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MINIMAL WAVE SPEED OF A NONLOCAL DIFFUSIVE EPIDEMIC MODEL WITH TEMPORAL DELAY

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ABSTRACT. This note is concerned with a nonlocal version of the man-environment-man epidemic model in which the dispersion of infectious agents is assumed to follow a nonlocal diffusion law modeled by a convolution operator. The purpose of this note is to show that the minimal wave speeds of properly re-scaled nonlocal diffusion equations can approximate the corresponding one of the classical diffusion equation for this model. As a byproduct, our results indicate that the temporal delay in an epidemic model can reduce the speed of epidemic spread while the nonlocal effect can increase the speed.

1. Introduction. The spatial spread of epidemics is an important subject in mathematical epidemiology, and much attention has been paid to the traveling wave fronts in various diffusive epidemic models, especially to the minimal wave speeds due to the significant sense in epidemic phenomena, see the monographs of Murray [**8**] and Rass and Radcliffe [**10**], a survey paper by Ruan [**11**] and the references cited therein.

In [**2**], the following man-environment-man epidemic model was proposed to model the cholera epidemic which spread in the European Mediterranean regions in 1973

$$\begin{cases} \frac{\partial}{\partial t}u_1(x,t) = d\frac{\partial^2}{\partial x^2}u_1(x,t) - a_{11}u_1(x,t) + a_{12}u_2(x,t), \\ \frac{\partial}{\partial t}u_2(x,t) = -a_{22}u_2(x,t) + g(u_1(x,t)), \end{cases}$$

where d, a_{11}, a_{12} and a_{22} are positive constants, $u_1(x,t)$ denotes the spatial density of infectious agent at a point x in the habitat at time

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$t > 0$ and $u_2(x,t)$ denotes the spatial density of the infective human population at time $t > 0$. Hence, the term $-a_{11}u_1(x,t)$ describes the natural growth rate of the bacterial population, while the term $a_{12}u_2(x,t)$ is the contribution of infective humans to the growth rate of bacteria. In the second equation, the term $-a_{22}u_2(x,t)$ describes the natural damping of the infective population due to the finite mean duration of the infectiousness of humans. It was assumed that the bacteria diffuse randomly in the habitat due to the particular habits in the regions where these kinds of epidemics usually spread. As far as the human population is concerned, the random diffusion may actually be neglected with respect to that of bacteria. The last term $g(u_1(x,t))$ is the infection rate of humans under the assumption that the total susceptible human population is constant during the evolution of the epidemic. Mathematically it suffices to study the following dimensionless system

$$\begin{cases} \frac{\partial}{\partial t}u_1(x,t) = d\frac{\partial^2}{\partial x^2}u_1(x,t) - u_1(x,t) + \alpha u_2(x,t), \\ \frac{\partial}{\partial t}u_2(x,t) = -\beta u_2(x,t) + g(u_1(x,t)), \end{cases} \tag{1.1}$$

where

$$\alpha = \frac{a_{12}}{a_{11}^2}, \qquad \beta = \frac{a_{22}}{a_{11}}.$$

In view of epidemiology, there should be $\tau > 0$ period of latency, after which the infected human population becomes infectious. Taking the latent period into account, (1.1) becomes the reaction-diffusion system with delay

$$\begin{cases} \frac{\partial}{\partial t}u_1(x,t) = d\frac{\partial^2}{\partial x^2}u_1(x,t) - u_1(x,t) + \alpha u_2(x,t), \\ \frac{\partial}{\partial t}u_2(x,t) = -\beta u_2(x,t) + g(u_1(x,t-\tau)). \end{cases} \tag{1.2}$$

In [**15**], Zhao and Wang proved the existence of the minimal wave speed in the man-environment-man epidemic model (1.1). Subsection 4.2 in [**13**] handled both (1.1) and (1.2), and the effect of delays on the minimal wave speed was discussed in [**14**].

It is noted that, in the above system, diffusion of the infectious agent is described by the classical Laplacian operator. However, as mentioned in Murray [**8**] and Fife [**6**], the Laplacian operator is not sufficiently precise in modeling the spatial diffusion of individuals in some spatial-temporal evolutionary processes. In this note, we consider the nonlocal version of (1.2), namely,

$$\begin{cases} \frac{\partial}{\partial t}u_1(x,t) = \int_{-\infty}^{+\infty} J(x-y)[u_1(y,t)-u_1(x,t)]\,dy \\ \quad -u_1(x,t)+\alpha u_2(x,t), \\ \frac{\partial}{\partial t}u_2(x,t) = -\beta u_2(x,t)+g(u_1(x,t-\tau)), \end{cases} \tag{1.3}$$

where the kernel $J : \mathbf{R} \to \mathbf{R}$ is a nonnegative, even, continuous and compactly supported function. As stated in [**6**], $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, and the integral $\int_{-\infty}^{+\infty} J(x-y)[u_1(y,t)-u_1(x,t)]\,dy$ takes into account the infectious agent arriving or leaving position x from or to the other places.

It is well known that traveling wave fronts also play an important role in nonlocal reaction diffusion equations since they can describe phenomena in many practical fields, see e.g., Bates et al. [**1**] for a model in phase transition, Coville and Dupaigne [**5**] for a population model science. For the recent work concerning the traveling wave fronts of nonlocal reaction diffusion equations, we refer to [**3–5, 9**] and the references therein.

Since the classical Laplacian operator is the approximation of the nonlocal convolution operator (see [**5, 8, 9**]), it is natural to ask whether the minimal wave speed of (1.2) can be approximated by that of properly re-scaled nonlocal diffusion equations of the form of (1.3). In this note we give a positive answer. More precisely, for given J, we consider the re-scaled kernel $J_\varepsilon(x) = (1/\varepsilon^3)J(x/\varepsilon)$ and the minimal wave speed c_ε^* of

$$\begin{cases} \frac{\partial}{\partial t}u_1(x,t) = \int_{-\infty}^{+\infty} J_\varepsilon(x-y)(u_1(y,t)-u_1(x,t))\,dy \\ \quad -u_1(x,t)+\alpha u_2(x,t), \\ \frac{\partial}{\partial t}u_2(x,t) = -\beta u_2(x,t)+g(u_1(x,t-\tau)), \end{cases} \tag{1.4}$$

and show that $c_\varepsilon^* \to c^*$ as $\varepsilon \to 0$, where c^* is the minimal wave speed of (1.2) with $d = \int_0^{+\infty} J(z)z^2\,dz$.

