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THE STRONG LAW OF LARGE NUMBERS FOR LINEAR RANDOM FIELDS GENERATED BY NEGATIVELY ASSOCIATED RANDOM VARIABLES ON $\mathbf{Z}^d$

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ABSTRACT. We establish the strong law of large numbers for linear multi-parameter stochastic processes generated by identically distributed and negatively associated random fields. Our result generalizes the classical strong law of large numbers for the identically distributed and negatively associated random field to the linear random field by using the Beveridge-Nelson decomposition.

1. Introduction. Let $\mathbf{Z}^d$ $(d \geq 2)$ denote the integer d-dimensional lattice with coordinatewise partial ordering $\leq$. The notation $\mathbf{m} \leq \mathbf{n}$, where $\mathbf{m} = (m_1.m_2, \ldots, m_d)$ and $\mathbf{n} = (n_1, n_2, \ldots, n_d)$, thus means that $m_k \leq n_k$ for $k = 1, 2, \ldots, d$. We also use $|\mathbf{n}| = \prod_{k=1}^d n_k$, $\mathbf{n} \to \infty$, is to be interpreted as $n_k \to \infty$ for $k = 1, 2, \ldots, d$ and $\mathbf{1} = (1, 1, \ldots, 1)$. Let $\{\xi_{\mathbf{n}}, \mathbf{n} \in \mathbf{Z}^d\}$ be a random field on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Set $S_{\mathbf{n}} = \sum_{\mathbf{1} \leq \mathbf{j} \leq \mathbf{n}} \xi_{\mathbf{j}}$. Then, $S_{\mathbf{n}}$ is simply a sum of $|\mathbf{n}|$ random variables. The concept of negative association was introduced by Alam and Saxena [**1**] and Joag-Dev and Proschan [**4**]. This concept can be extended to the random field on $\mathbf{Z}^d$ $(d \geq 2)$ as follows.

The field $\{\xi_{\mathbf{n}}, \mathbf{n} \in \mathbf{Z}^d\}$ is called negatively associated (NA) if, for every pair of disjoint subsets A, B of $\mathbf{Z}^d$ and for any pair of coordinatewise nondecreasing functions $f : \mathbf{R}^{|A|} \to \mathbf{R}$ and $g : \mathbf{R}^{|B|} \to \mathbf{R}$, where $|A|$ and $|B|$ stand for the cardinalities of A and B, respectively, $\operatorname{Cov}(f(\xi_{\mathbf{i}}, \mathbf{i} \in A), g(\xi_{\mathbf{j}}, \mathbf{j} \in B)) \leq 0$. We refer to [**4**] for fundamental properties. In the case of $d = 1$, we refer to Newman [**10**] for the central limit theorem, Matula [**9**] for the strong law of large numbers and Li and Zhang [**7**] for the complete moment convergence of moving average processes generated by NA random variables. In the case of

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$d \geq 2$, Roussas [**13**] studied the central limit theorem for weak stationary NA random fields, Zhang and Wen [**14**] investigated maximal moment inequality and the weak convergence for a centered stationary NA random field under finite second moments and Ko [**6**] proved the complete convergence for identically distributed NA random field. See also Bulinski and Shashkin [**3**] for more details.

Define a linear random field

$$X(\mathbf{t}) = \sum_{\mathbf{k} \geq \mathbf{0}} a(\mathbf{k}) \xi(\mathbf{t} - \mathbf{k}), \quad \mathbf{t} \in \mathbf{Z}^d, \tag{1.1}$$

where the coefficients $\{a(\mathbf{k}), \mathbf{k} \in \mathbf{Z}^d\}$ and a field $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbf{Z}^d\}$ are such that the random field $\{X(\mathbf{t}), \mathbf{t} \in \mathbf{Z}^d\}$ is well defined and strictly stationary. Phillips and Solo [**12**] demonstrated that the so called Beveridge-Nelson decomposition (BND) presents a rather simple method for proving limit theorems (central limit theorem, strong law of large numbers and invariance principle) for sums of values of linear processes.

