ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 4, 2013

MULTIPLIERS ON L^p-SPACES FOR HYPERGROUPS

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ABSTRACT. Let K be a commutative hypergroup. At first, we characterize the space of multipliers on $L^p(K,m)$. Then, we investigate the multipliers on $L^1(\mathcal{S},\pi)$ and $L^2(\mathcal{S},\pi)$, where $\mathcal{S}$ is the dual space of K, i.e., $\mathcal{S} = \operatorname{supp}\pi$, π is the Plancherel measure of K.

1. Introduction. There are a lot of results on multipliers defined on translation-invariant Banach spaces on a locally compact group G. The standard reference for that is the book by Larsen [**6**]. In this paper we investigate multipliers in the hypergroup setting. Hypergroups generalize locally compact groups. For the theory of hypergroups, we refer to [**1, 5**]. A hypergroup K is a locally compact Hausdorff space with a convolution, i.e., a map $K\times K\to M^1(K)$, $(x,y)\mapsto \delta_x*\delta_y$, ($M^1(K)$ is the space of probability measures on K) and an involution, i.e., $K\to K$, $x\mapsto \widetilde{x}$, satisfying certain axioms, see [**1**].

Many results of harmonic analysis can be shown for hypergroups, in particular, for commutative hypergroups. In the following, we assume throughout that K is a commutative hypergroup. For a locally compact Hausdorff space X let $C(X)$, $C^b(X)$, $C_0(X)$ and $C_{00}(X)$ be the spaces of all continuous functions on X, those that are bounded, those that vanish at infinity and those that have compact support. $M(X)$ denotes the space of all regular complex Borel measures on X which can be identified with $C_0(X)^*$, the dual space of $C_0(X)$.

The convolution allows us to define a translation operator on $C(K)$ by setting

$$T_xf(y) = \int_K f(z)\,d(\delta_x*\delta_y)(z)$$

2010 AMS *Mathematics subject classification.* Primary 43A22, 43A62, Secondary 43A15.
Keywords and phrases. Hypergroups, multiplier, Fourier-Stieltjes transform.
The first author supported by the TUM Graduate School's Faculty Graduate Center International School of Applied Mathematics (ISAM) at Technische Universität München, Germany.
Received by the editors on September 28, 2010, and in revised form on January 18, 2011.
DOI:10.1216/RMJ-2013-43-4-1115 Copyright ©2013 Rocky Mountain Mathematics Consortium

for $f \in C(K)$. Spector [**12**] has proved that each commutative hypergroup possesses a Haar measure m, which is characterized by

$$\int_K T_x f(y)\, dm(y) = \int_K f(y)\, dm(y)$$

for all $x \in K$ and $f \in C_{00}(K)$. The Banach spaces $L^p(K,m)$, $1 \le p \le \infty$, are invariant under the translation actions T_x, $x \in K$, and under the convolution operators T_μ, $\mu \in M(K)$, where T_μ is defined by

$$T_\mu(f)(x) = \int_K T_{\tilde{y}} f(x)\, d\mu(y).$$

$f \in L^p(K,m)$, T_x and T_μ are bounded linear operators on $L^p(K,m)$, $1 \le p \le \infty$, and $\|T_\mu(f)\|_p \le \|\mu\|\, \|f\|_p$. Moreover, $L^1(K,m)$ is a Banach $*$-algebra which is an ideal in $M(K)$, where we embed $L^1(K,m)$ into $M(K)$ by $f \mapsto fm$. In particular, $L^1(K,m)$ acts on $L^p(K,m)$ via

$$g * f(x) = T_{gm}(f)(x) = \int_K g(y) T_{\tilde{y}} f(x)\, dm(y)$$

for $g \in L^1(K,m)$, $f \in L^p(K,m)$, $1 \le p \le \infty$.

There exists a net $(g_i)_{i \in I}$ of functions $g_i \in C_{00}(K)$, $g_i \ge 0$,

$$\int_K g_i(x)\, dm(x) = 1$$

such that $\lim_i \|g_i * f - f\|_p = 0$ for all $f \in L^p(K,m)$, $1 \le p < \infty$, see [**11**, Lemma 1].

The symmetric structure space of the commutative Banach $*$-algebra $L^1(K,m)$ can be identified with

$$\begin{aligned}\widehat{K} = \{\alpha \in C^b(K) : \alpha(e) = 1,\ T_x\alpha(y) = \alpha(x)\alpha(y) \\ \text{and } \alpha(\widetilde{x}) = \overline{\alpha(x)} \text{ for all } x, y \in K\},\end{aligned}$$

where $\widehat{K}$ is equipped with the compact-open topology which is equal to the Gelfand topology. The Fourier transform of $f \in L^1(K,m)$ (the Fourier-Stieltjes transform of $\mu \in M(K)$) is defined by

$$\widehat{f}(\alpha) = \int_K f(x)\overline{\alpha(x)}\, dm(x) \qquad \left(\widehat{\mu}(\alpha) = \int_K \overline{\alpha(x)}\, d\mu(x) \text{ respectively}\right)$$

for $\alpha \in \widehat{K}$. $\widehat{f}$ and $\widehat{\mu}$ are bounded continuous complex-valued functions on $\widehat{K}$ and $\widehat{f}$ vanishes at infinity. Considering the Hilbert space $L^2(K,m)$ there exists a positive Borel measure π on $\widehat{K}$ (called a *Plancherel measure*), such that

$$\int_K |f(x)|^2\, dm(x) = \int_{\hat{K}} |\widehat{f}(\alpha)|^2\, d\pi(\alpha)$$

for all $f \in L^1(K,m) \cap L^2(K,m)$. We emphasize that (in contrast to the group case) $\widehat{K}$ does, in general, not bear a dual hypergroup structure. Moreover, whereas $\operatorname{supp} m = K$, the support of π can be a proper closed subset of $\widehat{K}$. We denote $\operatorname{supp}\pi$ by $\mathcal{S}$. The extension of the Fourier transform from $L^1(K,m) \cap L^2(K,m)$ to $L^2(K,m)$ is called the *Plancherel transform*. We denote the Plancherel transform of $f \in L^2(K,m)$ by $\mathcal{P}(f)$. The Plancherel transform is an isometric isomorphism from $L^2(K,m)$ onto $L^2(\mathcal{S},\pi)$ and, for $f,g \in L^2(K,m)$, the following holds:

$$\int_K f(x)\overline{g(x)}\, dm(x) = \int_{\mathcal{S}} \mathcal{P}f(\alpha)\overline{\mathcal{P}g(\alpha)}\, d\pi(\alpha),$$

and hence,

$$\int_K f(x)g(x)\, dm(x) = \int_{\mathcal{S}} \mathcal{P}f(\alpha)\mathcal{P}g(\overline{\alpha})\, d\pi(\alpha)$$

(Parseval's formula).

The inverse Fourier transform of $f \in L^1(\mathcal{S},\pi)$ (the inverse Fourier-Stieltjes transform of $\mu \in M(\widehat{K})$) is defined by

$$\check{f}(x) = \int_{\mathcal{S}} f(\alpha)\alpha(x)\, d\pi(\alpha) \qquad \left(\check{\mu}(x) = \int_{\hat{K}} \alpha(x)\, d\mu(\alpha) \text{ respectively}\right)$$

for $x \in K$.

$\check{f}$ and $\check{\mu}$ are bounded continuous complex-valued functions on K, and $\check{f}$ vanishes at infinity. An inversion theorem holds. That means if $f \in L^1(K,m)$ and $\widehat{f} \in L^1(\mathcal{S},\pi)$, then $f = (\widehat{f})^\vee$ with equality in $L^1(K,m)$. We will also use an inverse uniqueness theorem: If $\mu \in M(\widehat{K})$ and $\check{\mu} = 0$, then $\mu = 0$.

As a special class of hypergroups we will deal with polynomial hypergroups. In that case K is equal to $\mathbf{N}_0$ equipped with the discrete topology. The convolution is generated by an orthogonal polynomial system $(R_n(t))_{n\in\mathbf{N}_0}$ on the real axis, see [**8, 9**], which has nonnegative linearization coefficients $g(m,n;k)$ of the product $R_m(t)R_n(t)$, i.e.,

$$R_m(t)R_n(t) = \sum_{k=|n-m|}^{n+m} g(m,n;k)R_k(t).$$

Furthermore, we assume that $R_n(1) = 1$. Then, putting

$$\delta_m * \delta_n = \sum_{k=|n-m|}^{n+m} g(m,n;k)\,\delta_k,$$

a convex combination of the point measures δ_k, we get a convolution on $\mathbf{N}_0$. Together with $\widetilde{n} = n$ as the involution and $n = 0$ as the unit, this convolution defines a hypergroup structure on $\mathbf{N}_0$. In this way, every orthogonal polynomial system $(R_n(t))_{n\in\mathbf{N}_0}$ with $g(m,n;k) \geq 0$ and normalized by $R_n(1) = 1$ generates a hypergroup on $\mathbf{N}_0$.

