

MINIMAL PRIME IDEALS AND CYCLES IN ANNIHILATING-IDEAL GRAPHS

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ABSTRACT. Let R be a commutative ring with identity, and let $\mathbf{A}(R)$ be the set of ideals with non-zero annihilator. The *annihilating-ideal graph* of R is defined as the graph $\mathbf{AG}(R)$ with the vertex set $\mathbf{A}(R)^* = \mathbf{A}(R) \setminus \{0\}$, and two distinct vertices I and J are adjacent if and only if $IJ = 0$. In this paper, we study some connections between the graph theoretic properties of this graph and some algebraic properties of a commutative ring. We prove that if $\mathbf{AG}(R)$ is a tree, then either $\mathbf{AG}(R)$ is a star graph or a path of order 4 and, in the latter case, $R \cong F \times S$, where F is a field and S is a ring with a unique non-trivial ideal. Moreover, we prove that if R has at least three minimal prime ideals, then $\mathbf{AG}(R)$ is not a tree. It is shown that, for every reduced ring R , if R has at least three minimal prime ideals, then $\mathbf{AG}(R)$ contains a triangle. Also, we prove that, for every non-reduced ring R , if $|\text{Min}(R)| = 2$, then either $\mathbf{AG}(R)$ contains a cycle or $\mathbf{AG}(R) \cong P_4$. Finally, it is proved that, if $|\text{Min}(R)| = 1$ and $\mathbf{AG}(R)$ is a bipartite graph, then $\mathbf{AG}(R)$ is a star graph.

1. Introduction. The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past 20 years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring, for instance, see [1–4, 7, 10, 11, 14]. Throughout this paper, all rings are assumed to be non-domain commutative rings with identity. A *multiplicative closed subset* of a commutative ring R is a subset S of R such that $1 \in S$ and, for every $x, y \in S$, $xy \in S$. By $\text{Min}(R)$ and $Z(R)$, we denote the set of all minimal prime ideals of R and the set of all zero-divisors of R , respectively. A ring R is said to be *reduced*, if it has no non-zero

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nilpotent element or equivalently $\bigcap_{P \in \text{Min}(R)} P = 0$. A non-zero ideal I of a ring R is said to be *minimal* if there is no non-trivial ideal of R contained in I . For every graph G , we denote by $V(G)$, the vertex set of G . For some $U \subseteq V(G)$, we denote by $N(U)$ the set of all vertices of $G \setminus U$ adjacent to at least one vertex of U . An *independent set* is a subset of the vertices of a graph such that no vertices are adjacent. We denote by P_n , the path of order n . The *complete graph* of order n , denoted by K_n , is a graph whose any two distinct vertices are adjacent. A *bipartite graph* is a graph all of whose vertices can be partitioned into two parts U and V such that every edge joins a vertex in U to one in V . It is well known that a graph is bipartite if and only if it has no odd cycle. A *complete bipartite graph* is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. If the size of one of the parts is 1, then the graph is said to be a *star graph*. The *center* of a star graph is a vertex that is adjacent to all of the other vertices. The *distance* between two vertices in a graph is the number of edges in a shortest path connecting them. The *diameter* of a connected graph G , denoted by $\text{diam}(G)$, is the maximum distance between any pair of the vertices of G . The *girth* of a graph G , denoted by $\text{girth}(G)$, is the order of a shortest cycle contained in G . If G does not contain cycle, $\text{girth}(G)$ is defined to be infinity. We call an ideal I of R , an *annihilating-ideal* if there exists a non-zero ideal J of R such that $IJ = 0$. We use the notation $\mathbf{A}(R)$ for the set of all annihilating-ideals of R . By the *annihilating-ideal graph* of R , $\mathbf{AG}(R)$, we mean the graph with the vertex set $\mathbf{A}(R)^* = \mathbf{A}(R) \setminus \{0\}$ such that two distinct vertices I and J are adjacent if and only if $IJ = 0$. The annihilating-ideal graph was first introduced in [6], and some interesting properties of this graph have been studied.

The following useful theorems will be used frequently in this paper.

Theorem A [6, Theorem 1.4]. *Let R be a ring. Then the following statements are equivalent:*

- (1) $\mathbf{AG}(R)$ is a finite graph.
- (2) R has finitely many ideals.
- (3) Every vertex of $\mathbf{AG}(R)$ has finite degree. Moreover, $\mathbf{AG}(R)$ has n vertices, $n \geq 1$, if and only if R has exactly n non-zero proper ideals.