The purpose of this note is to show that the minimal wave speed of (1.4) approximates that of (1.2) as $\varepsilon \to 0$. The note is organized as follows. In Section 2, we transform the existence of traveling wave fronts of (1.3) to the existence of the fixed point of an operator, and further reduce the problem to the construction of a pair of super and sub-solutions of an integral-differential equation with the wave speed c being a parameter. In Section 3, inspired by the super and sub-solutions introduced in [**5, 14**], we construct a pair of appropriate super and sub-solutions by some delicate analysis of the characteristic equation $P_{c,\tau}(\lambda) = 0$ of the linearized problem at zero point, and prove the existence of traveling wave fronts and the minimal wave speed of (1.3). Finally, based on the previous results, we can prove the convergence result. As a byproduct, we find that the temporal delay in the epidemic model can reduce the speed of epidemic spread while the nonlocal effect can increase the speed.

2. Preliminaries. The objective of this section is to reduce the existence of traveling wave solutions of system (1.3) to that of a pair of super- and sub-solutions of a differential equation with infinite delay.

Throughout the note, we also make the following assumptions on the function g:

(G_1) $g \in C^1(\mathbf{R}_+, \mathbf{R}_+)$, $g(0) = 0$, $g''(0)$ exists, and $g'(z) \geq 0$ for $z \geq 0$. where $\mathbf{R}_+ = [0, +\infty)$;

(G_2) $(\alpha g'(0))/\beta > 1$, and there is a $\overline{z} > 0$ such that $g(\overline{z})/\overline{z} \leq (\beta/\alpha)$;

(G_3) $g(z)/z$ is a decreasing function on $\mathbf{R}_+$. Hence, $g(sz) > sg(z)$ for all $z > 0, s \in (0, 1)$.

It is easy to see that the reaction system of system (1.3)

$$\begin{cases} u_1' = -u_1 + \alpha u_2, \\ u_2' = -\beta u_2 + g(u_1) \end{cases} \tag{2.1}$$

admits only the disease-free equilibrium $(0, 0)$ and an endemic equilibrium (u_1^*, u_2^*), and (u_1^*, u_2^*) is globally asymptotically stable. We are interested in the traveling wave fronts of (1.3) connecting these two equilibria.

Let $(u_1(x,t), u_2(x,t)) = (U_1(z), U_2(z))$, $z = x + ct$, be the traveling wave solution of (1.3) with positive wave speed c. Substituting this special solution into (1.3), we then obtain

$$
\begin{cases} cU_1'(z) = \int_{-\infty}^{+\infty} J(y-z)[U_1(y) - U_1(z)]\,dy \\ \quad -U_1(z) + \alpha U_2(z), \\ cU_2'(z) = -\beta U_2(z) + g(U_1(z - c\tau)). \end{cases} \tag{2.2}
$$

Since we are interested in the traveling waves connecting $(0,0)$ and (u_1^*, u_2^*), we impose the following boundary condition on (U_1, U_2)

$$
\begin{aligned} U_i(-\infty) &:= \lim_{z\to-\infty} U_i(z) = 0, \\ U_i(+\infty) &:= \lim_{z\to+\infty} U_i(z) = u_i^*, \quad 1 \le i \le 2. \end{aligned} \tag{2.3}
$$

By the second equation of (2.2), for any $z_0 \in \mathbf{R} = (-\infty, +\infty)$, we have

$$
U_2(z) = e^{-\beta(z-z_0)/2} U_2(z_0) + \frac{1}{c}\int_{z_0}^{t} e^{-\beta(z-s)/2} g(U_1(s-c\tau))\,ds.
$$

Since $U_2(z)$ and $g(U_1(z))$ are bounded functions on $\mathbf{R}$, by taking $z_0 \to -\infty$, we obtain

$$
U_2(z) = \frac{1}{c}\int_{-\infty}^{z} e^{-\beta(z-s)/c} g(U_1(s-c\tau))\,ds. \tag{2.4}
$$

By substituting (2.4) into the first equation of (2.2), we get

$$
\begin{aligned} cU_1'(z) = &\int_{-\infty}^{+\infty} J(y) U_1(y+z)\,dy - (1+a)U_1(z) \\ &+ \frac{\alpha}{c}\int_{-\infty}^{z} e^{-\beta(z-s)/c} g(U_1(s-c\tau))\,ds, \end{aligned} \tag{2.5}
$$

where $a = \int_{-\infty}^{+\infty} J(y)\,dy$.

If $U_1(z)$ is a monotone increasing solution of (2.5) with

$$
U_1(-\infty) = 0, \qquad U_1(+\infty) = u_1^*, \tag{2.6}
$$

and $U_2(z)$ is defined by (2.4), then $U_2(z)$ is also a monotone increasing function with

$$
U_2(-\infty) = 0, \qquad U_2(+\infty) = u_2^*, \tag{2.7}
$$

and thus $(U_1(z), U_2(z))$ is a solution of (2.2)–(2.3).

Consequently, it is sufficient to consider monotonic solutions of equation (2.5) subject to (2.6).

Let

$$X = \begin{cases} \phi(z) \in C(\mathbf{R}, \mathbf{R}_+) \mid \text{(i) } \phi(z) \text{ is a continuous function in } \mathbf{R}; \\ \text{(ii) } 0 \leq \phi(z) \leq u_1^*. \end{cases}$$

So X is a bounded and closed convex subset of $C(\mathbf{R}, \mathbf{R}_+)$ with the usual supremum norm. Define the operator $H : X \subset C(\mathbf{R}, \mathbf{R}_+) \to C(\mathbf{R}, \mathbf{R}_+)$ by

$$H(\phi)(z) = \int_{-\infty}^{+\infty} J(y)\phi(y+z)\,dy + \frac{\alpha}{c}\int_{-\infty}^{z} e^{-\frac{\beta}{c}(z-s)} g(\phi(s-c\tau))\,ds.$$

From the definition of H and the monotonicity of g on $(-\infty, +\infty)$, H has the following properties:

(H_1) If $\phi \in X$ is monotone increasing on $(-\infty, +\infty)$, then $H(\phi(z))$ is also increasing on $(-\infty, +\infty)$;

(H_2) If $\phi \in X$, then $0 \leq H(\phi(z)) \leq (1+a)u_1^*$;

(H_3) If $0 \leq \psi(z) \leq \phi(z) \leq u_1^*$, then $H(\psi(z)) \leq H(\phi(z))$.

