ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 6, 2013

A QUINTIC DIOPHANTINE EQUATION WITH APPLICATIONS TO TWO DIOPHANTINE SYSTEMS CONCERNING FIFTH POWERS

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ABSTRACT. In this paper we obtain a parametric solution of the quintic diophantine equation $ab(a+b)(a^2+ab+b^2) = cd(c+d)(c^2+cd+d^2)$. We use this solution to obtain parametric solutions of two diophantine systems concerning fifth powers, namely, the system of simultaneous equations $x_1+x_2+x_3 = y_1+y_2+y_3 = 0$, $x_1^5+x_2^5+x_3^5 = y_1^5+y_2^5+y_3^5$, and the system of equations given by $\sum_{i=1}^{5} x_i^k = \sum_{i=1}^{5} y_i^k$, $k = 1, 2, 4, 5$.

1. Introduction. This paper is concerned with the quintic diophantine equation

$$ab(a+b)\left(a^2+ab+b^2\right) = cd(c+d)\left(c^2+cd+d^2\right), \tag{1}$$

where a, b, c and d are positive integers. We in fact obtain a parametric solution of degree 6 for the more general equation

$$ab(a+b)\left(a^2+hab+b^2\right) = cd(c+d)\left(c^2+hcd+d^2\right), \tag{2}$$

where h is a fixed rational number. This solution, however, reduces to a trivial solution when $h = 1$. We therefore obtained numerical solutions of (1) by performing computer trials and, using these numerical results, we obtain a parametric solution of degree 16 for this equation.

We use the solution of equation (1) to obtain parametric solutions of the following two diophantine systems:

$$\begin{aligned} x_1+x_2+x_3 &= y_1+y_2+y_3 = 0,\\ x_1^5+x_2^5+x_3^5 &= y_1^5+y_2^5+y_3^5; \end{aligned} \tag{3}$$

2010 AMS *Mathematics subject classification.* Primary 11D41.
Keywords and phrases. Quintic equation, fifth powers, equal sums of powers.
Received by the editors on June 25, 2011.

DOI:10.1216/RMJ-2013-43-6-1893

and

$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k = y_1^k + y_2^k + y_3^k + y_4^k + y_5^k, \quad k = 1, 2, 4, 5. \tag{4}$$

In Section 2 we obtain the parametric solutions of equations (1) and (2). Sections 3 and 4 contain the solutions of the diophantine systems (3) and (4).

2. The quintic equation $ab(a+b)(a^2+ab+b^2) = cd(c+d)(c^2+cd+d^2)$**.** We note that the general equation (2) where $h \neq 1$ is a fixed rational number can be solved by a method described in [**3**, Section 2.6]. In fact, following this method, we readily obtain the following parametric solution of this equation:

$$\begin{aligned}
a &= k\{-(h-1)(h-2)^2p^5q + (h-1)^6q^6\}, \\
b &= k(h-1)^2(h-2)^2p^5q, \\
c &= -k\{(h-2)^3p^6 + (h-1)^5pq^5\}, \\
d &= k(h-1)^6pq^5,
\end{aligned} \tag{5}$$

where k, p and q are arbitrary parameters. Taking $h = 1$ or $h = 2$ in this solution gives a trivial solution of (2). For $h = 2$ an approach presented in [**3**, Section 2.1] leads to the complete rational solution $a = p\,q^3\,(p-q)$, $b = p\,q^4 - 1$, $c = p - q$, $d = p\,(p\,q^4 - 1)$, where p and q are arbitrary rational parameters. However, when $h = 1$, the general method of solving equation (2) described in [**3**] fails and so equation (1) has to be solved by other means.

We first give numerical results which enabled us to produce a parametric solution of the equation (1).

An exhaustive search determined that there are only five primitive solutions with terms below 10,000,000, which are listed in Table I.

TABLE I. Solutions of equation (1).

	a	b	c	d
(S1)	1153	105	582	455
(S2)	47242	31	9717	5681
(S3)	51858	3914	25403	19264
(S4)	59070	12292	37983	27745
(S5)	5313954	132861	1830610	1756799

Before the numerical search had been extended to so high a range, only the first four solutions were known. Numerical analysis did not produce any interesting observations, except for a single curiosity; namely, the equality $a(a+b) = 4cd$ satisfied by solution (S4).