Theorem B [6, 8]. *For every ring R , the annihilating-ideal graph $\mathbf{AG}(R)$ is connected and $\text{diam}(\mathbf{AG}(R)) \leq 3$. Moreover, if $\mathbf{AG}(R)$ contains a cycle, then $\text{girth}(\mathbf{AG}(R)) \leq 4$.*

In this paper, first we prove that if $\mathbf{AG}(R)$ is a tree, then either $\mathbf{AG}(R)$ is a star graph or is the path P_4 and, in this case, $R \cong F \times S$, where F is a field and S is a ring with a unique non-trivial ideal. Next, we study the bipartite annihilating-ideal graphs of Artinian rings. Moreover, we give some relations between the existence of cycles in the annihilating-ideal graph of a ring and the number of its minimal prime ideals. For instance, if $|\text{Min}(R)| \geq 3$, then it is shown that $\mathbf{AG}(R)$ contains a cycle. Also we prove that if $|\text{Min}(R)| = 1$ and $\mathbf{AG}(R)$ is a triangle-free graph, then $\mathbf{AG}(R)$ is a star graph.

2. Cycles in the annihilating-ideal graphs. We start by the following lemma. Using this lemma, we prove a theorem that characterizes all annihilating-ideal graphs which are trees.

Lemma 1. *Let R be a ring such that $R = R_1 \times R_2$, where R_i is a ring for $i = 1, 2$. If $\mathbf{AG}(R)$ is a triangle-free graph, then one of the following statements holds.*

- (i) *Both R_1 and R_2 are integral domains.*
- (ii) *One R_i is an integral domain and the other one is a ring with a unique non-trivial ideal.*

Moreover, $\mathbf{AG}(R)$ has no cycle if and only if either $R \cong F \times S$ or $R \cong F \times D$, where F is a field, S is a ring with a unique non-trivial ideal and D is an integral domain.

Proof. If none of R_1 and R_2 is an integral domain, then each R_i contains a non-trivial annihilating-ideal, say I_i , $1 \leq i \leq 2$. So, $I_1 \times 0$, $0 \times I_2$ and $\text{Ann}(I_1) \times \text{Ann}(I_2)$ form a triangle in $\mathbf{AG}(R)$, a contradiction. So, with no loss of generality, one can suppose that R_1 is an integral domain. We prove that $\mathbf{AG}(R_2)$ has at most one vertex. To the contrary, suppose that $\{I, J\}$ is an edge of $\mathbf{AG}(R_2)$. Therefore, $R_1 \times 0$, $0 \times I$ and $0 \times J$ form a triangle, a contradiction. If $\mathbf{AG}(R_2)$ has no vertex, then R_2 is an integral domain and so part (i) occurs. If

$\mathbf{AG}(R_2)$ has exactly one vertex, then by Theorem A, R_2 has a unique non-trivial ideal and so we obtain part (ii).

Now, suppose that $\mathbf{AG}(R)$ has no cycle. If none of R_1 and R_2 is a field, then one can choose non-trivial ideals I_i in R_i , for some $i = 1, 2$. So, four ideals $I_1 \times 0$, $0 \times I_2$, $R_1 \times 0$ and $0 \times R_2$ form a cycle, a contradiction. The converse is clear. \square

In the next theorem, we determine which trees can occur as the annihilating-graph of a ring.

Theorem 2. *Let R be a ring. If $\mathbf{AG}(R)$ is a tree, then either $\mathbf{AG}(R)$ is a star graph or $\mathbf{AG}(R) \cong P_4$. Moreover, $\mathbf{AG}(R) \cong P_4$ if and only if $R \cong F \times S$, where F is a field and S is a ring with a unique non-trivial ideal, and indeed $S \cong K[x]/(x^2)$, where K is a field, or $S \cong A/p^2A$, where A is a discrete valuation ring of characteristic 0 with the residue field of characteristic p , for some prime number p .*