Thus, (2.5) can be rewritten as

$$c\phi'(z) + (1+a)\phi(z) = H(\phi(z)). \tag{2.8}$$

We define the continuous map $S : X \to X$ as follows

$$S(\phi)(z) = \frac{1}{c}\int_{-\infty}^{z} e^{-[(1+a)/c](z-s)} H(\phi(s))\,ds. \tag{2.9}$$

Direct calculations show that the first order derivative of $S(\phi(z))$ with respect to z is bounded on $(-\infty, +\infty)$, and any fixed point of S in X is a solution of (2.5). By the monotonicity of $g(z)$ on $(-\infty, +\infty)$ and the properties of the operator H, S has the following properties:

(S_1) If $\phi \in X$ is increasing on $(-\infty, +\infty)$, so is $S(\phi)$;

(S_2) If $0 \leq \psi(z) \leq \phi(z) \leq u_1^*$, then $0 \leq S(\psi(z)) \leq S(\phi(z)) \leq u_1^*$.

Definition 2.1. A function $\phi \in X$ is called a super solution of (2.5) if $S(\phi)(z) \leq \phi(z)$, for any $z \in (-\infty, +\infty)$.

Note that if $\phi \in X$ is a continuously differentiable on $\mathbf{R}$ except finite many point z_i with $\phi'(z_i+) \leq \phi'(z_i-)$, $1 \leq i \leq m$, and satisfies

$$\begin{aligned} cU_1'(z) \geq & \int_{-\infty}^{+\infty} J(y)U_1(y+z)\,dy - (1+a)U_1(z) \\ & + \frac{\alpha}{c}\int_{-\infty}^{z} e^{-\beta(z-s)/c} g(U_1(s-c\tau))\,ds. \end{aligned} \tag{2.10}$$

By the definition of the operator H and S, ϕ is a super solution of (2.5). A similar note applies to sub-solutions of (2.5) if we reverse the aforementioned inequalities.

Based on the properties of S, we can establish the result guaranteeing the existence of traveling wave solutions of (2.5)–(2.6).

Theorem 2.1. *Suppose that* (2.5) *admits a super-solution* $\overline{\rho}(z)$ *and a sub-solution* $\underline{\rho}(z)$ *such that*

(1) $\overline{\rho}(z)$ *is monotone increasing on* $(-\infty, +\infty)$*, and* $\overline{\rho}(-\infty) = 0, \overline{\rho}(+\infty) = u_1^*$;

(2) $\underline{\rho}(z) \not\equiv 0$ *and* $\underline{\rho}(z) \leq \overline{\rho}(z),\ z \in (-\infty, +\infty)$.

Then (2.5)–(2.6) *has a monotone increasing solution on* $(-\infty, +\infty)$.

Proof. Let $\phi_n(z) = S^n(\overline{\rho}(z))$, $n = 1, 2, \ldots$. By Definition 2.1 and the property (S_2), it follows that

$$\underline{\rho}(z) \leq \phi_n(z) \leq \phi_{n-1}(z) \leq \overline{\rho}(z), \quad \text{for all } z \in \mathbf{R},\ n \geq 1.$$

In particular, for each $z \in \mathbf{R}$, the sequence $\{\phi_n(z)\}_{n=1}^{+\infty}$ is decreasing; thus, $\phi(z) = \lim_{n\to+\infty} \phi_n(z)$ exists and $\underline{\rho}(z) \leq \phi(z) \leq \overline{\rho}(z)$.

By property (S_1), we see that $\phi_n(z)$ is increasing in $z \in \mathbf{R}$, and thus $\phi(z)$ is increasing which satisfies $0 = \phi(-\infty) \leq \phi(+\infty) \leq u_1^*$.

For each fixed $z \in \mathbf{R}$, since $\phi_n(z) = S(\phi_{n-1}(z))$, by Lebesgue's convergence theorem, we obtain $\phi(z) = S(\phi(z))$. Hence, ϕ is a fixed point of S. This means that $\phi(z)$ is a monotone solution of (2.5). So we are left to show $\phi(+\infty) = u_1^*$.

Note that $\underline{\rho}(z) \le \phi(z) \le \phi(+\infty)$, for all $z \in \mathbf{R}$ and $\underline{\rho}(z) \not\equiv 0$, we have $0 < \phi(+\infty) \le u_1^*$. In view of (2.5), it is easy to see that $\phi(+\infty) = (\alpha/\beta)g(\phi(+\infty))$. Thus, the uniqueness of positive equilibrium of (1.3) implies that $\phi(+\infty) = u_1^*$. ◻

3. Existence of minimal wave speed. In this section, we will prove the existence of monotone waves and the minimal wave speed of system (1.3). In order to construct the appropriate super- and sub-solutions to (2.5), we linearize the function $g(U_1)$ at $U_1 = 0$ in (2.5) to obtain

$$\begin{aligned} cU_1'(z) &= \int_{-\infty}^{+\infty} J(y)U_1(y+z)\,dy - (1+a)U_1(z) \\ &\quad + \frac{\alpha g'(0)}{c}\int_{-\infty}^{z-c\tau} e^{-\beta(z-s-c\tau)/2}U_1(s)\,ds. \end{aligned} \tag{3.1}$$

Since J is a nonnegative, even, continuous and compactly supported function, by substituting $U_1(z) = e^{\lambda z}$ into (3.1), we get a characteristic equation of (2.5)

$$P_{c,\tau}(\lambda) := \int_{-\infty}^{+\infty} J(y)e^{\lambda y}dy - c\lambda - (1+a) + \frac{\alpha g'(0)}{c\lambda+\beta}e^{-c\tau\lambda} = 0, \tag{3.2}$$

where $P_{c,\tau}(\lambda)$ is considered to be a function of λ and c and τ are relatively fixed parameters.

Through direct calculations and some mathematical analysis, we get the following facts:

(P_1) For $c > 0$ and $\lambda > 0$, $P_{c,\tau}(\lambda)$ is a decreasing function with respect to $\tau \in [0,+\infty)$.