When the solution (S5) appeared, a quick numerical analysis revealed the equality $a(a+b) = 9cd$, which in view of a similar equality satisfied by (S4), could not be accidental. Prime factorization of all eight factors in equation (1) applied to numerical solutions (S4) and (S5) indicated similar multiplicative structure of both solutions. This led to the following parametrization:

$$
\begin{aligned}
a &= p^{16} - 3p^{11}q^5 - 5p^6q^{10} - pq^{15} \\
b &= 6p^{11}q^5 + 2pq^{15} \\
c &= p^{15}q + 5p^{10}q^6 + 3p^5q^{11} - q^{16} \\
d &= p^{15}q - 5p^{10}q^6 + 3p^5q^{11} + q^{16}.
\end{aligned} \tag{6}
$$

Solution (S4) corresponds to $(p,\ q) = (2,\ 1)$, and solution (S5) is obtained for $(p,\ q) = (3,\ 1)$ after dividing a, b, c and d by their common factor 8.

The smallest solution not discovered by a direct numerical search is obtained for $(p,\ q) = (3,\ 2)$:

$$(a,\ b,\ c,\ d) = (22209825,\ 34208832,\ 49020950,\ 11360662). \tag{7}$$

Note that integers a, b, c and d given directly by formula (6) do not need to be positive. This is not a problem, since we have

$$
\begin{aligned}
Q(a,\ b) &= Q(a,\ -a-b) = Q(b,\ -a-b) \\
&= Q(-a,\ -b) = Q(-a,\ a+b) = Q(-b,\ a+b),
\end{aligned} \tag{8}
$$

where

$$Q(a,\ b) = ab(a+b)\left(a^2+ab+b^2\right) \tag{9}$$

and, for nonzero $a,\ b$, exactly one of the six expressions in (8) has both arguments positive.

We were unable to produce a parametric solution generating any of the numerical solutions (S1), (S2), (S3).

3. The diophantine system $\sum_{i=1}^3 x_i^5 = \sum_{i=1}^3 y_i^5$, $\sum_{i=1}^3 x_i = \sum_{i=1}^3 y_i = 0$. We note that parametric solutions of the diophantine system given by

$$\begin{aligned} x_1+x_2+x_3 &= y_1+y_2+y_3,\\ x_1^5+x_2^5+x_3^5 &= y_1^5+y_2^5+y_3^5, \end{aligned} \tag{10}$$

have been obtained by several mathematicians [**1, 2, 4, 5**]. Until now, however, no solutions of this system have been published with the additional condition $x_1+x_2+x_3=0$.

To solve the diophantine system (3), we write,

$$\begin{aligned} x_1 &= a, & x_2 &= b, & x_3 &= -a-b,\\ y_1 &= c, & y_2 &= d, & y_3 &= -c-d, \end{aligned} \tag{11}$$

when the first equation of the system is identically satisfied while the second one reduces to (1). Thus, a parametric solution of (3) is given by (11) where a, b, c and d are given by (6).

4. The diophantine system $\sum_{i=1}^5 x_i^k = \sum_{i=1}^5 y_i^k$, $k = 1,\ 2,\ 4,\ 5$. To solve the diophantine system (4), we will solve the system of equations

$$\sum_{i=1}^{6} x_i^k = \sum_{i=1}^{6} y_i^k, \tag{12}$$

where $k = 1,\ 2,\ 4$ and 5 together with the additional condition $x_6 = y_6$.

We write

(13)

$$\begin{aligned} x_1 &= (m+2n)a + (-m+n)b, & y_4 &= (m-n)a + (-m-2n)b,\\ x_2 &= (-2m-n)a + (-m-2n)b, & y_5 &= (-2m-n)a + (-m+n)b,\\ x_3 &= (m-n)a + (2m+n)b, & y_6 &= (m+2n)a + (2m+n)b, \end{aligned}$$

when it is readily verified that

$$\sum_{i=1}^{3} x_i^k = \sum_{i=4}^{6} y_i^k, \quad k = 1,\ 2,\ 4. \tag{14}$$

Similarly we write

(15)

$$\begin{aligned} x_4 &= (m-n)c + (-m-2n)d, & y_1 &= (m+2n)c + (-m+n)d,\\ x_5 &= (-2m-n)c + (-m+n)d, & y_2 &= (-2m-n)c + (-m-2n)d,\\ x_6 &= (m+2n)c + (2m+n)d, & y_3 &= (m-n)c + (2m+n)d, \end{aligned}$$

when we have the relations

$$\sum_{i=4}^{6} x_i^k = \sum_{i=1}^{3} y_i^k, \quad k = 1,\ 2,\ 4. \tag{16}$$

It follows from (14) and (16) that the values of x_i, y_i, $i = 1, 2, \ldots, 6$, given by (13) and (15) satisfy equation (12) for $k = 1,\ 2,\ 4$. Equation (12) will also be satisfied for $k = 5$ if

$$x_1^5 + x_2^5 + x_3^5 - y_4^5 - y_5^5 - y_6^5 = y_1^5 + y_2^5 + y_3^5 - x_4^5 - x_5^5 - x_6^5. \tag{17}$$