Marinucci and Poghosyan [**8**] proved the invariance principle and strong convergence for linear random fields generated by independent and identically distributed random fields and Kim et al. [**5**] also investigated the invariance principle for the linear random field under association by the similar method. Paulauskas [**11**] showed that an analogue of the Beveridge-Nelson decomposition can be applied to limit theorems for sums of linear random fields and Banys, Davydov and Paulauskas [**11**] proved a strong law of large numbers for linear random field generated by a strictly stationary centered ergodic or mixing random field by using ergodic theory.

We say that strong law of large numbers holds for $S_{\mathbf{n}}$, if

$$|\mathbf{n}|^{-1} S_{\mathbf{n}} \longrightarrow 0 \text{ a.s., as } \mathbf{n} \to \infty \tag{1.2}$$

where $S_{\mathbf{n}} = \sum_{\mathbf{1} \leq \mathbf{t} \leq \mathbf{n}} Y(\mathbf{t})$ and $\{Y(\mathbf{t}), \mathbf{t} \in \mathbf{Z}^d\}$ is a random field with $EY(\mathbf{t}) = 0$ for all $\mathbf{t} \in \mathbf{Z}^d$. There are several interpretations of the growth of $\mathbf{n}$. In our paper we shall use two possibilities.

We shall write $\mathbf{n} \to \infty$ if

$$n_i \longrightarrow \infty, \quad i = 1, \ldots, d. \tag{1.3}$$

The second possibility of growth of $\mathbf{n}$ is to assume that

$$|\mathbf{n}| \to \infty. \tag{1.4}$$

Evidently, (1.4) follows from (1.3) but not converse.

In this paper we consider the conditions which assert the convergence

$$\begin{aligned} |\mathbf{n}|^{-1} \sum_{\mathbf{1}\le\mathbf{t}\le\mathbf{n}} X(\mathbf{t}) \\ = |\mathbf{n}|^{-1} \sum_{\mathbf{1}\le\mathbf{t}\le\mathbf{n}} \sum_{\mathbf{k}\ge\mathbf{0}} a(\mathbf{k})\xi(\mathbf{t}-\mathbf{k}) \longrightarrow 0 \text{ a.s. as } \mathbf{n}\to\infty, \end{aligned} \tag{1.5}$$

where $\{a(\mathbf{k}), \mathbf{k} \in \mathbf{Z}^d\}$ is a sequence of positive constants and $\{\xi(\mathbf{i}), \mathbf{i} \in \mathbf{Z}^d\}$ is a centered identically distributed NA random field.

2. Preliminaries. First we consider the decomposition of multivariate polynomials in [**8**]; put

$$A(\mathbf{x}) = \sum_{\mathbf{k}\ge\mathbf{0}} a(\mathbf{k})\mathbf{x}^{\mathbf{k}}, \tag{2.1}$$

where $\mathbf{k} = (k_1, \dots, k_d) \in \mathbf{Z}_+^d$, $\mathbf{x} = (x_1, \dots, x_d) \in \mathbf{R}^d$ with $|x_i| \le 1$ for $i = 1, \dots, d$, $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} \cdots x_d^{k_d}$ and $a(\mathbf{k})$'s are positive real numbers satisfying

$$\sum_{\mathbf{j}\ge\mathbf{0}} \sum_{\mathbf{k}\ge\mathbf{j}+\mathbf{1}} a(\mathbf{k}) < \infty, \tag{2.2}$$

where $\mathbf{j} = (j_1, \dots, j_d)$ and $\mathbf{j}+\mathbf{1} = (j_1+1, \dots, j_d+1)$.

Clearly, (2.2) implies

$$A(\mathbf{1}) = \sum_{\mathbf{k}\ge\mathbf{0}} a(\mathbf{k}) < \infty. \tag{2.3}$$

The following lemma generalizes a result known for the case of $d = 1$ as the Beveridge-Nelson decomposition (cf. [**12**]).

