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ON THE DISTRIBUTIONS OF $\sigma(n)/n$ AND $n/\varphi(n)$

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ABSTRACT. We prove that the distribution functions of $\sigma(n)/n$ and $n/\varphi(n)$ both have super-exponential asymptotic decay when n ranges over certain subsets of integers, which, in particular, can be taken as the set of l-free integers not divisible by a thin subset of primes.

1. Introduction. Let $\sigma(n)$ be the sum of divisors function, and let $\varphi(n)$ denote Euler's totient function. The existence and continuity of the limiting distribution of $\sigma(n)/n$ were first established independently by Behrend [**3**], Chowla [**4**], Davenport [**5**] and Erdös [**7**]. Precisely, they proved that the density

$$F(t) := \lim_{x\to\infty} \frac{1}{x}\left|\left\{n \le x : \frac{\sigma(n)}{n} \ge t\right\}\right|$$

is a continuous function for all values of t. The analogous statement for the close relative $n/\varphi(n)$ was obtained earlier by Schoenberg [**13**] who showed that the density

$$G(t) := \lim_{x\to\infty} \frac{1}{x}\left|\left\{n \le x : \frac{n}{\varphi(n)} \ge t\right\}\right|$$

is also continuous. Historically, these two results are special cases of a general phenomenon and led to the celebrated theorem of Erdös and Wintner [**10**] which completely determines the real additive (and also multiplicative) functions with continuous distributions. Erdös and Wintner proved that a real additive function f has a continuous limiting distribution only when the three series over primes

$$\sum_{|f(p)|>1} \frac{1}{p}, \quad \sum_{|f(p)|\le 1} \frac{f(p)}{p}, \quad \sum_{|f(p)|\le 1} \frac{f^2(p)}{p}$$

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all converge and the series

$$\sum_{f(p)\neq 0} \frac{1}{p}$$

diverges. Erdös [**8**] studied the density of integers satisfying $\sigma(n)/n \geq t$ asymptotically and showed that

$$F(t) = \exp\left\{-e^{te^{-\gamma}}(1+o(1))\right\}$$

as t tends to infinity, where γ is Euler's constant. This was recently sharpened to

$$F(t) = \exp\left\{ - e^{te^{-\gamma}}\left(1 + O\left(\frac{1}{t^2}\right)\right)\right\}$$

by Weingartner [**19**]. Concerning short interval estimates, Erdös [**9**] proved that the number of integers $n \leq x$ for which $0 < a \leq \sigma(n)/n < a + 1/t$ for $2 \leq t < x$ is $< c_1 x/\log t$ for a suitable constant $c_1 > 0$. For the current record on short interval estimates for the densities, we refer the reader to the work of Toulmonde [**15, 16**]. As a consequence of the Erdös-Wintner theorem, the functions $\log(n/\varphi(n))$, $\log(\sigma(n)/n)$ and $\log(\sigma(n)\varphi(n)/n^2)$ all have continuous distributions, and one can use the values of $n/\varphi(n)$, $\sigma(n)/n$ and $\sigma(n)\varphi(n)/n^2$ for Diophantine approximation in the appropriate ranges. In connection with this, the author, Ford and Zaharescu [**1**] obtained strong Diophantine approximation results concerning the values of a general family of additive and multiplicative functions including $n/\varphi(n)$ and $\sigma(n)/n$. By a completely different approach, the author, Ford and Zaharescu [**2**] showed that Diophantine approximation with normalized Fourier coefficients of newforms is possible. Inspired by the approach of Erdös [**8**] and modifying the method of Weingartner [**19**], in this paper we give upper and lower bounds for the distribution of $\sigma(n)/n \geq t$ when n is restricted to certain subsets of integers which in particular can be taken as the set of l-free numbers for every $l \geq 2$ not divisible by a thin subset of primes and at the same time n ranges over specially chosen arithmetic progressions. Our results confirm a super-exponential rate of decay for the distribution function of $\sigma(n)/n$ (and also for $n/\varphi(n)$) asymptotically as t tends to infinity. Recall that a set of integers A is divisor closed, if all divisors of elements of A belong to A. The support of A refers to the set of primes that appear in the prime factorization of elements of A. We also say that A is multiplicative closed if, for any two relatively prime integers $m, n \in A$, $mn \in A$ holds. A typical example of a set A which is both divisor closed and multiplicative closed is the set of l-free integers for $l \geq 2$.