There are a lot of orthogonal polynomial systems enjoying $g(m,n;k) \geq 0$, see [**1, 8, 9**]. The Haar measure on the polynomial hypergroup $\mathbf{N}_0$ is the counting measure with weights $h(n) = g(n,n;0)^{-1}$ at the points $n \in \mathbf{N}_0$. The symmetric structure space of the Banach $*$-algebra $l^1(\mathbf{N}_0, h)$ can be identified with the set

$$D_s = \{t \in \mathbf{R} : |R_n(t)| \leq 1 \text{ for all } n \in \mathbf{N}_0\}$$

via the mapping $t \mapsto \alpha_t$, $\alpha_t(n) = R_n(t)$, see [**1, 8**]. Hence, we consider $\widehat{\mathbf{N}_0}$ as a compact subset of $\mathbf{R}$ that contains $t = 1 \in \mathbf{R}$. The Plancherel measure π on D_s is exactly the orthogonalization measure of the orthogonal polynomial system $(R_n(t))_{n\in\mathbf{N}_0}$. Notice that $\mathcal{S} = \operatorname{supp}\pi \subseteq D_s$, and so the orthogonalization measure is (up to a multiplicative constant) uniquely determined. Important examples of polynomial hypergroups are generated by Jacobi polynomials $(R_n^{(\alpha,\beta)}(t))_{n\in\mathbf{N}_0}$, which are orthogonal with respect to $d\pi(t) = (1-t)^\alpha(1+t)^\beta\chi_{[-1,1]}(t)\,dt$. If $\alpha \geq \beta > -1$ and $\alpha+\beta+1 \geq 0$, then $g(m,n;k) \geq 0$ and hence, in that case, $(R_n^{(\alpha,\beta)}(t))_{n\in\mathbf{N}_0}$ generates a polynomial hypergroup on

$\mathbf{N}_0$. In fact, the parameter region for the parameters (α, β) such that $g(m, n; k)$ are nonnegative is a little bit larger. We notice that, for $\alpha \geq \beta > -1$ and either $\beta \geq -1/2$ or $\alpha + \beta \geq 0$, the symmetric structure space $D_s = [-1, 1] = \mathcal{S}$ bears a dual hypergroup structure, see [**1, 8**]. Furthermore the Jacobi polynomials (with (α, β) from the region above) are exactly those orthogonal polynomials with this property, see [**1**, Corollary 3.6.3]. The case $\alpha = \beta$ corresponds to the ultraspherical (or Gegenbauer) polynomials.

2. Multipliers on $L^p(K, m)$. A *multiplier* $T : L^p(K, m) \to L^p(K, m)$, $1 \leq p < \infty$, is a bounded linear operator from $L^p(K, m)$ into $L^p(K, m)$ (i.e., $T \in B(L^p(K, m))$, that commutes with all translation operators T_x, $x \in K$, i.e., $T \circ T_x = T_x \circ T$. We denote the space of multipliers on $L^p(K, m)$ by $M(L^p(K, m))$. We briefly recall results already shown for multipliers on hypergroups. In [**7**] one of the authors already generalized Wendel's classical result and Helson's result for multipliers on $L^1(K, m)$ in the case of K being commutative, see [**7**, Corollary 2.2]. For the sake of completeness, the characterizations of $T \in M(L^1(K, m))$ are formulated here again.

Theorem 1. *Let K be a commutative hypergroup, $T \in B(L^1(K, m))$. The following conditions are equivalent*:

(i) $T \in M(L^1(K, m))$.

(ii) $T(f) * g = T(f * g) = f * T(g)$ *for all* $f, g \in L^1(K, m)$.

(iii) *There exists a unique measure* $\mu \in M(K)$ *such that*

$$T(f) = \mu * f \quad \textit{for all} f \in L^1(K, m).$$

(iv) *There exists a unique measure* $\mu \in M(K)$ *such that*

$$\widehat{T(f)}|\mathcal{S} = \widehat{\mu}\widehat{f}|\mathcal{S} \quad \textit{for all } f \in L^1(K, m).$$

(v) *There exists a unique function* $\varphi \in C(\mathcal{S})$ *such that*

$$\widehat{T(f)}|\mathcal{S} = \varphi\widehat{f}|\mathcal{S} \quad \textit{for all } f \in L^1(K, m).$$

Moreover,

$$\|\varphi\|_{\mathcal{S}} \leq \|\mu\| = \|T\|.$$

The correspondence between T and μ defines an isometric algebra isomorphism from $M(L^1(K,m))$ onto $M(K)$.

The proof of the equivalence of (i)–(v) is in [**7**]. The statement on the norm is standard, compare Corollary 0.1.1. of [**6**]. It should be noted that the implication (v) $\Rightarrow$ (iv) actually establishes that φ is a bounded function.

In [**3**] the equivalence of (i) and (iii) in Theorem 1 above is shown for general hypergroups and weights on the measure algebra of K, see [**3**, Proposition 1]. The main topic of [**3**] is on compact multipliers. In [**11**], multipliers from $L^1(K,m)$ into $L^p(K,m)$, $1 < p \leq \infty$, are characterized as convolutors by some $f \in L^p(K,m)$, see [**11**, Theorem 6]. Moreover, compact multipliers are investigated in [**11**], too.

Now we consider $p > 1$. Denote by

$$A(K) := \{\check{\varphi} : \varphi \in L^1(\mathcal{S},\pi)\}.$$

By the uniqueness theorem for the inverse Fourier transformation $\|\check{\varphi}\|_A := \|\varphi\|_1$ is a norm on $A(K)$, see [**1**, Theorem 2.2.35]. With this norm, $A(K)$ is a Banach space. The space of all continuous linear functionals on $A(K)$ is denoted by $P(K)$, the elements σ of $P(K)$ are called pseudomeasures on K and $\|\sigma\|_P = \sup\{|\sigma(\check{\varphi})| : \|\check{\varphi}\|_A \leq 1\}$ is the norm on the dual space $P(K)$. $P(K)$ is isometrically isomorphic to $L^\infty(\mathcal{S},\pi)$, the space of π-essentially bounded Borel functions on K, where the functions are identified which differ only on locally π-zero sets. (We recall that the dual space of $L^1(\mathcal{S},\pi)$ is isometrically isomorphic to $L^\infty(\mathcal{S},\pi)$, see [**4**, Theorem 12.18].) The mapping $\Psi : P(K) \to L^\infty(\mathcal{S},\pi)$ where, for each $\sigma \in P(K)$, the element $\Psi(\sigma) \in L^\infty(\mathcal{S},\pi)$ is uniquely determined by

$$\int_{\mathcal{S}} \varphi(\alpha)\Psi(\sigma)(\overline{\alpha})\, d\pi(\alpha) = \sigma(\check{\varphi}) \quad \text{for } \varphi \in L^1(\mathcal{S},\pi),$$

defines an isometric isomorphism Ψ from the Banach space $P(K)$ onto $L^\infty(\mathcal{S},\pi)$. A convolution of $\sigma_1, \sigma_2 \in P(K)$ is determined by $\sigma_1 * \sigma_2 = \Psi^{-1}(\Psi(\sigma_1)\Psi(\sigma_2))$, so that Ψ is also an algebra isomorphism from $P(K)$ onto $L^\infty(\mathcal{S},\pi)$. The proof of these facts is exactly as in [**6**, Theorem 4.2.2]. We shall call $\Psi(\sigma)$ the Fourier transform of $\sigma \in P(K)$. If $\mu \in M(K)$, then

$$\begin{aligned}\int_K \check{\varphi}(x)\,d\mu(x) &= \int_K \int_{\mathcal{S}} \alpha(x)\,\varphi(\alpha)\,d\pi(\alpha)\,d\mu(x)\\ &= \int_{\mathcal{S}} \varphi(\alpha)\widehat{\mu}(\overline{\alpha})\,d\pi(\alpha)\end{aligned}$$

for all $\varphi \in L^1(\mathcal{S},\pi)$. Hence, each measure $\mu \in M(K)$ is a pseudomeasure and $\widehat{\mu} = \Psi(\mu)$. Moreover, we have $\|\mu\|_P = \|\widehat{\mu}\|_\infty \le \|\mu\|$. Furthermore,

$$\int_{\mathcal{S}} \varphi(\alpha)\Psi(\mu_1{*}\mu_2)(\overline{\alpha})\,d\pi(\alpha) = \int_K \check{\varphi}(x)\,d\mu_1{*}\mu_2(x) \quad \text{for all } \varphi \in L^1(\mathcal{S},\pi).$$

Hence, the convolution $\mu_1 * \mu_2$ of two measures $\mu_1, \mu_2 \in M(K)$ agrees with the convolution of μ_1 and μ_2 seen as pseudomeasures. Obviously, we have

$$\Psi(\sigma_1 * \sigma_2) = \Psi(\sigma_1)\Psi(\sigma_2) \quad \text{for } \sigma_1, \sigma_2 \in P(K).$$

The above conclusions are summarized as follows.