(P_2) For $\tau \ge 0$ and $\lambda \ge 0$, $P_{c,\tau}(\lambda)$ is a strictly decreasing function with respect to $c \in [0,+\infty)$, $P_{0,\tau}(\lambda) > 0$, and $\lim_{c\to+\infty} P_{c,\tau}(\lambda) = -\infty$.

(P_3) If $c \ge 0, \tau \ge 0$, then $P_{c,\tau}(\lambda)$ is a strictly convex function with respect to $\lambda \in (0,+\infty)$, $P_{c,\tau}(0) > 0$.

(P_4) If $c \ge 0$, $\tau \ge 0$, then $[\partial P_{c,\tau}(\lambda)]/(\partial\lambda)$ is a strictly increasing function with respect to $\lambda \in (0,+\infty)$, and there exists a unique

$\lambda^*_{c,\tau} > 0$, such that

$$\frac{\partial P_{c,\tau}(\lambda)}{\partial \lambda} \begin{cases} < 0 & \text{for } 0 < \lambda < \lambda^*_{c,\tau}, \\ = 0 & \text{for } \lambda = \lambda^*_{c,\tau}, \\ > 0 & \text{for } \lambda > \lambda^*_{c,\tau}. \end{cases}$$

(P_5) $P_{c,\tau}(\lambda)$ is a continuous function with respect to $(c, \tau, \lambda) \in \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_+$.

Consequently, from the facts (P_1)–(P_5), we obtain the following lemma which plays an important role in the construction of an order sub-super solutions of (2.5)–(2.6).

Lemma 3.1. *For each fixed* $\tau \in [0, +\infty)$, *there exists a unique* $c^*_\tau > 0$ *such that*

(i) *If* $c = c^*_\tau$, *then* $P_{c^*_\tau,\tau}(\lambda^*_{c^*_\tau,\tau}) = 0$, $\partial/(\partial\lambda) P_{c^*_\tau,\tau}(\lambda^*_{c^*_\tau,\tau}) = 0$.

(ii) *If* $0 \le c < c^*_\tau$, *then* $P_{c,\tau}(\lambda) > 0$ *for* $\lambda \ge 0$.

(iii) *If* $c > c^*_\tau$, *then* $P_{c,\tau}(\lambda) = 0$ *has just two positive real roots* λ_1, λ_2 *such that*

$$P_{c,\tau}(\lambda) \begin{cases} > 0 & \text{for } 0 < \lambda < \lambda_1, \\ < 0 & \text{for } \lambda_1 < \lambda < \lambda_2, \\ > 0 & \text{for } \lambda > \lambda_2. \end{cases}$$

Now we are in a position to prove the existence of traveling wave fronts of (1.3) by applying Theorem 2.1.

Theorem 3.1. *Assume that* (G_1)–(G_3) *hold,* $\tau \ge 0$, *and let* c^*_τ *be defined as in Lemma* 3.1. *Then* (1.3) *has a monotone traveling wave connecting* $(0, 0)$ *and* (u^*_1, u^*_2) *with speed* c *if* $c \ge c^*_\tau$, *and no such wave if* $0 < c < c^*_\tau$.

Proof. In the case of $c > c^*_\tau$, according to Lemma 3.1, the characteristic equation $P_{c,\tau}(\lambda) = 0$ has two positive roots $\lambda_1 < \lambda_2$. Now we define

$$\overline{\rho}(z) = \min\{u^*_1 e^{\lambda_1 z}, u^*_1\}$$

and verify that $\overline{\rho}(z)$ is a super-solution of (2.5).

If $z < 0$, then $\overline{\rho}(z) = u_1^* e^{\lambda_1 z}$. We obtain that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} J(y)\overline{\rho}(y+z)\,dy - c\overline{\rho}'(z) - (1+a)\overline{\rho}(z) \\
&\qquad + \frac{\alpha}{c}\int_{-\infty}^{z} e^{-\beta(z-s)/c} g(\overline{\rho}(s-c\tau))\,ds \\
&\quad \le \int_{-\infty}^{\infty} J(y)u_1^* e^{\lambda_1(y+z)}dy - (c\lambda_1 + 1 + a)u_1^* e^{\lambda_1 z} \\
&\qquad + \frac{\alpha g'(0)}{c}\int_{-\infty}^{z} e^{-\beta(z-s)/c} u_1^* e^{\lambda_1(s-c\tau)}ds \\
&\quad = \left(\int_{-\infty}^{\infty} J(y)e^{\lambda_1 y}dy - c\lambda_1 - a - 1\right)u_1^* e^{\lambda_1 z} \\
&\qquad + \frac{\alpha g'(0)e^{-c\tau\lambda_1}}{c\lambda_1 + \beta}u_1^* e^{\lambda_1 z} = 0,
\end{aligned} \tag{3.3}$$

where the first inequality is due to the fact that $g(u) \le g'(0)u$, which is implied by assumption (G_3).

If $t > 0$, then $\overline{\rho}(t) = u_1^*$. By (G_1) and (G_2), we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} J(y)\overline{\rho}(y+z)\,dy - c\overline{\rho}'(z) - (1+a)\overline{\rho}(z) \\
&\qquad + \frac{\alpha}{c}\int_{-\infty}^{z} e^{-\beta(z-s)/c} g(\overline{\rho}(s-c\tau))\,ds \\
&\quad \le \int_{-\infty}^{\infty} J(y)u_1^* dy - (1+a)u_1^* + \frac{\alpha}{c}\int_{-\infty}^{z} e^{-\beta(z-s)/c} g(u_1^*)\,ds \\
&\quad = -u_1^* + \frac{\alpha g(u_1^*)}{\beta} = 0.
\end{aligned} \tag{3.4}$$

Hence, by Definition 2.1, $\overline{\rho}(z)$ is a super-solution of (2.5).