Theorem 1. *Let A be a divisor closed set of integers with positive density such that*

$$N_A(x) = \sum_{\substack{n \leq x \\ n \in A}} 1 = \alpha x + O(x^{1-\varepsilon})$$

for some $0 < \varepsilon < 1$, where $0 < \alpha \leq 1$ is the density of A. Assume that the characteristic function of A, denoted as $\chi_A(n)$, satisfies

$$\chi_A(n) = \sum_{d^s | n} \nu\left(\frac{n}{d^s}\right) h(d)$$

for some $s \geq 2$, where h is multiplicative with $|h(n)| \leq 1$ for all n and ν is the characteristic function of the set of all integers (including 1) supported on a subset U of primes with the property that

$$\sum_{\substack{p \leq x \\ p \notin U}} 1 \leq C \log x$$

for some constant $0 < C < 1/\log 2(1 - (1/s))$. Let $k \in A$ be fixed. If $(b, k) = 1$ and

$$E > \frac{\pi^2 e^{-\gamma}}{6} \prod_{\substack{p \notin A \text{or} \\ p | k}} \left(1 + \frac{1}{p}\right)$$

(empty products are taken to be 1), then as t tends to infinity,

$$\left\{ n \in A : n \equiv b \pmod{k}, \, \frac{\sigma(n)}{n} \geq t \right\}$$

contains a set whose density is at least $\exp\{-e^{Et}(1 + o(1))\}$.

We first remark that, if A is divisor closed and multiplicative closed, then χ_A is forced to be a multiplicative function. Indeed, taking relatively prime integers m, n and assuming first $mn \in A$, one has $\chi_A(mn) = 1$. Since A is divisor closed, $m \in A$, $n \in A$, and consequently $\chi_A(mn) = 1 = \chi_A(m)\chi_A(n)$ follows. On the other hand, if $mn \notin A$, then $\chi_A(mn) = 0$. Assuming $m \in A$ and $n \in A$ gives $mn \in A$, since A is multiplicative closed, this gives $\chi_A(mn) = 1$, which is not possible. Therefore, at least one of m or n is not in A and $\chi_A(mn) = 0 = \chi_A(m)\chi_A(n)$ follows in this case as well. Conversely, if χ_A is a multiplicative function, then A is multiplicative closed since, taking relatively prime integers $m, n \in A$, $\chi_A(mn) = \chi_A(m)\chi_A(n) = 1$ and $mn \in A$ follow. Nevertheless, it is not in general true that A is divisor closed. In particular, choosing $A = \{2^j : j \geq 2\} \cup \{1\}$, it is easy to see that χ_A is multiplicative but A is not divisor closed since $2 \notin A$. Note that, despite the restrictive conditions imposed on set A in Theorem 1, one can give plenty of examples of such sets meeting the requirements. Precisely, let U be a subset of primes, and let A be the set of square-free integers with support U. Using the identity

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\chi_A(n)}{n^s} &= \prod_{p \in U} \left(1 + \frac{1}{p^s}\right) \\ &= \left(\sum_{m=1}^{\infty} \frac{\nu(m)}{m^s}\right)\left(\sum_{j=1}^{\infty} \frac{\mu_U(j)}{j^{2s}}\right) \\ &= \frac{\zeta(s)}{\zeta(2s)} \prod_{p \notin U} \left(1 + \frac{1}{p^s}\right)^{-1},\end{aligned}$$

among Dirichlet series (ζ denotes the Riemann zeta function) for $\mathrm{Re}(s) > 1$, where $\mu_U(n)$ is the Möbius function with support U and ν is the characteristic function of the set of integers with support U, we obtain that

$$\chi_A(n) = \sum_{d^2 | n} \nu\left(\frac{n}{d^2}\right) \mu_U(d).$$

If we assume that

$$\sum_{\substack{p \leq x \\ p \notin U}} 1 \leq C \log x,$$

then the product

$$\prod_{p \notin U} \left(1 + \frac{1}{p^s}\right)$$

is convergent and defines an analytic function for $\mathrm{Re}\,(s) \geq 1/2$. Employing a standard shift of contour argument using a truncated version of Perron's formula (see the proof of Theorem 2 below), one can show that

$$N_A(x) = \sum_{n \leq x} \chi_A(n) = \alpha x + O\left(x^{1-\varepsilon}\right)$$

for some $0 < \varepsilon < 1$, where

$$\alpha = \frac{6}{\pi^2} \prod_{p \notin U} \left(1 - \frac{1}{p+1}\right) > 0$$

is the density of A. A similar construction can be given for the set of l-free integers, all of whose prime factors are in U for every $l \geq 2$. In order to obtain the desired super-exponential type upper bounds for the densities, unlike Theorem 1, we need to exploit the finer structure of set A and have convenient Euler products for the Dirichlet series of χ_A which makes an analytic approach possible to the problem when χ_A is multiplicative. This is exactly the reason for us to state the result below only for the set of l-free numbers supported on a subset of primes having a thin complement rather than using the more general setting of Theorem 1. Precisely, the following theorem holds.