Proposition 1. *Let K be a commutative hypergroup. Then the Fourier transform $\Psi : P(K) \to L^\infty(\mathcal{S},\pi)$ determined by*

$$\int_{\mathcal{S}} \varphi(\alpha)\Psi(\sigma)(\overline{\alpha})\,d\pi(\alpha) = \sigma(\check{\varphi}) \quad \textit{for all } \varphi \in L^1(\mathcal{S},\pi),$$

is an isometric algebra isomorphism of $P(K)$ onto $L^\infty(\mathcal{S},\pi)$.

We shall say that a pseudomeasure $\sigma \in P(K)$ belongs to $L^2(K,m)$ if there is a $g \in L^2(K,m)$ such that

$$\sigma(\check{\varphi}) = \int_K \check{\varphi}(x)g(x)\,dm(x) \quad \text{for all } \varphi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi).$$

Since $\{\check{\varphi} : \varphi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)\}$ is dense in $L^2(K,m)$, the g is uniquely determined. If $\sigma \in P(K)$ belongs to $L^2(K,m)$ and g is the corresponding element from $L^2(K,m)$, then Parseval's formula yields, for $\varphi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$,

$$\begin{aligned}\int_{\mathcal{S}} \varphi(\alpha)\Psi(\sigma)(\overline{\alpha})\, d\pi(\alpha) &= \sigma(\check{\varphi}) \\ &= \int_K \check{\varphi}(x)g(x)\, dm(x) \\ &= \int_{\mathcal{S}} \varphi(\alpha)\mathcal{P}g(\overline{\alpha})\, d\pi(\alpha).\end{aligned}$$

Hence, we conclude $\Psi(\sigma) = \mathcal{P}g \in L^2(\mathcal{S},\pi) \cap L^\infty(\mathcal{S},\pi)$, that is, the Fourier transform of the pseudomeasure σ agrees with the Plancherel transform of g.

Conversely, let $\sigma \in P(K)$ be such that $\Psi(\sigma) \in L^2(\mathcal{S},\pi) \cap L^\infty(\mathcal{S},\pi)$. Then σ belongs to $L^2(K,m)$. Indeed, putting $g = \mathcal{P}^{-1}(\Psi(\sigma)) \in L^2(K,m)$, we obtain, using Parseval's formula,

$$\sigma(\check{\varphi}) = \int_{\mathcal{S}} \varphi(\alpha)\Psi(\sigma)(\overline{\alpha})\, d\pi(\alpha) = \int_K \check{\varphi}(x)g(x)\, dm(x)$$

for all $\varphi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$.

We may summarize the latter discussion in the following proposition.

Proposition 2. *Let K be a commutative hypergroup. A pseudomeasure $\sigma \in P(K)$ belongs to $L^2(K,m)$ if and only if $\Psi(\sigma) \in L^2(\mathcal{S},\pi) \cap L^\infty(\mathcal{S},\pi)$. Moreover, the Fourier transform of σ as a pseudomeasure coincides with the Plancherel transform of the corresponding $g \in L^2(K,m)$.*

It should be noted that every $g \in L^1(K,m) \cap L^2(K,m)$ determines a pseudomeasure $\sigma \in P(K)$ such that $\sigma(\check{\varphi}) = \int_K \check{\varphi}(x)g(x)\, dm(x)$ is true for all $\varphi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$. Indeed, put $\sigma = \Psi^{-1}(\widehat{g})$. Then

$$\begin{aligned}\Psi^{-1}(\widehat{g})(\check{\varphi}) &= \int_{\mathcal{S}} \varphi(\alpha)\Psi(\Psi^{-1}(\widehat{g}))(\overline{\alpha})\, d\pi(\alpha) \\ &= \int_{\mathcal{S}} \varphi(\alpha)\mathcal{P}g(\overline{\alpha})\, d\pi(\alpha) \\ &= \int_K \check{\varphi}(x)g(x)\, dm(x).\end{aligned}$$

In particular, the convolution $\sigma * g = \Psi^{-1}(\Psi(\sigma)\widehat{g})$ of $\sigma \in P(K)$ and $g \in L^1(K,m) \cap L^2(K,m)$ is well defined as a convolution of pseudomeasures.

If $\sigma \in P(K)$ belongs to $L^2(K,m)$ we will further designate the corresponding element of $L^2(K,m)$ also by σ.

In order to establish a complete characterization of the multipliers for $L^p(K,m)$, we derive that the space $M(L^p(K,m))$ coincides with the space of convolutors of $L^p(K,m)$, where $T \in B(L^p(K,m))$ is called *convolutor* if $T(f * g) = f * T(g)$ for all $f, g \in L^1(K,m) \cap L^p(K,m)$.

Proposition 3. *Let K be a commutative hypergroup and $T \in B(L^p(K,m))$. Then T is an element of $M(L^p(K,m))$ if and only if*

$$T(f) * g = T(f * g) = f * T(g) \quad \text{for all } f, g \in L^1(K,m) \cap L^p(K,m).$$

Proof. Basically, with the arguments used for the proof of Theorem 4.1.1 in [**6**], we obtain that for $T \in M(L^p(K,m))$ we have $T(f) * g = T(f * g) = f * T(g)$ for all $f, g \in L^1(K,m) \cap L^p(K,m)$.

Conversely, choose $f_i \in C_{00}(K)$, $i \in I$, with $\lim_i \|f_i * g - g\|_p = 0$. Since $T_x f_i * g = T_x(f_i * g)$, we obtain for a convolutor T and any $g \in L^1(K,m) \cap L^p(K,m)$

$$T(T_x f_i * g) = T_x f_i * T(g) = f_i * T_x T(g) \longrightarrow T_x T(g)$$

and, by the continuity of T and T_x,

$$T(T_x f_i * g) = T(T_x(f_i * g)) \longrightarrow T\,T_x(g)$$

for all $g \in L^1(K,m) \cap L^p(K,m)$. Now it is obvious that $T \in M(L^p(K,m))$. $\square$

Now we can give the following description of the multipliers for $L^2(K,m)$.

Theorem 2. *Let K be a commutative hypergroup, $T \in B(L^2(K,m))$. The following conditions are equivalent*:

(i) $T \in M(L^2(K,m))$.

(ii) $T(f)*g = T(f*g) = f*T(g)$ *for all* $f, g \in L^1(K,m) \cap L^2(K,m)$.

(iii) *There exists a unique pseudomeasure* $\sigma \in P(K)$ *such that* $\sigma * f$ *belongs to* $L^2(K,m)$ *and*

$$T(f) = \sigma * f \quad \textit{for all } f \in L^1(K,m) \cap L^2(K,m).$$

(iv) *There exists a unique pseudomeasure* $\sigma \in P(K)$ *such that*

$$\mathcal{P}(T(f)) = \Psi(\sigma)\mathcal{P}(f) \quad \textit{for all } f \in L^2(K,m).$$

(v) *There exists a unique* $\varphi \in L^\infty(\mathcal{S},\pi)$ *such that*

$$\mathcal{P}(T(f)) = \varphi\mathcal{P}(f) \quad \textit{for all } f \in L^2(K,m).$$

Moreover, $\|\varphi\|_{\mathcal{S}} = \|\sigma\|_P = \|T\|$.

The correspondence between T, σ and φ defines isometric algebra isomorphisms between $M(L^2(K,m))$, $P(K)$ and $L^\infty(\mathcal{S},\pi)$.

Proof. (i) $\Leftrightarrow$ (ii) is already shown in Proposition 3.

(v) $\Rightarrow$ (i). We have

$$\widehat{T_x f}(\alpha) = \alpha(x)\widehat{f}(\alpha) = \widehat{\varepsilon_{\tilde{x}}}(\alpha)\widehat{f}(\alpha) \quad \text{for all } f \in L^1(K,m),\ \alpha \in \widehat{K},$$

and hence $\mathcal{P}(T_x f) = \widehat{\varepsilon_{\tilde{x}}}\mathcal{P}(f)$ for all $f \in L^2(K,m)$. Therefore, it follows by (v) that

$$\begin{aligned} \mathcal{P}(T_x(T(f))) &= \widehat{\varepsilon_{\tilde{x}}}\mathcal{P}(T(f)) = \widehat{\varepsilon_{\tilde{x}}}\varphi\mathcal{P}(f) \\ &= \varphi\mathcal{P}(T_x(f)) = \mathcal{P}(T(T_x(f))) \end{aligned}$$

for all $f \in L^2(K,m)$, and hence $T \in M(L^2(K,m))$.