Lemma 2.1 [**8**]. *Let Γ_d be the class of all 2^d subsets γ of $\{1,2,\dots,d\}$. Let $y_k = x_k$ if $k \in \gamma$ and $y_k = 1$ if $k \notin \gamma$. Then we have*

$$A(x_1,\dots,x_d) = \sum_{\gamma\in\Gamma_d}\{\Pi_{k\in\gamma}(x_k-1)\}A_\gamma(y_1,\dots,y_d), \tag{2.4}$$

where it is assumed that $\Pi_{k\in\phi}=1$, and

$$A_\gamma(y_1,\dots,y_d) = \sum_{k_1=0}^{\infty}\cdots\sum_{k_d=0}^{\infty} a_\gamma(k_1,\dots,k_d)\prod_{i=1}^{d} y_i^{k_i}, \tag{2.5}$$

$$a_\gamma(k_1,\dots,k_d) = \sum_{s_1=k_1+1}^{\infty}\cdots\sum_{s_d=k_d+1}^{\infty} a(s_1,\dots,s_d), \tag{2.6}$$

where $s_i \ge k_j+1$ if $j\in\gamma$, and $s_j = k_j$ if $j\notin\gamma$.

Next we consider the partial backshift operator satisfying

$$B_i\xi(t_1,\dots,t_i,\dots,t_d) = \xi(t_1,\dots,t_i-1,\dots,t_d). \tag{2.7}$$

From (2.1) and (2.7), we write as

$$\begin{aligned} X(\mathbf{t}) &= \sum_{k_1=0}^{\infty}\cdots\sum_{k_d=0}^{\infty} a(k_1,\dots,k_d)\left(\prod_{i=1}^{d} B_i^{k_i}\right)\xi(\mathbf{t}) \\ &= A(B_1,\dots,B_d)\xi(\mathbf{t}), \end{aligned} \tag{2.8}$$

where $A(B_1,\dots,B_d) = \sum_{k_1=0}^{\infty}\cdots\sum_{k_d=0}^{\infty} a(k_1,\dots,k_d)\prod_{i=1}^{d} B_i^{k_i}$

The above ideas shall be exploited to establish the strong law of large numbers for the linear negatively associated random fields.

To this aim, we write

$$\xi_\gamma(\mathbf{t}) = A_\gamma(L_1,\dots,L_d)\xi(\mathbf{t}), \tag{2.9}$$

where the operator L_k is defined as $L_k = B_k$ for $k\in\gamma$, $L_k = 1$ otherwise. For instance, when $d=2$,

$$\begin{aligned} \xi_1(t_1,t_2) &= A_1(B_1,1)\xi(t_1,t_2), \\ \xi_2(t_1,t_2) &= A_2(1,B_2)\xi(t_1,t_2), \\ \xi_{12}(t_1,t_2) &= A_{12}(B_1,B_2)\xi(t_1,t_2). \end{aligned}$$

It follows from (2.5), (2.7) and (2.9) that

$$\xi_\gamma(\mathbf{t}) = \sum_{\mathbf{k}\geq \mathbf{0}} a_\gamma(\mathbf{t})\xi(\mathbf{t}-\mathbf{k}). \tag{2.10}$$

Moreover, from (2.2), (2.3) and (2.6)

$$0 \leq \sum_{\mathbf{k}\geq \mathbf{0}} a_\gamma(\mathbf{k}) < \infty. \tag{2.11}$$

By the property of negative association (see [**3**]) and (2.10) we have the following lemma:

Lemma 2.2. *Let* $\{\xi(\mathbf{t}), t \in \mathbf{Z}^d\}$ *be an* NA *random field and* $\{a(\mathbf{k}), \mathbf{k} \in \mathbf{Z}_+^d\}$ *a collection of positive numbers. Then* $\{\xi_\gamma(\mathbf{t})\}$ *defined in* (2.9) *is also an* NA *random field.*

Lemma 2.3 [**9**]. *Let* $\{\xi_n, n \geq 1\}$ *be a sequence of identically distributed negatively associated random variables with* $E\xi_1 = 0$ *and* $E\xi_1^2 < \infty$*. Then*

$$\frac{1}{n}\sum_{i=1}^{n} \xi_i \longrightarrow 0 \text{ a.s. as } n \to \infty.$$

Theorem 2.4 [**9**]. *Let* $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbf{Z}^d\}$ *be a field of centered pairwise* NQD *random variables with the same distribution. Then*

$$E|\xi(\mathbf{0})|(\log^+ |\xi(\mathbf{0})|)^{d-1} < \infty \tag{2.12}$$

implies

$$|\mathbf{n}|^{-1} \sum_{\mathbf{1}\leq \mathbf{t}\leq \mathbf{n}} \xi(\mathbf{t}) \longrightarrow 0 \text{ a.s.} \tag{2.13}$$

From Theorem 2.4, we have the following result.