Proof. Suppose that $\mathbf{AG}(R)$ is not a star graph. Then, by Theorem B, $\mathbf{AG}(R)$ has at least four vertices. Clearly, there are two adjacent vertices I and J of $\mathbf{AG}(R)$ such that $|N(I) \setminus \{J\}| \geq 1$ and $|N(J) \setminus \{I\}| \geq 1$. Let $N(I) \setminus \{J\} = \{I_i\}_{i \in \Lambda}$ and $N(J) \setminus \{I\} = \{J_j\}_{j \in \Gamma}$. Since $\mathbf{AG}(R)$ is a tree, we have $N(I) \cap N(J) = \emptyset$. By Theorem B, $\text{diam}(\mathbf{AG}(R)) \leq 3$, and so every edge of $\mathbf{AG}(R)$ is of the form $\{I, J\}$, $\{I, I_i\}$ or $\{J, J_j\}$, for some $i \in \Lambda$ and $j \in \Gamma$. Now, we have the following claims:

Claim 1. Either $I^2 = 0$ or $J^2 = 0$. Assume that $p \in \Lambda$ and $q \in \Gamma$. Since $\mathbf{AG}(R)$ is a tree, $I_p J_q$ is a vertex of $\mathbf{AG}(R)$. If $I_p J_q = I_u$, for some $u \in \Lambda$, then $J I_u = 0$, a contradiction. If $I_p J_q = J_v$, for some $v \in \Gamma$, then $I J_v = 0$, a contradiction. If $I_p J_q = I$ or $I_p J_q = J$, then $I^2 = 0$ or $J^2 = 0$, respectively, and the claim is proved. \square

From now on, without loss of generality, we suppose that $I^2 = 0$. Clearly, $I \not\subseteq J$ and $J \not\subseteq I$.

Claim 2. I is a minimal ideal of R and $J^2 \neq 0$. First we show that, for every $0 \neq x \in I$, $(x) = I$. Assume that $0 \neq x \in I$ and $(x) \neq I$. If $(x) = J$, then $J \subseteq I$, a contradiction. Thus $(x) \neq J$, and the induced

subgraph of $\mathbf{AG}(R)$ on I, J and (x) is K_3 , a contradiction. So $(x) = I$. This implies that I is a minimal ideal of R . Now, if $J^2 = 0$, then the induced subgraph on I, J and $I + J$ is K_3 , a contradiction. Thus, $J^2 \neq 0$, and the claim is proved. \square

Claim 3. For every $i \in \Lambda$ and every $j \in \Gamma$, $I_i \cap J_j = I$. Suppose that $i \in \Lambda$ and $j \in \Gamma$. Since $I_i \cap J_j$ is a vertex and $I(I_i \cap J_j) = J(I_i \cap J_j) = 0$, either $I_i \cap J_j = I$ or $I_i \cap J_j = J$. If $I_i \cap J_j = J$, then $J^2 = 0$, a contradiction. Hence $I_i \cap J_j = I$, and the claim is proved. \square

Claim 4. R has exactly two minimal ideals I and J . Let K be a non-zero ideal properly contained in J . Since $IK \subseteq IJ = 0$, either $K = I$ or $K = I_i$, for some $i \in \Lambda$. So, by Claim 3, $I \subseteq K \subseteq J$, a contradiction. Hence, J is a minimal ideal of R . Suppose that L is another minimal ideal of R . Since I and J both are minimal ideals, we deduce that $IL = JL = 0$, a contradiction. Hence, claim is proved. \square

By Claims 2 and 4, $J^2 \neq 0$ and J is a minimal ideal of R . By Brauer’s lemma (see [9, subsection 10.22]), there exists an idempotent $e \in R$ such that $J = Re$ and $R \cong Re \times R(1 - e)$. Now, by Lemma 1, we deduce that either $R \cong F \times S$ and $\mathbf{AG}(R) \cong P_4$ or $R \cong F \times D$ and $\mathbf{AG}(R)$ is a star graph. Now, by [12], $S \cong K[x]/(x^2)$, where K is a field, or $S \cong A/p^2A$, where A is a discrete valuation ring of characteristic 0 with the residue field of characteristic p , for some prime number p . Conversely, assume that $R \cong F \times S$. Hence, R has exactly four nontrivial annihilating-ideals $0 \times S, F \times 0, 0 \times I$ and $F \times I$, where $I = (x)/(x^2)$ or $I = pA/p^2A$. Thus, $\mathbf{AG}(R) \cong P_4$ with the vertices $0 \times S, F \times 0, 0 \times I, F \times I$. \square

As a consequence of the next theorem, one can see that a finite bipartite annihilating-ideal graph is a star graph or the path P_4 . Moreover, we show that there is a ring whose annihilating-ideal graph is a complete bipartite graph with infinite parts.