(i) $\Rightarrow$ (v). (The proof follows the lines of [**6**, Theorem 4.1.1] for groups.) (i) is equivalent to (ii). Hence, we have

$$\mathcal{P}(T(f))\widehat{g} = \widehat{f}\mathcal{P}(T(g)) \tag{$*$}$$

for all $f, g \in C_{00}(K)$. For $\alpha \in \mathcal{S}$, choose $f \in C_{00}(K)$ such that $\widehat{f}(\alpha)$ is nonzero on a neighborhood of α, and define on this neighborhood $\varphi = \mathcal{P}(T(f))/\widehat{f}$. By identity $(*)$, φ is independent of the choice of f, and it follows that there is a unique locally measurable function φ on $\mathcal{S}$ such that $\mathcal{P}(T(f)) = \varphi\widehat{f}$ for all $f \in C_{00}(K)$. If $f \in L^2(K,m)$, there exists a sequence $(f_n)_{n\in\mathbf{N}}$, $f_n \in C_{00}(K)$, such that $\lim_{n\to\infty} \|f - f_n\|_2 = 0$. Then $\lim_{n\to\infty} \|\mathcal{P}(T(f)) - \mathcal{P}(T(f_n))\|_2 = 0$ and, replacing $(f_n)_{n\in\mathbf{N}}$ by a subsequence, we can suppose that $\widehat{f_n} \to \mathcal{P}(f)$ π-almost everywhere and $\mathcal{P}(T(f_n)) \to \mathcal{P}(T(f))$ π-almost everywhere. It follows that $\mathcal{P}(T(f)) = \varphi\mathcal{P}(f)$ π-almost everywhere.

It remains to prove that $\varphi \in L^\infty(\mathcal{S},\pi)$. (We will even show that $\|\varphi\|_{\mathcal{S}} \le \|T\|$). Assume, on the contrary, that there is a compact subset $C \subseteq \mathcal{S}$ such that $\pi(C) > 0$ and $|\varphi(\alpha)| > \|T\|$ for π-almost all $\alpha \in C$. Put $g \in L^2(K,m)$ such that $\mathcal{P}(g) = \chi_C$. Then $\|\varphi\chi_C\|_2 > \|T\|(\pi(C))^{1/2}$ and, on the other hand,

$$\begin{aligned}\|\varphi\chi_C\|_2 &= \|\varphi\mathcal{P}(g)\|_2 = \|\mathcal{P}(T(g))\|_2 \\ &= \|T(g)\|_2 \le \|T\|\|g\|_2 = \|T\|(\pi(C))^{1/2},\end{aligned}$$

a contradiction. Hence, $\varphi \in L^\infty(\mathcal{S},\pi)$ and $\|\varphi\|_{\mathcal{S}} \le \|T\|$. In addition, we have $\|T(g)\|_2 = \|\mathcal{P}(T(g))\|_2 = \|\varphi\mathcal{P}(g)\|_2 \le \|\varphi\|_{\mathcal{S}}\|g\|_2$, and we get $\|T\| \le \|\varphi\|_{\mathcal{S}}$.

(iv) $\Leftrightarrow$ (v). The equivalence of (iv) and (v) is true by Proposition 1.

(iv) $\Rightarrow$ (iii). We have already shown that the convolution $\sigma * f$ of $\sigma \in P(K)$ and $f \in L^1(K,m) \cap L^2(K,m)$ yields a pseudomeasure. For pseudomeasures σ enjoying property (iv), we have

$$\Psi(\sigma * f) = \Psi(\sigma)\widehat{f} = \mathcal{P}(T(f))$$

for all $f \in L^1(K,m) \cap L^2(K,m)$. Hence, $\Psi(\sigma * f) \in L^2(\mathcal{S},\pi) \cap L^\infty(\mathcal{S},\pi)$ and (iii) follows by Proposition 2.

(iii) $\Rightarrow$ (iv). Using Proposition 2 once again, we conclude

$$\mathcal{P}(T(f)) = \mathcal{P}(\sigma * f) = \Psi(\sigma * f) = \Psi(\sigma)\widehat{f}$$

for all $f \in L^1(K,m) \cap L^2(K,m)$. Since T is continuous and $L^1(K,m) \cap L^2(K,m)$ is dense in $L^2(K,m)$, statement (iv) is shown. ◻

Now we investigate multipliers on $L^p(K,m)$, $p \neq 1$, $p \neq 2$. Basically, with the same arguments used for abelian groups, we obtain inclusion results for $M(L^p(K,m))$. Following the proof of [**6**, Theorem 4.1.2], we get:

Proposition 4. *Let K be a commutative hypergroup, $1 < p < \infty$, $1/p + 1/q = 1$. Then there exists an isometric algebra isomorphism of $M(L^p(K,m))$ onto $M(L^q(K,m))$.*

Using a form of the Riesz-Thorin convexity theorem we can derive just as in [**6**, Theorem 4.1.3, Corollary 4.1.3] the next result.

Proposition 5. *Let K be a commutative hypergroup, $1 < p \leq r \leq 2$. Then there exists a norm-decreasing algebra isomorphism J from $M(L^p(K,m))$ into $M(L^r(K,m))$. Moreover, J can be interpreted as an inclusion, since $J(T)$, $T \in M(L^p(K,m))$, is the unique extension to $L^r(K,m)$ of the restriction $T|L^p(K,m) \cap L^r(K,m)$.*

With a slight abuse of terminology, we have the following inclusions for $1 \leq p < r \leq 2$:

$$\begin{aligned} M(K) &\cong M(L^1(K,m)) \subseteq M(L^p(K,m)) \\ &\subseteq M(L^r(K,m)) \subseteq M(L^2(K,m)) \\ &\cong P(K), \end{aligned}$$

and the inclusion mappings are norm-decreasing.

We now apply Theorem 1 and Theorem 2, respectively, to polynomial hypergroups. In that case $K = \mathbf{N}_0$ is discrete. Hence, the measure space $M(\mathbf{N}_0)$ can be identified with $l^1(\mathbf{N}_0)$. Moreover, $\mathcal{S} = \operatorname{supp} \pi$ is compact. Thus, $L^\infty(\mathcal{S},\pi) \subseteq L^2(\mathcal{S},\pi)$, and so every pseudomeasure $\sigma \in P(\mathbf{N}_0)$ belongs to $l^2(\mathbf{N}_0, h)$.

Let $(\tau(m,n))_{m,n\in\mathbf{N}_0}$ be an infinite matrix of complex numbers. Consider the linear transformations T defined by

$$Tf(m) = g(m) = \sum_{n=0}^{\infty} \tau(m,n) f(n) h(n). \tag{1}$$

Then $g = (g(m))_{m\in\mathbf{N}_0}$ is defined at least whenever $f = (f(n))_{n\in\mathbf{N}_0} \in l_{fin}$, i.e., $f(n) \neq 0$ for at most finitely many $n \in \mathbf{N}_0$. We begin with a simple observation.

Lemma 1. *Define T by* (1) *for $f \in l_{fin}$. Then we have $T\circ T_n = T_n\circ T$ for all $n \in \mathbf{N}_0$ if and only if $\tau(m,n) = T_n\sigma(m)$ for all $m, n \in \mathbf{N}_0$, where $\sigma(k) = \tau(k,0)$.*

Proof. Let $\varepsilon_l(k) = \delta_{k,l}$ for $k, l \in \mathbf{N}_0$. Then $T_n\varepsilon_0(k) = 0$ for $k \neq n$ and $T_n\varepsilon_0(n) = 1/h(n)$, and $T\varepsilon_0(k) = \tau(k,0) = \sigma(k)$. Therefore, $T \circ T_n\varepsilon_0(m) = \tau(m,n)$ and $T_n \circ T\varepsilon_0(m) = T_n\sigma(m)$. It follows that $T \circ T_n = T_n \circ T$ implies $\tau(m,n) = T_n\sigma(m)$. Conversely, $\tau(m,n) = T_n\sigma(m)$ yields $T \circ T_n\varepsilon_0 = T_n \circ T\varepsilon_0$, and then

$$T \circ T_n\varepsilon_l = h(l)T \circ T_n \circ T_l\varepsilon_0 = h(l)T_l \circ T_n \circ T\varepsilon_0 = T_n \circ T\epsilon_l,$$

for each $l \in \mathbf{N}_0$. Thus. the converse implication is also true. $\square$

In view of Lemma 1, we confine our attention to the case $\tau(m,n) = T_n\sigma(m)$, where $\sigma = (\sigma(k))_{k\in\mathbf{N}_0}$. Notice that $\tau(m,n) = T_n\sigma(m) = T_m\sigma(n) = \tau(n,m)$. From Theorems 1 and 2 we can conclude the following characterizations.