By assumption (G_1), it follows that there exist $k > 0$ and $\delta \in (0, u_1^*)$ such that

$$g(z) \ge g'(0)z - kz^2, \quad \text{for } z \in [0, \delta]. \tag{3.5}$$

Now we fix an $\varepsilon \in (0, \lambda_1)$ such that $\lambda_1 + \varepsilon < \lambda_2$, and define

$$\underline{\rho}(z) = \max\{0, \delta(1 - Me^{\varepsilon z})e^{\lambda_1 z}\},$$

where

$$M \geq \max\left\{1, -\frac{\alpha k(u_1^*)^2}{\delta[c(\lambda_1+\varepsilon)+\beta]P_{c,\tau}(\lambda_1+\varepsilon)}\right\}.$$

Since $\delta < u_1^*$ and $t_0 := -(\ln M)/\varepsilon \leq 0$, we obtain that

$$0 \leq \underline{\rho}(z) \leq \overline{\rho}(z), \qquad \underline{\rho}^2(z) \leq (u_1^*)^2 e^{(\lambda_1+\varepsilon)z}. \tag{3.6}$$

If $t \geq z_0$, then $\underline{\rho}(z) = 0$. It follows that

$$\begin{aligned}
&\int_{-\infty}^{\infty} J(y)\underline{\rho}(y+z)\,dy - c\underline{\rho}'(z) - (1+a)\underline{\rho}(z) \\
&\quad + \frac{\alpha}{c}\int_{-\infty}^{z-c\tau} e^{-\beta(z-c\tau-s)/c} g(\underline{\rho}(s))\,ds \\
&= \int_{-\infty}^{\infty} J(y)\underline{\rho}(y+z)\,dy + \frac{\alpha}{c}\int_{-\infty}^{z-c\tau} e^{-\beta(z-c\tau-s)/c} g(\underline{\rho}(s))\,ds \geq 0.
\end{aligned} \tag{3.7}$$

If $t < z_0$, then $\underline{\rho}(z) = \delta e^{\lambda_1 z} - \delta M e^{(\lambda_1+\varepsilon)z}$. It follows that

$$\begin{aligned}
&\int_{-\infty}^{\infty} J(y)\underline{\rho}(y+z)\,dy - c\underline{\rho}'(z) - (1+a)\underline{\rho}(z) \\
&\quad + \frac{\alpha}{c}\int_{-\infty}^{z-c\tau} e^{-\beta(z-c\tau-s)/c} g(\underline{\rho}(s))\,ds \\
&\geq \int_{-\infty}^{\infty} J(y)(\delta e^{\lambda_1(y+z)} - \delta M e^{(\lambda_1+\epsilon)(y+z)})\,dy \\
&\quad - c(\delta e^{\lambda_1 z} - \delta M e^{(\lambda_1+\varepsilon)z})' \\
&\quad - (1+a)(\delta e^{\lambda_1 z} - \delta M e^{(\lambda_1+\varepsilon)z}) \\
&\quad + \frac{\alpha}{c}\int_{-\infty}^{z-c\tau} e^{-\beta(z-c\tau-s)/c} g(\underline{\rho}(s))\,ds \\
&= \delta e^{\lambda_1 z} P_{c,\tau}(\lambda_1) - \delta M e^{(\lambda_1+\epsilon)z} P_{c,\tau}(\lambda_1+\varepsilon)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\alpha g'(0)e^{-c\tau\lambda_1}}{c\lambda_1+\beta}\delta e^{\lambda_1 z} + \frac{\alpha g'(0)e^{-c\tau(\lambda_1+\varepsilon)}}{c(\lambda_1+\varepsilon)+\beta}\delta M e^{(\lambda_1+\varepsilon)z} \\
&+ \frac{\alpha}{c}\int_{-\infty}^{z-c\tau} e^{-\beta(z-c\tau-s)/c} g(\underline{\rho}(s))\,ds.
\end{aligned}
$$

By (G_1), (3.5) and (3.6), we obtain that

$$
\begin{aligned}
&\int_{-\infty}^{z-c\tau} e^{-\beta(z-c\tau-s)/c} g(\underline{\rho}(s))\,ds \\
&\qquad \geq \int_{-\infty}^{z-c\tau} e^{-\beta(z-c\tau-s)/c}(g'(0)\underline{\rho}(s) - k\underline{\rho}(s)^2)\,ds \\
&\qquad \geq \int_{-\infty}^{z-c\tau} e^{-\beta(z-c\tau-s)/c}(g'(0)\underline{\rho}(s) - k(u_1^*)^2 e^{(\lambda_1+\varepsilon)s})\,ds \\
&\qquad = \int_{-\infty}^{z-c\tau} e^{-\beta(z-c\tau-s)/c}[g'(0)(\delta e^{\lambda_1 s} - \delta M e^{(\lambda_1+\varepsilon)s}) \\
&\qquad\qquad\qquad - k(u_1^*)^2 e^{(\lambda_1+\varepsilon)s}]\,ds.
\end{aligned}
\tag{3.8}
$$

Noticing that (3.8) and $P_{c,\tau}(\lambda_1)=0$, we get

$$
\begin{aligned}
&\int_{-\infty}^{\infty} J(y)\underline{\rho}(y+z)\,dy - c\underline{\rho}'(z) - (1+a)\underline{\rho}(z) \\
&\quad + \frac{\alpha}{c}\int_{-\infty}^{z-c\tau} e^{-\beta(z-c\tau-s)/c} g(\underline{\rho}(s))\,ds \\
&\geq -\delta M e^{(\lambda_1+\varepsilon)z} P_{c,\tau}(\lambda_1+\varepsilon) - \frac{\alpha g'(0)e^{-c\tau\lambda_1}}{c\lambda_1+\beta}\delta e^{\lambda_1 z} \\
&\quad + \frac{\alpha g'(0)e^{-c\tau(\lambda_1+\varepsilon)}}{c(\lambda_1+\varepsilon)+\beta}\delta M e^{(\lambda_1+\varepsilon)z} \\
&\quad + \frac{\alpha}{c}\int_{-\infty}^{z-c\tau} e^{-\beta(z-c\tau-s)/c}[g'(0)(\delta e^{\lambda_1 s} - \delta M e^{(\lambda_1+\varepsilon)s}) \\
&\qquad\qquad\qquad\qquad - k(u_1^*)^2 e^{(\lambda_1+\varepsilon)s}]\,ds \\
&= \Big(-\delta M P_{c,\tau}(\lambda_1+\varepsilon) - \frac{\alpha k(u_1^*)^2}{c(\lambda_1+\varepsilon)+\beta} e^{-c\tau(\lambda_1+\varepsilon)} \Big) e^{(\lambda_1+\varepsilon)z} \\
&\geq \Big(-\delta M P_{c,\tau}(\lambda_1+\varepsilon) - \frac{\alpha k(u_1^*)^2}{c(\lambda_1+\varepsilon)+\beta} \Big) e^{(\lambda_1+\varepsilon)z} \geq 0.
\end{aligned}
$$

Hence, by Definition 2.1, $\underline{\rho}(z)$ is a sub-solution of (2.5). By Theorem 2.1, the existence of traveling waves of (1.3) with speed c follows.