Theorem 3. *Let R be an Artinian ring such that $\mathbf{AG}(R)$ is a bipartite graph. Then either $\mathbf{AG}(R)$ is a star graph or $\mathbf{AG}(R) \cong P_4$. Moreover, $\mathbf{AG}(R) \cong P_4$ if and only if $R \cong F \times S$, where F is a field and S is a ring with a unique non-trivial ideal.*

Proof. First suppose that R is not a local ring. Hence, by [5, Theorem 8.7], $R \cong R_1 \times R_2$, where R_1 and R_2 are two Artinian rings. Since every Artinian integral domain is a field, by Lemma 1, we deduce that either $\mathbf{AG}(R)$ is isomorphic to P_2 or P_4 . Now, assume that R is an Artinian local ring. Let \mathfrak{m} be the unique maximal ideal of R and k be a natural number such that $\mathfrak{m}^k = 0$ and $\mathfrak{m}^{k-1} \neq 0$. Clearly, \mathfrak{m}^{k-1} is adjacent to every other vertex of $\mathbf{AG}(R)$ and so $\mathbf{AG}(R)$ is a star graph. \square

Corollary 4. *If $\mathbf{AG}(R)$ is a finite bipartite graph, then either $\mathbf{AG}(R)$ is a star graph or $\mathbf{AG}(R) \cong P_4$.*

Proof. This follows from Theorem A and Theorem 3. \square

The above theorem does not necessarily hold for non-Artinian rings whose annihilating-ideal graphs are bipartite. To see this, consider the ring $R = K[x, y]/(xy)$, where K is an arbitrary field. It is not hard to see that $\mathbf{AG}(R)$ is a complete bipartite graph with two infinite parts $V_1 = \{(\bar{f}) \mid f \in K[x], f(0) = 0\}$ and $V_2 = \{(\bar{g}) \mid g \in K[y], g(0) = 0\}$.

By Theorem B, if $\mathbf{AG}(R)$ is not a tree, then $\mathbf{AG}(R)$ contains a cycle. In the sequel, we provide a sufficient condition under which the annihilating-ideal graph contains a cycle.

Theorem 5. *For every ring R , if $|\text{Min}(R)| \geq 3$, then $\mathbf{AG}(R)$ contains a cycle.*

Proof. If $\mathbf{AG}(R)$ is a tree, then by Theorem 2, either $\mathbf{AG}(R)$ is a star graph or $R \cong F \times S$, such that S has a unique non-trivial ideal. The latter case is impossible, because $|\text{Min}(F \times S)| = 2$. Suppose that $\mathbf{AG}(R)$ is a star graph and I is the center of star. Clearly, one can assume that I is a minimal ideal of R . If $I^2 \neq 0$, then by Brauer's lemma (see [9, subsection 10.22]), there exists an idempotent $e \in R$ such that $I = Re$ and $R \cong Re \times R(1 - e)$. Now, by Lemma 1, we conclude that $|\text{Min}(R)| = 2$, a contradiction. Hence $I^2 = 0$. Thus, one may assume that $I = (x)$ and $x^2 = 0$. If $y \in Z(R) \setminus (x)$, then $(x)(y) = 0$, and so $y \in \text{Ann}(x)$. Thus, $Z(R) = \text{Ann}(x)$. Suppose that P_1 and P_2 are two distinct minimal prime ideals of R . Since $x^2 = 0$, we deduce that $x \in P_1 \cap P_2$. Choose $z \in P_1 \setminus P_2$, and set $S_1 = R \setminus P_1$

and $S_2 = \{1, z, z^2, \dots\}$. If $0 \notin S_1S_2$, then by [13, Theorem 3.44], there exists a prime ideal P in R such that $P \cap S_1S_2 = \emptyset$, and hence $P = P_1$, a contradiction. So, $0 \in S_1S_2$. Therefore, there exist positive integer k and $y \in R \setminus P_1$ such that $yz^k = 0$. Now, consider the ideals (x) , (y) and (z^k) . It is clear that $(x) \neq (y)$ and $(y) \neq (z^k)$. If $(x) = (z^k)$, then z is a nilpotent element and so $z \in P_2$, a contradiction. Thus (x) , (y) and (z^k) form a triangle in $\mathbf{AG}(R)$, a contradiction. So, $\mathbf{AG}(R)$ contains a cycle. \square

Theorem B and the previous theorem show that if $|\text{Min}(R)| \geq 3$, then $\text{girth}(\mathbf{AG}(R)) \leq 4$. Now, we show that if R is a reduced ring with $|\text{Min}(R)| \geq 3$, then $\mathbf{AG}(R)$ contains a triangle. Before stating the next result, we need the following lemma.