Corollary 2.5. *Let $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbf{Z}^d\}$ be a centered and identically distributed* NA *random field. Then, $E|\xi_{\mathbf{1}}|(\log^+|\xi_{\mathbf{1}}|)^{d-1} < \infty$ implies that $|\mathbf{n}|^{-1}\sum_{\mathbf{1}\le\mathbf{t}\le\mathbf{n}}\xi_{\mathbf{t}} \to 0$ a.s. as $\mathbf{n}\to\infty$, where $\log^+ x = \max\{1, \log x\}$.*

3. Results.

Theorem 3.1. *Let $\{X(\mathbf{t}), t \in \mathbf{Z}^d\}$ be defined as in* (1.1), *where $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbf{Z}^d\}$ is the field of identically distributed* NA *random variables with $E\xi(\mathbf{t}) = 0$, $E(\xi(\mathbf{t}))^2 < \infty$ and $\{a(\mathbf{k})\}$ is a collection of real numbers such that $a(\mathbf{k}) \ge 0$ for all $\mathbf{k} \ge \mathbf{0}$ and it satisfies* (2.2). *Then $E|\xi(\mathbf{1})|(\log^+|\xi(\mathbf{1})|)^{d-1} < \infty$ implies*

$$|\mathbf{n}|^{-1}\sum_{\mathbf{1}\le\mathbf{t}\le\mathbf{n}} X(\mathbf{t}) \longrightarrow 0 \text{ a.s. as } \mathbf{n}\to\infty. \tag{3.1}$$

To prove Theorem 3.1, we need the following lemma.

Lemma 3.2. *Let $\{\xi(\mathbf{t})\}$ be an identically distributed* NA *random field with $E\xi(\mathbf{t}) = 0$, and $E(\xi(\mathbf{t}))^2 < \infty$, and let $a(\mathbf{k})$'s be positive real numbers satisfying* (2.2). *Then*

$$E(\xi_\gamma(\mathbf{t}))^2 < \infty \quad \text{for } \gamma \in \Gamma. \tag{3.2}$$

Proof.

$$\begin{aligned}\xi_\gamma(0,\dots,0) &= \sum_{k_1=0}^{\infty}\cdots\sum_{k_d=0}^{\infty} a_\gamma(k_1,\dots,k_d)\xi(-k_1,\dots,-k_d)\\ &= \sum_{k=0}^{\infty} a_\gamma(\phi(k))\xi(-\phi(k)),\end{aligned}$$

where $\phi : \mathbf{Z} \to \mathbf{Z}^d$ and $\{\xi(-\phi(k))\}$ is a sequence of identically distributed NA random variables. Hence,

$$\begin{aligned}E(\xi_\gamma(\mathbf{t}))^2 = E(\xi_\gamma(0,\dots,0))^2 &= E\bigg(\sum_{k=0}^{\infty} a_\gamma(\phi(k))\xi(-\phi(k))\bigg)^2\\ &\le \bigg[\sum_{k=0}^{\infty} a_\gamma(\phi(k))(E(\xi(-\phi(k)))^2)^{1/2}\bigg]^2\\ &\le C\bigg[\sum_{k=0}^{\infty} a_\gamma(\phi(k))\bigg]^2 < \infty,\end{aligned}$$

the first bound following from Minkowski's inequality and the second bound from (2.11).

Remark. In Theorem 3.1, by taking $a(\mathbf{k}) = 1$ for $\mathbf{k} = \mathbf{0}$, $a(\mathbf{k}) = 0$ otherwise, we obtain Corollary 2.5.