In the case of $c = c^*$, we choose a sequence $\{c_n\}$ such that $c_n \in (c_\tau^*, c_\tau^*+1]$ and $\lim_{n\to+\infty} c_n = c^*$. Let $U_{1n}(z)$ be the monotone solution of (2.5)–(2.6) with $c = c_n$. Since each $U_{1n}(z+h)$, $h \in \mathbf{R}$, is also a solution, we may assume that $U_{1n}(0) = u_1^*/2$. Clearly, $|U_{1n}(z)| \le u_1^*$, and $U_{1n}(z)$ satisfies

(3.9)

$$\begin{aligned} c_nU_{1n}'(z) = &\int_{-\infty}^{\infty} J(y)U_{1n}(y+z)\,dy - (1+a)U_{1n}(z) \\ &+ \frac{\alpha}{c_n}\int_{-\infty}^{z-c_n\tau} e^{-\beta/c_n(z-c_n\tau-s)}g(U_{1n}(s))\,ds, \quad \text{for all } z \in \mathbf{R}. \end{aligned}$$

By assumption (G_1), we obtain that

(3.10)

$$\begin{aligned} c_nU_{1n}'(z) &= \int_{-\infty}^{+\infty} J(y)U_{1n}(y+z)\,dy - (1+a)U_{1n}(z) \\ &\quad + \frac{\alpha}{c_n}\int_{-\infty}^{z-c_n\tau} e^{-\beta(z-c_n\tau-s)/c_n}g(U_{1n}(s))\,ds \\ &< \int_{-\infty}^{+\infty} J(y)u_{1n}^*dy + \frac{\alpha}{c_n}\int_{-\infty}^{z-c_n\tau} e^{-\beta(z-c_n\tau-s)/c_n}g(u_1^*)\,ds \\ &= (1+a)u_1^*, \end{aligned}$$

and thus $|U_{1n}'(z)| \le [(1+a)u_1^*]/c_\tau^*$.

Differentiating both sides of (3.9) with respect to z, we then get

(3.11)

$$\begin{aligned} c_nU_{1n}''(z) &= \int_{-\infty}^{+\infty} J(y)U_{1n}'(y+z)\,dy - (1+a)U_{1n}'(z) \\ &\quad + \frac{\alpha}{c_n}\bigg(g(U_{1n}(z-c_n\tau)) \\ &\qquad - \frac{\beta}{c_n}\int_{-\infty}^{z-c_n\tau} e^{-\beta(z-c_n\tau-s)/c_n}g(U_{1n}(s))\,ds\bigg) \\ &\le \int_{-\infty}^{+\infty} J(y)\frac{(1+a)u_1^*}{c_\tau^*}\,dy + \frac{(1+a)^2u_1^*}{c_\tau^*} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha}{c_n}g(U_{1n}(z-c_n\tau))\\
&\le \frac{(1+a)au_1^*}{c_\tau^*}+\frac{(1+a)^2u_1^*}{c_\tau^*}+\frac{\alpha g(u_1^*)}{c_\tau^*}\\
&\le \frac{\beta+2(1+a)^2}{c_\tau^*}u_1^*
\end{aligned}
$$

and

(3.12)

$$
\begin{aligned}
c_nU_{1n}''(z) &= \int_{-\infty}^{+\infty} J(y)U_{1n}'(y+z)\,dy-(1+a)U_{1n}'(z)\\
&\quad+\frac{\alpha}{c_n}\biggl(g(U_{1n}(z-c_n\tau))\\
&\qquad\qquad -\frac{\beta}{c_n}\int_{-\infty}^{z-c_n\tau} e^{-\beta(z-c_n\tau-s)/c_n}g(U_{1n}(s))\,ds\biggr)\\
&\ge -\frac{4u_1^*}{c_\tau^*}-\frac{\alpha\beta}{c_n^2}\int_{-\infty}^{z-c_n\tau} e^{-\beta(z-c_n\tau-s)/c_n}g(u_1^*)\,ds\\
&\ge -\frac{(1+a)^2u_1^*}{c_\tau^*}-\frac{\alpha g(u_1^*)}{c_\tau^*}\ge -\frac{\beta+2(1+a)^2}{c_\tau^*}u_1^*.
\end{aligned}
$$

Then $|U_{1n}''(z)|\le[\beta+2(1+a)^2/c_{\tau^*}^2]u_1^*$.

Consequently, $\{U_{1n}(z)\}_{n=1}^{+\infty}$ and $\{U_{1n}'(z)\}_{n=1}^{+\infty}$ are equicontinuous and uniformly bounded sequences of function on $\mathbf{R}$. By Ascoli's theorem and a nested subsequence argument, it follows that there exists a subsequence of $\{c_n\}$, still denoted by $\{c_n\}$, such that $\lim_{n\to\infty} c_n=c^*$, $\{U_{1n}(z)\}_{n=1}^{+\infty}$ and $\{U_{1n}'(z)\}_{n=1}^{+\infty}$ converge uniformly on every bounded interval, and hence pointwise on $\mathbf{R}$ to $V(z)$ and $V_1(z)$, respectively. Then $V(z)$ is differentiable, and $V'(z)=V_1(z)$, for all $z\in\mathbf{R}$. Letting $n\to\infty$ in (3.9) and using the dominated convergence theorem and (G_1), we obtain

$$
\begin{aligned}
c_\tau^*V'(z) &= \int_{-\infty}^{+\infty} J(y)V(y+z)\,dy-(1+a)V(z)\\
&\quad+\frac{\alpha}{c_\tau^*}\int_{-\infty}^{z-c_\tau^*\tau} e^{-\beta(z-c_\tau^*\tau-s)/c_\tau^*}g(V(s))\,ds.
\end{aligned}
\tag{3.13}
$$

Then $V(z)$ is a solution of (2.5) with $c=c_\tau^*$. Clearly, $V(z)$ is monotone increasing on $\mathbf{R}$ and $V(0)=u_1^*/2$. Since both $V(-\infty)$ and

$V(+\infty)$ exist, $V'(\pm\infty) = 0$ holds. Letting $z \to -\infty$ and $z \to +\infty$ in (3.13), respectively, we then obtain $V(-\infty) = 0$ and $V(+\infty) = (\alpha/\beta)g(V(+\infty))$. Since $u_1^*/2 \leq V(+\infty) \leq u_1^*$, uniqueness of the positive equilibrium of (2.1) in the order interval $[0, u_1^*]$ implies that $V(+\infty) = u_1^*$. Consequently, (1.3) has a monotone traveling wave connecting $(0,0)$ and (u_1^*, u_2^*) with speed c_τ^*.