Now, on substituting the values of x_i, y_i, $i = 1, 2, \ldots, 6$, given by (13) and (15), we observe that equation (17) reduces, on removing common factors, to equation (1). It follows that a solution of the simultaneous equations (12) with $k = 1,\ 2,\ 4$ and 5 is given by (13) and (15) where a, b, c and d are given by (6). As this solution is linear in the parameters m and n, we can easily satisfy the additional condition $x_6 = y_6$ by a suitable choice of m and n. This leads to a solution of the diophantine system (4) in terms of polynomials of degree 27 in the parameters p and q. Denoting the polynomial $c_0p^n + c_1p^{n-1}q + \cdots + c_{n-1}pq^{n-1} + c_nq^n$

briefly by $[c_0, c_1, \ldots, c_{n-1}, c_n]$, and defining four polynomials $f_i(p, q)$, $i = 1, 2, 3, 4$, by

(18)
$$\begin{aligned} f_1(p, q) = {} & [0, -1, 0, -2, -2, 8, -11, 12, -4, 4, 8, -2, 12, 16, \\ & 16, -8, 10, 4, -12, 12, -8, 3, 4, 2, 2, 0, 1, 0], \\ f_2(p, q) = {} & [1, -1, 2, 0, 2, -3, 9, -4, 12, 8, -18, 18, -16, -12, \\ & -4, -10, 6, -12, 4, -8, 1, -1, -2, -4, 2, -3, 1, 0], \\ f_3(p, q) = {} & [0, -1, 3, -2, 4, 2, -9, 9, -12, -4, -8, -6, 10, -16, \\ & 12, -4, -6, 6, -4, -12, 8, -9, 3, 2, 0, 2, -1, 1], \\ f_4(p, q) = {} & [0, 2, -3, 4, -2, 2, 2, 3, 4, 12, 12, -4, 2, 12, -16, \\ & 16, -4, -2, 12, 4, 4, 2, 1, 0, 2, -2, 2, -1], \end{aligned}$$

the aforementioned solution of (4) may be written as

$$\begin{array}{lll} x_1 = f_1(p, q), & x_2 = f_2(p, q), & x_3 = -f_1(p, q) - f_2(p, q), \\ x_4 = f_3(p, q), & x_5 = f_4(p, q), & \\ y_1 = f_2(q, p), & y_2 = f_1(q, p), & y_3 = -f_1(q, p) - f_2(q, p), \\ y_4 = f_4(q, p), & y_5 = f_3(q, p). & \end{array} \tag{19}$$

Taking $p = 2$, $q = 1$, we get the following numerical solution:

$$\begin{array}{lll} x_1 = -93728398, & x_2 = 159506374, & x_3 = -65777976, \\ x_4 = 23860807, & x_5 = 106362539, & \\ y_1 = -47848441, & y_2 = 136632334, & y_3 = -88783893, \\ y_4 = -15872456, & y_5 = 146095802. & \end{array} \tag{20}$$

We can try to obtain additional parametric solutions of the diophantine system (4) by solving the simultaneous equations (12) with $k = 1, 2, 4, 5$ together with other conditions of the type $x_i = y_j$ instead of the condition $x_6 = y_6$. For instance, imposing the condition $x_5 = y_6$, we get a parametric solution of degree 30 for the diophantine system (4). Further, we note that other possible conditions to cancel a pair of terms, one on each side, from the solution of $\sum_{i=1}^{6} x_i^k = \sum_{i=1}^{6} y_i^k$, $k = 1, 2, 4, 5$, lead either to a solution effectively equivalent to one of these two parametric solutions of (4) of degrees 27 and 30, or to trivial solutions. We thus get just two parametric solutions of the diophantine system (4). As the parametric solution of degree 30 is too cumbersome to write, it is not being given explicitly.

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