})^{\vee} \simeq \mathbf{R}\Gamma_{\mathfrak{m}}(X)$. Since R is complete and $X^{\dagger} \in \mathcal{D}_{\square}^f(R)$, this implies that $X^{\dagger} \simeq \mathbf{R}\Gamma_{\mathfrak{m}}(X)^{\vee}$. Thus, by Corollary 2.3, we have

$$\mathfrak{F}_{\mathfrak{a}}(X) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{m}}(X)^{\vee})^{\vee} \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(\Sigma^d H_{\mathfrak{m}}^d(X)^{\vee})^{\vee}.$$

Hence,

$$\mathfrak{F}_{\mathfrak{a}}^{i}(X) \simeq H_{-i}(\mathbf{R}\Gamma_{\mathfrak{a}}(\Sigma^{d}H_{\mathfrak{m}}^{d}(X)^{\vee})^{\vee}) \simeq H_{i-d}(\mathbf{R}\Gamma_{\mathfrak{a}}(H_{\mathfrak{m}}^{d}(X)^{\vee}))^{\vee}$$
$$= H_{\mathfrak{a}}^{d-i}(H_{\mathfrak{m}}^{d}(X)^{\vee})^{\vee}.$$

Also, we have

$$\mathfrak{F}_{\mathfrak{a}}(X) = \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{m}}(X)) = \mathbf{L}\Lambda^{\mathfrak{a}}(\Sigma^{-d}H_{\mathfrak{m}}^{d}(X)),$$

and so

$$\mathfrak{F}_{\mathfrak{a}}^{i}(X) \simeq H_{-i}(\mathbf{L}\Lambda^{\mathfrak{a}}(\Sigma^{-d}H_{\mathfrak{m}}^{d}(X))) \simeq H_{d-i}(\mathbf{L}\Lambda^{\mathfrak{a}}(H_{\mathfrak{m}}^{d}(X)))$$
$$= H_{d-i}^{\mathfrak{a}}(H_{\mathfrak{m}}^{d}(X)).$$

iii) Suppose that R is Cohen-Macaulay and complete and ω_R is a canonical module of R. Assume that $\mathfrak{a}$ is a cohomologically complete intersection (i.e., $\mathrm{cd}_{\mathfrak{a}}(R) = \mathrm{ht}\,\mathfrak{a}$), and let $t := \dim R/\mathfrak{a}$. Then $\mathrm{id}_R(\mathfrak{F}_{\mathfrak{a}}^{t}(\omega_R)) < \infty$. To this end, first note that $\mathrm{Supp}_R\omega_R = \mathrm{Spec}\,R$, and so by [**7**, Theorem 2.2], we have

$$\dim_R(\omega_R/\mathfrak{a}\omega_R) = \dim_R R/\mathfrak{a} = \dim R - \mathrm{cd}_{\mathfrak{a}}(R)$$
$$= \dim R - \mathrm{cd}_{\mathfrak{a}}(\omega_R) = \mathrm{fdepth}\,(\mathfrak{a}, \omega_R).$$

Hence, $\mathfrak{F}_{\mathfrak{a}}^{i}(\omega_R) = 0$ for all $i \neq t$, and so

$$\mathfrak{F}_{\mathfrak{a}}^{t}(\omega_R) \simeq \Sigma^{t}\mathfrak{F}_{\mathfrak{a}}(\omega_R) \simeq \Sigma^{t}\mathbf{R}\mathrm{Hom}_R(\check{C}(\underline{\mathfrak{a}}), \omega_R \otimes_R^{\mathbf{L}} \check{C}(\underline{\mathfrak{m}})).$$

Now, since $\mathrm{id}_R(\omega_R) < \infty$, the conclusion follows by [**5**, Proposition 2.4].

iv) The notion of Frobenius depth was defined by Hartshorne and Speiser in [**14**, page 60]. Suppose R is regular of prime characteristic. By [**16**, Theorem 4.3], it follows that Frobenius depth of $R/\mathfrak{a}$ is equal to $\dim R - \mathrm{cd}_{\mathfrak{a}}(R)$. Having [**18**, Lemma 4.8 d)] and Corollary 2.4 in mind, we conclude that Frobenius depth of $R/\mathfrak{a}$ is equal to $\mathrm{fdepth}\,(\mathfrak{a}, R)$.

v) Suppose R is a one-dimensional domain, and set $M := R/\mathfrak{m} \oplus R$. Clearly M is not Cohen-Macaulay, while $\mathrm{fdepth}\,(\mathfrak{b}, M) = \dim M - \mathrm{cd}_{\mathfrak{b}}(H(M)^{\sharp})$ for all non-zero ideals $\mathfrak{b}$ of R.

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