Proof of Theorem 3.1. We start from the case $d = 2$, where we provide full details; the extension to $d > 2$ is discussed afterwards. If we apply Lemma 2.1 to the backshift polynomial $A(B_1, \dots, B_d)$, we find that the following a.s. equality holds

$$\begin{aligned} X(t_1,t_2) = A(1,1)\xi(t_1,t_2) &+ (B_1-1)A_1(B_1,1)\xi(t_1,t_2) \\ &+ (B_2-1)A_2(1,B_2)\xi(t_1,t_2) \\ &+ (B_1-1)(B_2-1)A_{12}(B_1,B_2)\xi(t_1,t_2), \end{aligned}$$

which implies that

$$\begin{aligned} &\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2} X(t_1,t_2) \\ &\quad = \sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2} A(1,1)\xi(t_1,t_2) - \sum_{t_2=1}^{n_2}\xi_1(n_1,t_2) + \sum_{t_2=1}^{n_2}\xi_1(0,t_2) \\ &\qquad - \sum_{t_1=1}^{n_1}\xi_2(t_1,n_2) + \sum_{t_1=1}^{n_1}\xi_2(t_1,0) - \xi_{12}(0,n_2) + \xi_{12}(0,0) \\ &\qquad - \xi_{12}(n_1,0) + \xi_{12}(n_1,n_2) \\ &\quad = \sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2} A(1,1)\xi(t_1,t_2) + R_n(t_1,t_2), \end{aligned} \tag{3.3}$$

where $\mathbf{n} = (n_1, n_2)$.

First, we obtain

$$|\mathbf{n}|^{-1}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2} A(1,1)\xi(t_1,t_2) \longrightarrow 0 \text{ a.s. as } \mathbf{n}\to\infty, \tag{3.4}$$

by Corollary 2.5.

It follows from Lemma 2.3 and Lemma 3.2 that

$$(n_1n_2)^{-1}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}\xi_1(n_1,t_2)=n_2^{-1}\sum_{t_2=1}^{n_2}\xi_1(n_1,t_2)\longrightarrow 0 \quad \text{as } \mathbf{n}\to\infty,$$

$$(n_1n_2)^{-1}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}\xi_1(0,t_2)=n_2^{-1}\sum_{t_2=1}^{n_2}\xi_1(0,t_2)\longrightarrow 0 \quad \text{as } \mathbf{n}\to\infty,$$

$$(n_1n_2)^{-1}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}\xi_2(t_1,n_2)=n_1^{-1}\sum_{t_1=1}^{n_2}\xi_2(t_1,n_2)\longrightarrow 0 \quad \text{as } \mathbf{n}\to\infty,$$

and

$$(n_1n_2)^{-1}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}\xi_2(t_1,0)=n_1^{-1}\sum_{t_1=1}^{n_2}\xi_2(t_1,0)\longrightarrow 0 \quad \text{as } \mathbf{n}\to\infty.$$

Finally, we have $\xi_{12}(0,n_2)\to 0$ a.s., $\xi_{12}(0,0)\to 0$ a.s., $\xi_{12}(n_1,0)\to 0$ a.s. and $\xi_{12}(n_1,n_2)\to 0$ a.s. as $\mathbf{n}\to\infty$.

Hence,

$$|\mathbf{n}|^{-1}R_n(t_1,t_2)\longrightarrow 0 \text{ a.s.,} \tag{3.5}$$

which implies

$$|\mathbf{n}|^{-1}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}X(t_1,t_2)\longrightarrow 0 \text{ a.s. as } \mathbf{n}\to\infty,$$

together with (3.3) and (3.4).

In the general case where $d>2$, the argument is analogous; we have

$$\sum_{\mathbf{1}\le\mathbf{t}\le\mathbf{n}}X(\mathbf{t})=A(\mathbf{1})\sum_{\mathbf{1}\le\mathbf{t}\le\mathbf{n}}\xi(\mathbf{t})+R_n(\mathbf{t}), \tag{3.6}$$

where

$$R_n(\mathbf{t})=\sum_{\mathbf{1}\le\mathbf{t}\le\mathbf{n}}\sum_{\substack{\gamma\in\Gamma_d\\ \gamma\neq\phi}}\{\Pi_{j\in\gamma}(B_j-1)\}A_\gamma(L_1,\dots,L_d)\xi(\mathbf{t}) \tag{3.7}$$