Proposition 6. *Let $K = \mathbf{N}_0$ be a polynomial hypergroup.*

(i) *A necessary and sufficient condition that T is a bounded linear operator from $l^1(\mathbf{N}_0, h)$ into $l^1(\mathbf{N}_0, h)$ is that $\sigma = (\sigma(n))_{n\in\mathbf{N}_0} \in l^1(\mathbf{N}_0)$.*

(ii) *A necessary and sufficient condition that T is a bounded linear operator from $l^2(\mathbf{N}_0, h)$ into $l^2(\mathbf{N}_0, h)$ is that $\sigma = (\sigma(n))_{n\in\mathbf{N}_0}$ is given by $\sigma(n) = \check{f}(n)$ for some $f \in L^\infty(\mathcal{S}, \pi)$.*

One should compare Proposition 6 (ii) with [**13**, Chapter IV, Theorem 9.18].

Considering compact hypergroups, we want to point out that Theorem 2 generalizes Theorem 1.1, Corollary 3.1 and Lemma 4.1 of [**2**], where the dual hypergroup $K = [-1, 1]$ of the ultraspherical polynomials $R_n^{(\alpha,\alpha)}(t)$ is investigated.

3. Multiplier for $L^1(\mathcal{S},\pi)$ and $L^2(\mathcal{S},\pi)$. The dual spaces $\widehat{K}$ or $\mathcal{S}$, in general, do not bear a dual hypergroup structure, and hence hypergroup-translation-operators do not exist on $\mathcal{S}$. Nevertheless, using the inverse Fourier transform we can characterize those functions f on K such that $f\check{\varphi} \in L^1(\mathcal{S},\pi)^\vee$ for all $\varphi \in L^1(\mathcal{S},\pi)$. Even more, applying the Plancherel transform we can define a translation operator for $L^2(\mathcal{S},\pi)$ and derive five equivalent conditions for multipliers on $L^2(\mathcal{S},\pi)$.

Let f be a continuous function on K such that $f\check{\varphi} \in L^1(\mathcal{S},\pi)^\vee$ for all $\varphi \in L^1(\mathcal{S},\pi)$. Then f is called a *multiplier* for $L^1(\mathcal{S},\pi)$. Since the inverse Fourier transform is one-to-one, defining $T\varphi \in L^1(\mathcal{S},\pi)$ by the equation $(T\varphi)^\vee = f\check{\varphi}$ determines a well-defined linear operator from $L^1(\mathcal{S},\pi)$ to $L^1(\mathcal{S},\pi)$. Moreover, T is a closed mapping. In fact, if $\lim_{n\to\infty} \|\varphi_n - \varphi\|_1 = 0$ and $\lim_{n\to\infty} \|T\varphi_n - \kappa\|_1 = 0$ for $\varphi_n, \varphi, \kappa \in L^1(\mathcal{S},\pi)$, then $\check{\kappa}(x) = \lim_{n\to\infty}(T\varphi_n)^\vee(x) = f(x)\lim_{n\to\infty}\check{\varphi}_n(x) = f(x)\check{\varphi}(x)$ for all $x \in K$. From the closed graph theorem we conclude that T is continuous.

Lemma 2. *Let K be a commutative hypergroup, and assume that $1 \in \mathcal{S} = \operatorname{supp}\pi$. Then there exists a net $(k_i)_{i\in I}$ of functions $k_i \in C_{00}(\mathcal{S})$ such that $k_i \geq 0$, $\|k_i\|_1 = 1$ and $\check{k}_i(x) \to 1$ for all $x \in K$.*

Proof. Consider

$$V_{\varepsilon,C} = \{\alpha \in \mathcal{S} : |\alpha(x) - 1| < \varepsilon \quad \text{for all } x \in C\},$$

where $C \subseteq K$ is compact and $\varepsilon > 0$ is a member of the neighborhood basis of $1 \in \mathcal{S}$. Put $k_{\varepsilon,C} = \chi_{V_{\varepsilon,C}}/\pi(V_{\varepsilon,C})$, introduce a corresponding index set $i \in I$ with the usual order and observe that $k^\vee_{\varepsilon,C}(x) \to 1$, whenever $x \in C$ and $\varepsilon \to 0$. Now the statement follows. ◻

Theorem 3. *Let K be a commutative hypergroup, and assume that $1 \in \mathcal{S}$. If $f \in C(K)$ is a multiplier for $L^1(\mathcal{S},\pi)$, i.e., $f\check{\varphi} \in L^1(\mathcal{S},\pi)^\vee$ for all $\varphi \in L^1(\mathcal{S},\pi)$, then there is a unique measure $\mu \in M(\widehat{K})$ such that $\check{\mu} = f$.*

Proof. We already know that the linear operator T determined by $f\check{\varphi} = (T\varphi)^\vee$ is continuous, and so there is some $M \geq 0$ with

$\|T\varphi\|_1 \leq M\|\varphi\|_1$ for all $\varphi \in L^1(\mathcal{S},\pi)$. Consider a net $(k_i)_{i\in I}$ as in Lemma 2, and let $h \in C_{00}(K)$. Then

$$\int_K f(x)h(\widetilde{x})\,dm(x) = \lim_i \int_K (f\,\check{k}_i)(x)h(\widetilde{x})\,dm(x)$$

by Lebesgue's theorem of dominated convergence. We have with Fubini's theorem

$$\begin{aligned}\int_K (f\,\check{k}_i)(x)h(\widetilde{x})\,dm(x) &= \int_K (Tk_i)^{\vee}(x)h(\widetilde{x})\,dm(x)\\ &= \int_K\int_{\mathcal{S}} Tk_i(\alpha)\alpha(x)\,d\pi(\alpha)h(\widetilde{x})\,dm(x)\\ &= \int_{\mathcal{S}} Tk_i(\alpha)\widehat{h}(\alpha)\,d\pi(\alpha),\end{aligned}$$

and then

$$\left|\int_K (f\check{k}_i)(x)h(\widetilde{x})\,dm(x)\right| \leq \|\widehat{h}\|_{\mathcal{S}}\|Tk_i\|_1 \leq M\|\widehat{h}\|_{\mathcal{S}}.$$

Therefore,

$$\left|\int_K f(x)h(\widetilde{x})\,dm(x)\right| \leq M\|\widehat{h}\|_{\mathcal{S}}$$

for all $h \in C_{00}(K)$. Now it is routine to finish the proof in the following way. The space $(C_{00}(K))^{\wedge}$ is dense in $C_0(\widehat{K})$. Hence, the linear functional $F(\widehat{h}) = \int_K f(x)h(\widetilde{x})\,dm(x)$ defined on $(C_{00}(K))^{\wedge}$ can be continuously extended to a continuous linear functional on $C_0(\widehat{K})$. Riesz's representation theorem yields a unique measure $\mu \in M(\widehat{K})$ such that

$$\int_K f(x)h(\widetilde{x})\,dm(x) = F(\widehat{h}) = \int_{\hat{K}} \widehat{h}(\alpha)\,d\mu(\alpha)$$

for all $h \in C_{00}(K)$. By Fubini's theorem, it follows that

$$\int_K \check{\mu}(x)h(\widetilde{x})\,dm(x) = \int_{\hat{K}} \widehat{h}(\alpha)\,d\mu(\alpha) = \int_K f(x)h(\widetilde{x})\,dm(x)$$

for all $h \in C_{00}(K)$, and hence $\check{\mu} = f$. $\square$

Remark. Theorem 3 establishes that every multiplier f for $L^1(\mathcal{S},\pi)$ is a bounded function. The reader should notice that $f = \check{\mu}$, $\mu \in M(\widehat{K})$, is, in general, not a multiplier for $L^1(\mathcal{S},\pi)$. The reason is that there is, in general, no dual hypergroup structure on $\mathcal{S}$. So the space of multipliers for $L^1(\mathcal{S},\pi)$ is, in general, smaller than $M(\widehat{K})^\vee$.

The proof of Theorem 3 tells us that the operator $T \in B(L^1(\mathcal{S},\pi))$ induced by the multiplier f satisfies $\|\mu\| \le \|T\|$.

We investigate this situation briefly in the case of polynomial hypergroups, since (as we already know) the Jacobi polynomials are the only ones in that class which possess a dual hypergroup structure, and there are many others; for example, generalized Chebyshev polynomials, associated ultraspherical polynomials, Pollaczek polynomials, little q-Legendre polynomials, and so on, see [**1, 8, 9**]. Let $K = \mathbf{N}_0$ be a polynomial hypergroup generated by $(R_n(t))_{n\in\mathbf{N}_0}$. Applying the inversion theorem, we can easily show that each $f \in l^1(\mathbf{N}_0,h)$ is a multiplier for $L^1(\mathcal{S},\pi)$. In fact, for every $\varphi \in L^1(\mathcal{S},\pi)$, we have $\psi := (f\check{\varphi})^\wedge/\mathcal{S} \in C(\mathcal{S}) \subseteq L^1(\mathcal{S},\pi)$ and $f\check{\varphi} = \check{\psi}$.