Lemma 6. *Let R be a reduced ring and $P \in \text{Min}(R)$. If $a \in P$, then there exists $b \in R \setminus P$ such that $ab = 0$.*

Proof. Let $S = (R \setminus P)\{1, a, a^2, \dots\}$. We note that S is a multiplicative closed subset of R . If $0 \notin S$, then by [13, Theorem 3.44], there exists a prime ideal Q such that $Q \cap S = \emptyset$. This yields that $Q \subseteq P$, a contradiction. So, we can assume that $0 \in S$. Thus, there is a natural number i such that $ba^i = 0$, for some $b \in R \setminus P$. Thus, $ab = 0$ and we are done. \square

Theorem 7. *Let R be a reduced ring with $|\text{Min}(R)| \geq 3$. Then $\text{girth}(\mathbf{AG}(R)) = 3$.*

Proof. First we claim that R contains three elements $u, v, w \in Z(R) \setminus \{0\}$ such that $uv = vw = uw = 0$. To see this, let P_1, P_2 and P_3 be three distinct minimal prime ideals of R . We prove that $(P_1 \setminus P_2)P_2 \neq \{0\}$. By contradiction, assume that $(P_1 \setminus P_2)P_2 = \{0\} \subseteq P_3$. So, $P_1 \setminus P_2 \subseteq P_3$, and this implies that $P_1 \subseteq P_2 \cup P_3$, a contradiction, see [13, Theorem 3.61]. Hence, $(P_1 \setminus P_2)P_2 \neq \{0\}$. Assume that $h_1h_2 \neq 0$, $h_1 \in P_1 \setminus P_2$ and $h_2 \in P_2$. By Lemma 6, there are two elements $a \in R \setminus P_1$ and $b \in R \setminus P_2$ such that $h_1a = h_2b = 0$. Assume that $ab = 0$. Thus, h_1h_2, a, b are the desired elements in the claim. So, suppose that $ab \neq 0$. Since $h_1a = 0 \in P_2$, we have $a \in P_2$. Thus, $ab \in P_2$ and

$h_2ab = 0$. By Lemma 6, there exists $d \in R \setminus P_2$ such that $abd = 0$. Therefore, h_2, ab, bd are the desired elements, and the claim is proved. Now, consider three ideals (u) , (v) and (w) . Since R is a reduced ring, these ideals are distinct. Hence $\mathbf{AG}(R)$ contains a triangle and the proof is complete. \square

Let R be a reduced ring and $\text{Min}(R) = \{P_1, P_2\}$. Put $V_1 = \{I \in V(\mathbf{AG}(R)) \mid I \subseteq P_1\}$ and $V_2 = \{I \in V(\mathbf{AG}(R)) \mid I \subseteq P_2\}$. Behboodi and Rakeei proved that $\mathbf{AG}(R)$ is a complete bipartite graph with parts V_1 and V_2 . Moreover, Theorem 7 implies that, if $\mathbf{AG}(R)$ is a bipartite graph, then $|\text{Min}(R)| = 2$.

Theorem 8. *Let R be a non-reduced ring. If $|\text{Min}(R)| = 2$, then either $\mathbf{AG}(R)$ contains a cycle or $\mathbf{AG}(R) \cong P_4$.*

Proof. A similar argument to the proof of Theorem 5 shows that either $\mathbf{AG}(R)$ contains a cycle or $R \cong F \times S$, where F is a field and S is a ring with a unique non-trivial ideal. The latter case implies that $\mathbf{AG}(R) \cong P_4$. \square

Remark 9. The annihilating-ideal graph of a non-reduced ring R with $|\text{Min}(R)| = 2$, need not be a complete bipartite graph or does not necessarily contain a triangle. To see this, let $R = D \times S$, where D is an arbitrary integral domain and S is a ring with a unique non-trivial ideal. Then $\mathbf{AG}(R)$ is a bipartite graph which is not a complete bipartite graph.