with L_i defined as in (2.9); note that, for $j \in \gamma$,

$$(3.8)\quad \begin{aligned}\sum_{t_j=1}^{n_j} (B_j - 1) A_\gamma(L_1, \ldots, L_d)\xi(\mathbf{t})&\\ &= \sum_{t_j=1}^{n_j} A_\gamma(L_1, \ldots, L_d)\xi(t_1, \ldots, t_j - 1, \ldots, t_d)\\ &\quad - \sum_{t_j=1}^{n_j} A_\gamma(L_1, \ldots, L_d)\xi(t_1, \ldots, t_d)\\ &= A_\gamma(L_1, \ldots, L_d)\xi(t_1, \ldots, 0, \ldots, t_d)\\ &\quad - A_\gamma(L_1, \ldots, L_d)\xi(t_1, \ldots, n_j, \ldots, t_d).\end{aligned}$$

Thus, the right-hand side of (3.7) can be written more explicitly as

$$(3.9)\quad \begin{aligned}&\sum_{t_2=1}^{n_2}\sum_{t_3=1}^{n_3}\cdots\sum_{t_d=1}^{n_d} A_1(B_1, \ldots, 1)\xi(0, \ldots, t_d)\\ &- \sum_{t_2=1}^{n_2}\sum_{t_3=1}^{n_3}\cdots\sum_{t_d=1}^{n_d} A_1(B_1, \ldots, 1)\xi(n_1, \ldots, t_d)\\ &+ \sum_{t_1=1}^{n_1}\sum_{t_3=1}^{n_3}\cdots\sum_{t_d=1}^{n_d} A_2(1, B_2, \ldots, 1)\xi(t_1, 0, \ldots, t_d)\\ &- \sum_{t_1=1}^{n_1}\sum_{t_3=1}^{n_3}\cdots\sum_{t_d=1}^{n_d} A_2(1, B_2, \ldots, 1)\xi(t_1, n_2, \ldots, t_d) + \cdots\\ &+ A_{12\cdots d}(B_1, \ldots, B_d)\xi(0, \ldots, 0) - A_{12\cdots d}(B_1, \ldots, B_d)\xi(0, \ldots, n_d)\\ &- A_{12\cdots d}(B_1, \ldots, B_d)\xi(n_1, \ldots, 0) + \cdots\\ &+ A_{12\cdots d}(B_1, \ldots, B_d)\xi(n_1, \ldots, n_d),\end{aligned}$$

where, in view of (3.8), the sums corresponding to each $A_\gamma(\cdot, \ldots, \cdot)$ run over t_i such that $i \notin \gamma$. Now

$$(3.10)\qquad |\mathbf{n}|^{-1} A(\mathbf{1}) \sum_{\mathbf{1}\le\mathbf{t}\le\mathbf{n}} \xi(\mathbf{t}) \longrightarrow 0 \text{ as } \mathbf{n} \to \infty,$$

by Corollary 2.5, so it is sufficient to prove that

$$(3.11)\qquad |\mathbf{n}|^{-1} R_n(\mathbf{t}) \longrightarrow 0 \text{ a.s. as } \mathbf{n} \to \infty.$$

Considering, for instance, the first term on the right-hand side of (3.9), note that the $\xi(0, t_2, \dots, t_d)$ are NA for different values of $t_2, \dots, t_d$. Thus, it follows from the same argument as for $d = 2$ and Corollary 2.5 that

$$
\begin{aligned}
|\mathbf{n}|^{-1} \sum_{t_1=1}^{n_1} \cdots \sum_{t_d=1}^{n_d} & A(B_1, \dots, 1)\xi(0, \dots, t_d) \\
&= \left(\prod_{k=2}^{d} n_k \right)^{-1} \sum_{t_2=1}^{n_2} \cdots \sum_{t_d=1}^{n_d} A(B_1, \dots, 1)\xi(0, \dots, t_d) \\
&\longrightarrow 0 \text{ a.s. as } \mathbf{n} \to \infty.
\end{aligned}
$$

More generally, for γ each other term in (3.11) is $(n_1 \times \cdots \times n_d)^{-1}$ times a partial sum of $(n_1 \times \cdots \times n_d)(\prod_{i \in \gamma} n_i)^{-1}$ NA elements, and we can iteratively apply the same argument to complete the proof.

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