If the space of multipliers $M(L^1(\mathcal{S},\pi))$ is equal to $M(D_s)^\vee$ and if the correspondence between $M(L^1(\mathcal{S},\pi))$ and $M(D_s)$ is isometric, then dual product formulas for $R_n(t)$ are valid. We denote the multiplier operator corresponding to $\mu \in M(D_s)$ by T_μ. Recall that $\|\mu\| \le \|T_\mu\|$.

Proposition 7. *Let $K = \mathbf{N}_0$ be a polynomial hypergroup. Assume that $1 \in \mathcal{S}$. If the space of multipliers $M(L^1(\mathcal{S},\pi))$ is equal to $M(D_s)^\vee$, and if $\|T_\mu\| = \|\mu\|$, then $\mathcal{S} = D_s$ and for all $s,t \in D_s$, there exists a probability measure $\mu_{s,t} \in M^1(D_s)$ such that*

$$R_n(s)R_n(t) = \int_{D_s} R_n(u)\, d\mu_{s,t}(u) \quad \textit{for all } n \in \mathbf{N}_0.$$

Proof. Consider the points p_s and p_t of $s \in D_s$ and $t \in D_s$, respectively. There exist $T_{p_s}, T_{p_t} \in B(L^1(\mathcal{S},\pi))$ such that

$$T_{p_s}(\varphi)^\vee(n) = \check{p}_s(n)\check{\varphi}(n) = R_n(s)\check{\varphi}(n)$$

and

$$T_{p_t}(\varphi)^\vee(n) = R_n(t)\check{\varphi}(n)$$

for all $\varphi \in L^1(\mathcal{S},\pi)$ and $n \in \mathbf{N}_0$, and therefore

$$(T_{p_s} \circ T_{p_t}(\varphi))^\vee(n) = R_n(s)R_n(t)\check{\varphi}(n)$$

is valid for all $\varphi \in L^1(\mathcal{S},\pi)$, $n \in \mathbf{N}_0$. Now choose a net $(k_i)_{i\in I}$ with $k_i \in C(\mathcal{S}) \subseteq L^1(\mathcal{S},\pi)$, $\|k_i\|_1 = 1$ and $k_i^\vee(n) \to 1$ for all $n \in \mathbf{N}_0$, see Lemma 2.

Since $\|T_{p_s}\| = \|p_s\| = 1$ for all $s \in D_s$, $(T_{p_s} \circ T_{p_t}(k_i))_{i\in I}$ is a norm-bounded net in $L^1(D_s,\pi) \subseteq M(\mathcal{S})$. Thus, there is some subnet (we denote its index set again by I) such that $T_{p_s} \circ T_{p_t}(k_i)$ converges in the $\sigma(M(\mathcal{S}), C(\mathcal{S}))$-topology to some regular complex Borel measure $\mu_{s,t} \in M(\mathcal{S})$, where $\|\mu_{s,t}\| \le 1$. Hence, we have

$$\begin{aligned}\int_{\mathcal{S}} R_n(u)\, d\mu_{s,t}(u) &= \lim_i \int_{\mathcal{S}} R_n(u) T_{p_s} \circ T_{p_t}(k_i)(u)\, d\pi(u)\\ &= \lim_i (T_{p_s} \circ T_{p_t}(k_i))^\vee(n)\\ &= R_n(s)R_n(t) \lim_i k_i^\vee(n)\\ &= R_n(s)R_n(t)\end{aligned}$$

for all $n \in \mathbf{N}_0$. Putting $n = 0$, we have $\|\mu_{s,t}\| = \mu_{s,t}(\mathcal{S}) = 1$, and, in view of the Jordan-Hahn decomposition of $\mu_{s,t}$, it follows that $\mu_{s,t}$ is a probability measure on $\mathcal{S}$. Finally, notice that $\mu_{s,t} \in M(\mathcal{S})$ is uniquely determined by $R_n(s)R_n(t) = \mu_{s,t}^\vee(n)$ for all $n \in \mathbf{N}_0$. In particular, for $t = 1$, we have $\mu_{s,1}^\vee(n) = R_n(s) = p_s^\vee(n)$, and therefore each $s \in D_s$ is an element of $\mathcal{S} = \operatorname{supp} \pi$. $\square$

We shall say that the polynomial hypergroup $K = \mathbf{N}_0$ fulfills the continuous property (P) if, for all $s,t \in D_s$, there exists a probability measure $\mu_{s,t} \in M^1(D_s)$ such that

$$R_n(s)R_n(t) = \int_{D_s} R_n(u)\, d\mu_{s,t}(u).$$

If the continuous property (P) is satisfied we get a weak dual structure on D_s.

Proposition 7 states that continuous property (P) is necessarily satisfied if $M(L^1(\mathcal{S},\pi))$ and $M(D_s)$ are isometric and isomorphic (as Banach spaces), provided $1 \in \mathcal{S}$. We shall now prove the converse implication.

Lemma 3. *Let $K = \mathbf{N}_0$ be a polynomial hypergroup fulfilling the continuous property* (P). *Then $(s,t) \mapsto \mu_{s,t}$, $D_s \times D_s \to M^1(D_s)$, is continuous, where $M^1(D_s)$ bears the $\sigma(M^1(D_s), C(D_s))$-topology.*

Proof. Given any $\varphi \in C(D_s)$ and $\varepsilon > 0$, choose $f \in l^1(\mathbf{N}_0, h)$ with finite support such that $\|\varphi - \widehat{f}\|_\infty < \varepsilon$. Given $s_0, t_0 \in D_s$, we obtain

$$\begin{aligned}
&\left| \int_{D_s} \varphi(u)\, d\mu_{s,t}(u) - \int_{D_s} \varphi(u)\; d\mu_{s_0,t_0}(u) \right| \\
&\qquad \le 2\varepsilon + \left| \int_{D_s} \widehat{f}(u)\, d\mu_{s,t}(u) - \int_{D_s} \widehat{f}(u)\, d\mu_{s_0,t_0}(u) \right| \\
&\qquad \le 2\varepsilon + \left| \sum_{k \in \operatorname{supp} f} f(k) h(k) \int_{D_s} R_k(u)\, d(\mu_{s,t} - \mu_{s_0,t_0})(u) \right| \\
&\qquad \le 2\varepsilon + \sum_{k \in \operatorname{supp} f} |f(k)| |R_k(s) R_k(t) - R_k(s_0) R_k(t_0)| h(k).
\end{aligned}$$

Now it is obvious that we can find neighborhoods U_{s_0}, U_{t_0} of s_0 and t_0 such that

$$\left| \int_{D_s} \varphi(u)\, d\mu_{s,t}(u) - \int_{D_s} \varphi(u)\, d\mu_{s_0,t_0}(u) \right| \le 3\varepsilon,$$

for all $s \in U_{s_0}$, $t \in U_{t_0}$. $\square$

We can now define the translation of any function $\varphi \in C(D_s)$ by $s \in D_s$. We define $T_s\varphi(t) = \mu_{s,t}(\varphi)$. By Lemma 3, we have $T_s\varphi \in C(D_s)$. The orthogonalization measure π behaves like a Haar measure on D_s.

Proposition 8. *Let $K = \mathbf{N}_0$ be a polynomial hypergroup fulfilling the continuous property* (P). *Then*

$$\int_{D_s} T_s\varphi(t)\, d\pi(t) = \int_{D_s} \varphi(t)\, d\pi(t)$$

holds for all $\varphi \in C(D_s)$ and $s \in D_s$.

Proof. For $s \in D_s$, let $\alpha_s(n) = R_n(s)$. For $f \in l^1(\mathbf{N}_0, h)$ and $s, t \in D_s$, we have

$$\begin{aligned} T_s\widehat{f}(t) &= \int_{D_s} \widehat{f}(u)\, d\mu_{s,t}(u) \\ &= \int_{D_s} \sum_{n\in\mathbf{N}_0} f(n)R_n(u)h(n)\, d\mu_{s,t}(u) \\ &= \sum_{n\in\mathbf{N}_0} f(n)R_n(s)R_n(t)h(n) \\ &= (\alpha_s \cdot f)^\wedge(t), \end{aligned}$$

and then

$$\begin{aligned} \int_{D_s} T_s\widehat{f}(t)\, d\pi(t) &= \int_{D_s} \sum_{n\in\mathbf{N}_0} f(n)R_n(s)R_n(t)h(n)\, d\pi(t) \\ &= f(0) \\ &= \int_{D_s} \widehat{f}(u)\, d\pi(u). \end{aligned}$$

For any $\varphi \in C(D_s)$ and $\varepsilon > 0$, there exist $f \in l^1(\mathbf{N}_0, h)$ (even with finite support) so that $\|\varphi - \widehat{f}\|_\infty < \varepsilon$. Then $\|T_s\varphi - T_s\widehat{f}\|_\infty < \varepsilon$, and it follows that

$$\int_{D_s} T_s\varphi(t)\, d\pi(t) = \int_{D_s} \varphi(t)\, d\pi(t). \qquad \square$$

The next step is to consider Borel measurable functions on D_s. Based on Lemma 3 and Proposition 8 using the methods in [**5**], one can prove the following result.