In the case of $0 < c < c_\tau^*$, by the result of asymptotic spreading in [**7, 13**], it is easy to prove that system (1.3) has no monotone solution connecting $(0,0)$ and (u_1^*, u_2^*). □

We now consider the effect of time delay on the minimal wave speed of (1.3). It is clear that Theorem 3.1 implies that c_τ^* is the minimal wave speed in the sense that (1.3) has no nontrivial traveling wave fronts if $c < c_\tau^*$ while (1.3) has a monotone traveling wave front for $c \geq c_\tau^*$. From the definition of $P_{c,\tau}(\lambda)$, property (P_1) and Lemma 3.1, it is easy to see that the time delay reduces the minimal wave speed c_τ^*, which is the following conclusion.

Theorem 3.2. *If* $0 \leq \tau_1 < \tau_2 < +\infty$, *then* $0 < c_{\tau_2}^* < c_{\tau_1}^*$.

4. Convergence of minimal wave speeds. In this section, we show that the minimal wave speeds of (1.4) converges to the minimal wave speed c^* of (1.2) with $d = \int_0^{+\infty} J(y)y^2 dy$ as ε goes to zero.

Let's give an heuristic idea of the choice of the scaling involved in (1.4). For any fixed smooth function U, $\varepsilon \in (0,1)$ sufficiently small, the Taylor expansion gives that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} J_\varepsilon(y-x)(U(y)-U(x))\,dy \\
&\qquad = \int_{-\infty}^{+\infty} \frac{1}{\varepsilon^3} J(\frac{y-x}{\varepsilon})(U(y)-U(x))\,dy \\
&\qquad = \int_{-\infty}^{+\infty} \frac{1}{\varepsilon^2} J(y)(U(x+\varepsilon y)-U(x))\,dy \\
&\qquad = \int_{-\infty}^{+\infty} \frac{1}{\varepsilon^2} J(y)(\varepsilon y U'(x) + \frac{1}{2}\varepsilon^2 y^2 U''(x)
\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{6}\varepsilon^3 y^3 U'''(x) + o(\varepsilon^4))\, dy \\
&= U'(x)\int_{-\infty}^{+\infty} \frac{1}{\varepsilon} J(y) y\, dy + \frac{1}{2}U''(x)\int_{-\infty}^{+\infty} J(y)y^2 dy \\
&+ \frac{\varepsilon}{6}U'''(x)\int_{-\infty}^{+\infty} J(y)y^3 dy + o(\varepsilon^2) \\
&= \frac{1}{2}U''(x)\int_{-\infty}^{+\infty} J(y)y^2 dy + o(\varepsilon^2) \\
&\longrightarrow dU''(x) \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

where we use the fact that J is even in the last equality. Note that the above derivation implies that the Laplacian operator only averages the neighboring densities, which shows that the Laplacian operator is not sufficiently accurate in some cases.

Assume that (G_1)–(G_3) hold, $\tau \geq 0$ and $\varepsilon \in (0,1)$. Then, by the results of the previous sections and [**14**], there exist $c^* > 0$ and $c^*_\varepsilon > 0$, which are the minimal wave speeds of (1.2) and (1.4), respectively.

In this section, we prove the following convergence result.

Theorem 4.1. *Assume that* (G_1)–(G_3) *hold,* $\tau \geq 0$*,* $\varepsilon \in (0,1)$ *is sufficiently small and* c^* *and* c^*_ε *are the minimal wave speeds of* (1.2) *and* (1.4)*, respectively. Then* $c^* < c^*_\varepsilon$ *and* $c^* = \lim_{\varepsilon\to 0^+} c^*_\varepsilon$*.*

Proof. Let

$$P_c(\lambda) := \lambda^3 + \left(\frac{\beta}{c} - \frac{c}{d}\right)\lambda^2 - \frac{1+\beta}{d}\lambda + \frac{\alpha g'(0)e^{-c\tau\lambda} - \beta}{dc},$$

and

$$P_{c,\varepsilon}(\lambda) := \int_{-\infty}^{+\infty} J_\varepsilon(y)(e^{\lambda y} - 1)\, dy - c\lambda - 1 + \frac{\alpha g'(0)}{c\lambda + \beta}e^{-c\tau\lambda}.$$

Then

$$\widetilde{P_c}(\lambda) := d\lambda^2 - c\lambda - 1 + \frac{\alpha g'(0)}{c\lambda + \beta}e^{-c\tau\lambda} = \frac{dc}{c\lambda + \beta}P_c(\lambda). \tag{4.1}$$

From the results of the previous section and [**14**], we know that

(C_1) c^* is the minimal wave speed of (1.2) if and only if there exists $\lambda^* > 0$ such that $P_{c^*}(\lambda^*) = 0$, and $P_{c^*}(\lambda) > 0$ for $\lambda \in \mathbf{R}_+ \setminus \lambda^*$.

(C_2) c^*_ε is the minimal wave speed of (1.4) if and only if there exists $\lambda^*_\varepsilon > 0$ such that $P_{c^*_\varepsilon,\,\varepsilon}(\lambda^*_\varepsilon) = 0$, and $P_{c^*_\varepsilon,\varepsilon}(\lambda) > 0$ for $\lambda \in \mathbf{R}_+ \setminus \lambda^*_\varepsilon$.

(C_3) If $c > 0$ such that $P_{c,\,\varepsilon}(\lambda) > 0$ for $\lambda \in (0, +\infty)$, then $c < c^*_\varepsilon$.

(C_4) (c^*, λ^*) is the unique positive solution of $P_c(\lambda) = 0$, $\partial/(\partial\lambda)P_c(\lambda) = 0$.