Theorem 10. *Let R be a ring such that $|\text{Min}(R)| = 1$. If $\mathbf{AG}(R)$ is a triangle-free graph, then $\mathbf{AG}(R)$ is a star graph.*

Proof. Let P be the unique minimal prime ideal of R . If R is reduced, then $P = 0$ and so R is an integral domain, a contradiction. Thus, we may assume that R is a non-reduced ring and so there exists a non-zero element $x \in R$ such that $x^2 = 0$. Since $\mathbf{AG}(R)$ is triangle-free, there exists at most one non-zero ideal, properly contained in (x) . So, without loss of generality, we can assume that (x) is a minimal ideal of R such that $x^2 = 0$. We claim that (x) is the unique minimal ideal

of R . To the contrary, suppose that J is another minimal ideal of R . So either $J^2 = J$ or $J^2 = 0$. If $J^2 = J$, then by Brauer's lemma [9, subsection 10.22], $J = Re$ for some idempotent element $e \in R$ and $R \cong Re \times R(1 - e)$. This implies that $|\text{Min}(R)| > 1$, a contradiction. If $J^2 = 0$, then $J, (x)$ and $(x) + J$ form a triangle, a contradiction. So (x) is the unique minimal ideal of R . Let $V_1 = N((x)), V_2 = V(\mathbf{AG}(R)) \setminus V_1, A = \{K \in V_1 \mid (x) \subseteq K\}, B = V_1 \setminus A$ and $C = V_2 \setminus \{(x)\}$. We show that $\mathbf{AG}(R)$ is a bipartite graph with parts V_1 and V_2 . Since $\mathbf{AG}(R)$ is triangle-free, we deduce that V_1 is an independent set. We claim that one end of every edge of $\mathbf{AG}(R)$ is adjacent to (x) and another end contains (x) . To see this, suppose that $\{I, J\}$ is an edge of $\mathbf{AG}(R)$ and $(x) \neq I, (x) \neq J$. Since $I(x) \subseteq (x)$ and (x) is a minimal ideal of R , either $I(x) = 0$ or $(x) \subseteq I$. The latter case implies that $J(x) = 0$. If $I(x) = 0$, then $J(x) \neq 0$ and hence $(x) \subseteq J$. Therefore, the claim is proved. This implies that V_2 is an independent set and $N(C) \subseteq V_1$. Since every vertex of A contains (x) and $\mathbf{AG}(R)$ is triangle-free, all vertices in A are just adjacent to (x) and so by Theorem B, $N(C) \subseteq B$. Since one end of every edge is adjacent to (x) and another end contains (x) , we also deduce that every vertex of C contains (x) and so every vertex of $A \cup V_2$ contains (x) . Note that if $(x) = P$, then one end of each edge of $\mathbf{AG}(R)$ is contained in (x) and since (x) is a minimal ideal of R , $\mathbf{AG}(R)$ is a star graph with center $(x) = P$. Now, suppose that $P \neq (x)$. We claim that $P \in A$. Since $(x) \subseteq P$, it suffices to show that $(x)P = 0$. For this let $t \in P$. We prove that $tx = 0$. Clearly, $(tx) \subseteq (x)$. If $tx = 0$, then we are done. Thus, $(tx) = (x)$ and so $x = rtx$, for some $r \in R$. Therefore, $x(1 - rt) = 0$. Since $t \in P$ is nilpotent, $x = 0$, a contradiction. Hence, the claim is proved. Since $N(C) \subseteq B$, if $B = \emptyset$, then $C = \emptyset$ and so $\mathbf{AG}(R)$ is a star graph with center (x) . Now, we show that $B = \emptyset$. Suppose that $J \in B$, and consider the vertex $J \cap P$ of $\mathbf{AG}(R)$. Since every vertex of $A \cup V_2$ contains (x) , $J \cap P \in B$. Choose $0 \neq y \in J \cap P$. Since $\mathbf{AG}(R)$ is triangle-free, one can find an element $z \in (y)$ such that (z) is a minimal ideal of R and $z^2 = 0$. Since (x) is the unique minimal ideal of R , we have $(x) = (z) \subseteq (y)$. Thus, $(x) \subseteq J \cap P$, a contradiction. So $B = \emptyset$ and we are done. Hence, $\mathbf{AG}(R)$ is a star graph whose center is (x) and the proof is complete. \square

Corollary 11. *Let R be a ring and $|\text{Min}(R)| = 1$. If $\mathbf{AG}(R)$ is a bipartite graph, then $\mathbf{AG}(R)$ is a star graph.*

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