Proposition 9. *Let $K = \mathbf{N}_0$ be a polynomial hypergroup fulfilling the continuous property* (P). *Let $\psi : D_s \to [0, \infty]$ be a Borel measurable function. Then $(s,t) \mapsto \mu_{s,t}(\psi)$, $D_s \times D_s \to [0,\infty]$ is Borel measurable. For each complex-valued, Borel measurable function ψ with $\int_{D_s} |\psi(t)|\, d\pi(t) < \infty$ and $s \in D_s$, $T_s g(t) := \mu_{s,t}(g)$ satisfies*

$$\int_{D_s} T_s\psi(t)\, d\pi(t) = \int_{D_s} \psi(t)\, d\pi(t).$$

By Proposition 9, we know that $T_s\psi$ is a well-defined element of $L^1(D_s,\pi)$ for each $\psi \in L^1(D_s,\pi)$. Furthermore, we have $\|T_s\psi\|_1 \le \|\psi\|_1$. Let $\mu \in M(D_s)$ and $\psi \in L^1(D_s,\pi)$. It is routine to show that

$$\mu * \psi(t) := \int_{D_s} T_s\psi(t)\, d\mu(s)$$

is a well-defined element of $L^1(D_s,\pi)$ with $\|\mu * \psi\|_1 \le \|\mu\|\|\psi\|_1$.

Theorem 4. *Let $K = \mathbf{N}_0$ be a polynomial hypergroup, and suppose that $1 \in \mathcal{S}$. Then the following two conditions are equivalent.*

(i) *The space of multipliers $M(L^1(\mathcal{S},\pi))$ is equal to $M(D_s)^\vee$, and the correspondence between $M(L^1(\mathcal{S},\pi))$ and $M(D_s)$ is isometric, i.e., $\|T_\mu\| = \|\mu\|$.*

(ii) *$\mathcal{S} = D_s$ and the continuous property* (P) *are fulfilled.*

Proof. (i) $\Rightarrow$ (ii) is already shown in Proposition 7. If (ii) is satisfied, we have proven: if $\nu \in M(D_s)$ and $\psi \in L^1(\mathcal{S},\pi)$, then $\nu * \psi \in L^1(\mathcal{S},\pi)$ and $\|\nu * \psi\|_1 \le \|\nu\|\|\psi\|_1$. If $T \in M(L^1(\mathcal{S},\pi))$ and $\mu \in M(D_s)$, such that $(T\varphi)^\vee = \check{\mu}\check{\varphi}$ for all $\varphi \in L^1(\mathcal{S},\pi)$, then the inverse uniqueness theorem yields $T\varphi = \mu * \varphi$. Furthermore, we have $\|T\| \le \|\mu\|$. □

Now we investigate multipliers for $L^2(\mathcal{S},\pi)$. Applying the Plancherel transform, we can define a (rather weak) translation operator for $L^2(\mathcal{S},\pi)$. This translation is already introduced by one of the authors in [**10**]. For every $f \in L^\infty(K,m)$, define $M_f \in B(L^2(\mathcal{S},\pi))$ by means of

$$M_f(\varphi) = \mathcal{P}(\overline{f}\mathcal{P}^{-1}(\varphi)) \quad \text{for } \varphi \in L^2(\mathcal{S},\pi).$$

M_f is a bounded linear operator satisfying $\|M_f(\varphi)\|_2 \le \|f\|_K\|\varphi\|_2$.

Proposition 10. *Let K be a commutative hypergroup. If $f,g \in L^\infty(K,m)$, then $M_{fg} = M_f \circ M_g$, $M_{\bar{f}} = (M_f)^*$ and $\|M_f\| = \|f\|_K$. Furthermore, $M_f = 0$ if and only if $f = 0$.*

The proof is straightforward, see [**10**].

Lemma 4. *Let $\varphi \in L^2(\mathcal{S},\pi)$. The mapping $\alpha \mapsto M_\alpha(\varphi)$, $\widehat{K} \to L^2(\mathcal{S},\pi)$ is continuous.*

Proof. Let $\alpha_0 \in \widehat{K}$, $\varepsilon > 0$. Since $\mathcal{P}^{-1}(\varphi) \in L^2(K,m)$, there is a compact subset $C \subseteq K$ such that

$$\int_{K\setminus C} |\mathcal{P}^{-1}(\varphi)(z)|^2 \, dm(z) < \frac{\varepsilon}{8}.$$

Let $M = \int_C |\mathcal{P}^{-1}(\varphi)(z)|^2 \, dm(z)$ and

$$V(\alpha_0) = \left\{ \alpha \in \widehat{K} : |\alpha(z) - \alpha_0(z)|^2 < \frac{\varepsilon}{2M} \text{ for } z \in C \right\}.$$

For every $\alpha \in V(\alpha_0)$,

$$\begin{aligned} \|M_\alpha(\varphi) - M_{\alpha_0}(\varphi)\|_2^2 &= \|\overline{\alpha}\mathcal{P}^{-1}(\varphi) - \overline{\alpha}_0\mathcal{P}^{-1}(\varphi)\|_2^2 \\ &= \int_C |\alpha(z) - \alpha_0(z)|^2 |\mathcal{P}^{-1}(\varphi)(z)|^2 \, dm(z) \\ &\quad + \int_{K\setminus C} |\alpha(z) - \alpha_0(z)|^2 |\mathcal{P}^{-1}(\varphi)(z)|^2 \, dm(z) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square \end{aligned}$$

We call M_α, $\alpha \in \widehat{K}$, a translation operator on $L^2(\mathcal{S},\pi)$.

Now we can introduce an action of $L^1(\mathcal{S},\pi)$ on $L^2(\mathcal{S},\pi)$. Given $\psi \in C_{00}(\mathcal{S})$ and $\varphi \in L^2(\mathcal{S},\pi)$, we use an $L^2(\mathcal{S},\pi)$-valued integral to define

$$\psi * \varphi := \int_{\mathcal{S}} \psi(\alpha) \; M_{\overline{\alpha}}(\varphi) \, d\pi(\alpha) \in L^2(\mathcal{S},\pi).$$

We have $\|\psi * \varphi\|_2 \le \|\psi\|_1 \|\varphi\|_2$ and, for any $\psi \in L^1(\mathcal{S},\pi)$, choose a sequence $(\psi_n)_{n\in\mathbf{N}}$, $\psi_n \in C_{00}(\mathcal{S})$, with $\|\psi - \psi_n\|_1 \to 0$ as $n \to \infty$. It is easily shown that

$$\psi * \varphi := \lim_{n\to\infty} \psi_n * \varphi \in L^2(\mathcal{S},\pi)$$

is a well-defined action of $L^1(\mathcal{S},\pi)$ on $L^2(\mathcal{S},\pi)$ with $\|\psi * \varphi\|_2 \le \|\psi\|_1 \|\varphi\|_2$.

Now we proceed as before to introduce pseudomeasures on $\widehat{K}$. Denote by

$$A(\widehat{K}) := \{\widehat{f} : f \in L^1(K,m)\}.$$

Take $\|\widehat{f}\|_A := \|f\|_1$ as the norm on $A(\widehat{K})$. Then $A(\widehat{K})$ is a Banach space, and the dual space $A(\widehat{K})^*$ is denoted by $P(\widehat{K})$, the elements $s \in P(\widehat{K})$ are called pseudomeasures on $\widehat{K}$. The mapping $\Phi : P(\widehat{K}) \to L^\infty(K,m)$ where, for each $s \in P(\widehat{K})$, the element $\Phi(s) \in L^\infty(K,m)$ is uniquely determined by

$$\int_K f(x)\Phi(s)(\widetilde{x})\,dm(x) = s(\widehat{f}) \quad \text{for} \quad f \in L^1(K,m),$$

is an isometric isomorphism from the Banach space $P(\widehat{K})$ onto $L^\infty(K,m)$. A convolution of $s_1, s_2 \in P(\widehat{K})$ is determined by $s_1 * s_2 = \Phi^{-1}(\Phi(s_1)\Phi(s_2))$, and Φ is then an algebra isomorphism. $\Phi(s)$ is called inverse Fourier transform of $s \in P(\widehat{K})$. If $\mu \in M(\widehat{K})$, we have

$$\begin{aligned}\int_{\hat{K}} \widehat{f}(\alpha)\,d\mu(\alpha) &= \int_{\hat{K}}\int_K f(x)\overline{\alpha(x)}\,dm(x)\,d\mu(\alpha)\\ &= \int_K f(x)\check{\mu}(\widetilde{x})\,dm(x),\end{aligned}$$

for all $f \in L^1(K,m)$. Hence, each measure $\mu \in M(\widehat{K})$ is a pseudomeasure, $\check{\mu} = \Phi(\mu)$ and $\|\mu\|_P = \|\check{\mu}\|_\infty \le \|\mu\|$.