If $\varepsilon > 0$ is sufficiently small, by the assumption of $J_\varepsilon(y)$ and applying Taylor's formula, we have

(4.2)
$$\begin{aligned}
P_{c,\,\varepsilon}(\lambda) &= \int_{-\infty}^{+\infty} \frac{1}{\varepsilon^3} J\left(\frac{y}{\varepsilon}\right)(e^{\lambda y} - 1)\,dy - c\lambda - 1 + \frac{\alpha g'(0)}{c\lambda + \beta} e^{-c\tau\lambda} \\
&= \int_{-\infty}^{+\infty} \frac{1}{\varepsilon^2} J(y)(e^{\lambda\varepsilon y} - 1)\,dy - c\lambda - 1 + \frac{\alpha g'(0)}{c\lambda + \beta} e^{-c\tau\lambda} \\
&= \int_{-\infty}^{+\infty} \frac{1}{\varepsilon^2} J(y)\Big(\lambda\varepsilon y + \frac{1}{2}\lambda^2\varepsilon^2 y^2 + \frac{1}{3!}\lambda^3\varepsilon^3 y^3 \\
&\quad + \frac{1}{4!}\lambda^4\varepsilon^4 y^4 + \frac{1}{5!}\lambda^5\varepsilon^5 y^5 + \cdots\Big)\,dy \\
&\quad - c\lambda - 1 + \frac{\alpha g'(0)}{c\lambda + \beta} e^{-c\tau\lambda} \\
&= d\lambda^2 - c\lambda - 1 + \frac{\alpha g'(0)}{c\lambda + \beta} e^{-c\tau\lambda} + \mathscr{A}(\varepsilon) \\
&= \widetilde{P_c}(\lambda) + \mathscr{A}(\varepsilon),
\end{aligned}$$

where

$$\mathscr{A}(\varepsilon) = \int_{-\infty}^{+\infty} J(y)\bigg(\sum_{k=2}^{+\infty} \lambda^{2k}\varepsilon^{2k} y^{2k}(2k)!\bigg)\,dy > 0,$$
$$\mathscr{A}(\varepsilon) = o(\varepsilon) \text{ as } \varepsilon \to 0.$$

By (4.2) and (C_1), $P_{c^*,\,\varepsilon}(\lambda) > 0$ for $\lambda \in (0, +\infty)$, and thus (C_3) implies that $c^* < c^*_\varepsilon$. If $0 < \varepsilon_1 < \varepsilon_2 < 0$, then $P_{c,\varepsilon_1}(\lambda) < P_{c,\varepsilon_2}(\lambda)$ for $\lambda \in (0, +\infty)$. Thence, by (C_2) and (C_3), we have $c^*_{\varepsilon_1} < c^*_{\varepsilon_2}$, and thus $\lim_{\varepsilon\to 0^+} c^*_\varepsilon$ exists. Next we are left to show $\lim_{\varepsilon\to 0^+} c^*_\varepsilon = c^*$.

Let $\lim_{\varepsilon\to 0^+} c_\varepsilon^* = c_0^*$. Then $c_0^* \geq c^*$. Let $\varepsilon_0 \in (0,1)$ be sufficiently small. Then $c_\varepsilon^* \in [c_0^*, c_{\varepsilon_0}^*]$ for any $\varepsilon \in (0,\varepsilon_0)$. Moreover, since $P_{c_\varepsilon^*,\,\varepsilon}(\lambda_\varepsilon^*) = 0$, we have $\lambda_\varepsilon^* \leq 1 + (c_{\varepsilon_0}^* + 1)/d$. Hence, we can choose a sequence $\{\varepsilon_n\}_{n=1}^{+\infty}$ such that $\lim_{n\to+\infty} \varepsilon_n = 0$, $\lim_{n\to+\infty} c_{\varepsilon_n}^* = c_0^*$ and $\lim_{n\to+\infty} \lambda_{\varepsilon_n}^* = \lambda_0^*$, where $(c_{\varepsilon_n}^*, \lambda_{\varepsilon_n}^*)$ is the unique solution of the equations

$$(4.3) \qquad \begin{cases} P_{c,\,\varepsilon_n}(\lambda) = 0, \\ (\partial/\partial\lambda) P_{c,\,\varepsilon_n}(\lambda) = 0. \end{cases}$$

Similarly with (4.2), by the assumption of $J_\varepsilon(y)$ and applying Taylor's formula, we have

$$(4.4) \qquad \frac{\partial}{\partial\lambda} P_{c,\varepsilon}(\lambda) = \frac{\partial}{\partial\lambda} \widetilde{P_c}(\lambda) + \mathscr{B}(\varepsilon),$$

where

$$\mathscr{B}(\varepsilon) = \int_{-\infty}^{+\infty} J(y)\left(\sum_{k=1}^{+\infty} \frac{\lambda^{2k+1}\varepsilon^{2k}y^{2k+2}}{(2k+1)!}\right) dy > 0,$$
$$\mathscr{B}(\varepsilon) = o(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

By substituting $c_{\varepsilon_n}^*$ and $\lambda_{\varepsilon_n}^*$ into (4.2) and (4.4), we get

$$(4.5) \qquad \begin{cases} P_{c_{\varepsilon_n}^*,\varepsilon_n}(\lambda_{\varepsilon_n}^*) = \widetilde{P}_{c_{\varepsilon_n}^*}(\lambda_{\varepsilon_n}^*) + \mathscr{A}(\varepsilon_n), \\ \frac{\partial}{\partial\lambda} P_{c_{\varepsilon_n}^*,\,\varepsilon_n}(\lambda_{\varepsilon_n}^*) = \frac{\partial}{\partial\lambda} \widetilde{P}_{c_{\varepsilon_n}^*}(\lambda_{\varepsilon_n}^*) + \mathscr{B}(\varepsilon_n). \end{cases}$$

Letting $n \to \infty$ in (4.5), we then obtain

$$(4.6) \qquad \begin{cases} \widetilde{P}_{c_0^*}(\lambda_0^*) = 0, \\ \frac{\partial}{\partial\lambda} \widetilde{P}_{c_0^*}(\lambda_0^*) = 0. \end{cases}$$

From (4.1) and (4.6), we get

$$(4.7) \qquad \begin{cases} P_{c_0^*}(\lambda_0^*) = 0, \\ \frac{\partial}{\partial\lambda} P_{c_0^*}(\lambda_0^*) = 0. \end{cases}$$

Therefore, by (C_4), (4.7) implies $c_0^* = c^*$ and the proof is completed. □

It should be remarked that $c_\varepsilon^* > c^*$ indicates that the nonlocal effect in (1.3) can increase the speed of epidemic spread.

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