In conclusion, we have

Proposition 11. *Let K be a commutative hypergroup. Then the inverse Fourier transform $\Phi : P(\widehat{K}) \to L^\infty(K,m)$ determined by*

$$\int_K f(x)\Phi(s)(\widetilde{x})\,dm(x) = s(\widehat{f}) \quad \textit{for all } f \in L^1(K,m),$$

is an isometric algebra isomorphism of $P(\widehat{K})$ onto $L^\infty(K,m)$.

Remark. The convolution $\mu_1 * \mu_2$ of two measures $\mu_1, \mu_2 \in M(\widehat{K})$ makes sense if we interpret μ_1 and μ_2 as pseudomeasures, and so $\mu_1 * \mu_2 \in P(\widehat{K})$. However, in general, $\mu_1 * \mu_2$ is not a member of $M(\widehat{K})$.

Next, we shall say that a pseudomeasure $s \in P(\widehat{K})$ belongs to $L^2(\mathcal{S},\pi)$ if there is a $\psi \in L^2(\mathcal{S},\pi)$ such that

$$s(\widehat{f}) = \int_{\mathcal{S}} \widehat{f}(\alpha)\psi(\alpha)\,d\pi(\alpha) \quad \text{for all } f \in L^1(K,m) \cap L^2(K,m).$$

Similarly, as in Section 2, one can derive the following properties of the $\psi \in L^2(\mathcal{S}, \pi)$ belonging to $s \in P(\widehat{K}) : \psi$ is uniquely determined and

$$\begin{aligned}\int_K f(x)\Phi(s)(\widetilde{x})\, dm(x) = s(\widehat{f})& \\ &= \int_{\mathcal{S}} \widehat{f}(\alpha)\psi(\alpha)\, d\pi(\alpha) \\ &= \int_K f(x)\mathcal{P}^{-1}\psi(\widetilde{x})\, dm(x),\end{aligned}$$

for all $f \in L^1(K, m) \cap L^2(K, m)$.

In particular, we have $\Phi(s) = \mathcal{P}^{-1}(\psi) \in L^2(K, m) \cap L^\infty(K, m)$. Conversely, every $s \in P(\widehat{K})$ with $\Phi(s) \in L^2(K, m) \cap L^\infty(K, m)$ belongs to $L^2(\mathcal{S}, \pi)$. For that, put $\psi = \mathcal{P}(\Phi(s)) \in L^2(\mathcal{S}, \pi)$. Hence, the following "dual" statement of Proposition 2 is valid.

Proposition 12. *Let K be a commutative hypergroup. A pseudomeasure $s \in P(\widehat{K})$ belongs to $L^2(\mathcal{S}, \pi)$ if and only if $\Phi(s) \in L^2(K, m) \cap L^\infty(K, m)$. Moreover, the inverse Fourier transform of s as a pseudomeasure coincides with the inverse Plancherel transform of the corresponding $\psi \in L^2(\mathcal{S}, \pi)$.*

Furthermore, every $\psi \in L^1(\mathcal{S}, \pi) \cap L^2(\mathcal{S}, \pi)$ determines a pseudomeasure $s \in P(\widehat{K})$ such that

$$s(\widehat{f}) = \int_{\mathcal{S}} \widehat{f}(\alpha)\psi(\alpha)\, d\pi(\alpha)$$

is true for all $f \in L^1(K, m) \cap L^2(K, m)$. For that, put $s = \Phi^{-1}(\check{\psi})$. In particular, the convolution $s * \psi = \Phi^{-1}(\Phi(s)\check{\psi})$ of $s \in P(\widehat{K})$ and $\psi \in L^1(\mathcal{S}, \pi) \cap L^2(\mathcal{S}, \pi)$ is well defined as a convolution of pseudomeasures.

Now we have all the tools to give a complete characterization of multipliers on $L^2(\mathcal{S}, \pi)$. We say that $T \in B(L^2(\mathcal{S}, \pi))$ is a multiplier on $L^2(\mathcal{S}, \pi)$ whenever

$$\mathcal{P}^{-1}(T\varphi) = f\mathcal{P}^{-1}(\varphi)$$

for all $\varphi \in L^2(\mathcal{S}, \pi)$, where f is an element from $L^\infty(K, m)$. We denote the space of multipliers on $L^2(\mathcal{S}, \pi)$ by $M(L^2(\mathcal{S}, \pi))$.

Theorem 5. *Let K be a commutative hypergroup, and let $T \in B(L^2(\mathcal{S},\pi))$. The following conditions are equivalent*:

(i) $T \in M(L^2(\mathcal{S},\pi))$.

(ii) $T \circ M_\alpha = M_\alpha \circ T$ for all $\alpha \in \widehat{K}$.

(iii) $\psi * T\varphi = T(\varphi * \psi) = \varphi * T\psi$ for all $\varphi, \psi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$.

(iv) *There exists a unique pseudomeasure $s \in P(\widehat{K})$ such that $s * \varphi$ belongs to $L^2(\mathcal{S},\pi)$ and $T\varphi = s * \varphi$ for all $\varphi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$.*

(v) *There exists a unique pseudomeasure $s \in P(\widehat{K})$ such that*

$$\mathcal{P}^{-1}(T\varphi) = \Phi(s)\mathcal{P}^{-1}(\varphi) \quad \textit{for all } \varphi \in L^2(\mathcal{S},\pi).$$

Proof. The equivalence of (i) and (v) follows from Proposition 11. Proposition 12 yields (iv) $\Leftrightarrow$ (v).

(i) $\Rightarrow$ (ii). We have $\mathcal{P}^{-1}(M_\alpha\varphi) = \overline{\alpha}\mathcal{P}^{-1}\varphi$ for all $\varphi \in L^2(\mathcal{S},\pi)$. Therefore, assumption (i) implies

$$\begin{aligned}\mathcal{P}^{-1}(M_\alpha T\varphi) &= \overline{\alpha}\mathcal{P}^{-1}(T\varphi) = \overline{\alpha} f \mathcal{P}^{-1}(\varphi)\\ &= f\mathcal{P}^{-1}(M_\alpha\varphi) = \mathcal{P}^{-1}(TM_\alpha\varphi),\end{aligned}$$

for all $\varphi \in L^2(\mathcal{S},\pi)$. Hence, statement (ii) is valid.

(ii) $\Rightarrow$ (iii). Let $\psi \in C_{00}(\mathcal{S})$, $\varphi \in L^2(\mathcal{S},\pi)$. Since $\psi * T\varphi$ and $\psi * \varphi$ are defined as $L^2(\mathcal{S},\pi)$-valued integrals, and since T is continuous, we have

$$\begin{aligned}\psi * T\varphi &= \int_{\mathcal{S}} \psi(\alpha) M_{\overline{\alpha}}(T\varphi)\, d\pi(\alpha)\\ &= \int_{\mathcal{S}} \psi(\alpha) T(M_{\overline{\alpha}}\varphi)\, d\pi(\alpha)\\ &= T\bigg(\int_{\mathcal{S}} \psi(\alpha) M_{\overline{\alpha}}\varphi\, d\pi(\alpha)\bigg) = T(\psi * \varphi).\end{aligned}$$

If $\psi \in L^1(\mathcal{S},\pi)$, approximate ψ by a sequence $(\psi_n)_{n\in\mathbf{N}}$ in $C_{00}(\mathcal{S})$ such that $\|\psi - \psi_n\|_1 \to 0$. It follows that $\psi * T\varphi = T(\varphi * \psi)$ for $\psi \in L^1(\mathcal{S},\pi)$ and $\varphi \in L^2(\mathcal{S},\pi)$. Interchanging the roles of ψ and φ, it follows that for all $\varphi, \psi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$,

$$\psi * T\varphi = T(\varphi * \psi) = \varphi * T\psi.$$

(One should notice that $\varphi * \psi = \psi * \varphi$ for $\varphi, \psi \in L^1(\mathcal{S}, \pi) \cap L^2(\mathcal{S}, \pi)$, see [**5**].)

It remains to prove "(iii) $\Rightarrow$ (i)." One can proceed along the lines of the proof "(i) $\rightarrow$ (v)" of Theorem 2 to obtain $f \in L^\infty(K, m)$ such that $\mathcal{P}^{-1}(T\varphi) = f\mathcal{P}^{-1}(\varphi)$ for all $\varphi \in L^2(\mathcal{S}, \pi)$. $\square$

Remark. We want to point out that Theorem 5 generalizes results of [**13**] to a vastly bigger class of orthogonal polynomials, viz., those that generate a polynomial hypergroup on $\mathbf{N}_0$.

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