Theorem 2. *Let A be the set of all l-free integers supported on a subset U of primes such that*

$$\sum_{\substack{p \leq x \\ p \notin U}} 1 \leq B \log x$$

for some constant $0 < B < 1/\log 2(1 - (1/l))$. *Put*

$$C_A = \prod_{p \notin A} \left(1 - \frac{1}{p}\right) = \prod_{p \notin U} \left(1 - \frac{1}{p}\right) > 0,$$

where empty products are taken to be 1*. Let* $y = e^{te^{-\gamma}}$*, and assume that* $k \in A$ *is fixed. If* $(b,k) = 1$*, then as* t *tends to infinity,*

$$\left\{ n \in A : n \equiv b \pmod{k}, \frac{n}{\varphi(n)} \geq t \right\}$$

is contained in a set whose density is at most

$$\frac{C_A}{k} \prod_{\substack{p \leq y \\ p \in A \\ (p,k)=1}} \left(1 - \frac{1}{p^{l-1}}\right)$$

$$\exp\left(-(\log C_{A,k})[y \log y] - \sum_{\substack{p \leq y \\ p \in A \\ (p,k)=1}} \log p + O\left(\frac{y}{\log^2 y}\right)\right),$$

where

$$C_{A,k} := \exp\left(- \sum_{\substack{p \notin A \text{or} \\ p \mid k}} \log\left(1 - \frac{1}{p}\right)\right) \geq 1.$$

Let us remark that, since $\sigma(n)/n \leq n/\varphi(n)$ and $C_A > 0$, we have

$$\sum_{\substack{p \leq y \\ p \in A \\ (p,k)=1}} \log p \sim y$$

by the prime number theorem. Therefore, Theorems 1 and 2 provide the promised upper and lower bounds of super-exponential decay for the distributions of both of $\sigma(n)/n$ and $n/\varphi(n)$ when $n \in A$ is subject to the conditions given above. Moreover, in the case of square-free numbers, by taking $l = 2$ in Theorem 2 and using Mertens' estimate, the product

$$\prod_{\substack{p \leq y \\ p \in A \\ (p,k)=1}} \left(1 - \frac{1}{p^{l-1}}\right)$$

tends to zero at the rate of a constant multiple of $1/\log y$ as y tends to infinity since $C_A > 0$. On the contrary, this product is uniformly bounded below by a positive number for all $l \geq 3$. Lastly, the density of A in Theorem 2 is easily seen to be

$$C_A \prod_{p \in A} \left(1 - \frac{1}{p^l}\right) > 0.$$

2. Proof of Theorem 1. Since ν is multiplicative (in fact it is completely multiplicative) and h is multiplicative, it follows from the relation

$$\chi_A(n) = \sum_{d^s|n} \nu\left(\frac{n}{d^s}\right) h(d)$$

that χ_A is multiplicative and, therefore, that A is multiplicative closed. The basic idea of the proof is to find a relatively small square-free number in A which is almost t-abundant. To this end, let $k \in A$ be fixed, and put $y = e^t$ and

$$n = n(t) := \prod_{\substack{p \leq y \\ p \in A \\ (p,k)=1}} p. \tag{2.1}$$

Since A is multiplicative closed, $n \in A$ and $(n, k) = 1$ follow. Using (2.1), we have

$$\frac{\sigma(n)}{n} = \prod_{\substack{p \leq y \\ p \in A \\ (p,k)=1}} \left(1 + \frac{1}{p}\right). \tag{2.2}$$

As a consequence of Mertens' estimate, one has

$$\prod_{p \leq x} \left(1 + \frac{1}{p}\right) = \left(\frac{6e^\gamma}{\pi^2} + o(1)\right) \log x. \tag{2.3}$$

The fact that A has positive density forces $\prod_{p \notin A}(1 - (1/p)) > 0$ and consequently

$$\prod_{p \notin A} \left(1 + \frac{1}{p}\right)$$

converges. Therefore, for any constant

$$M < \frac{6e^{\gamma}}{\pi^2} \prod_{\substack{p \notin A \text{or} \\ p|k}} \left(1+\frac{1}{p}\right)^{-1},$$

one obtains from (2.2) and (2.3) that

$$\frac{\sigma(n)}{n} \geq M \log y = Mt \tag{2.4}$$

when t is large enough. Next we show that the set A is well distributed among arithmetic progressions with modulus in A. Precisely, using the orthogonality of Dirichlet characters modulo k, we have

$$\sum_{\substack{u \leq x \\ u \in A \\ u \equiv b \pmod{k}}} 1 = \frac{1}{\varphi(k)} \sum_{\substack{u \leq x \\ u \in A \\ (u,k)=1}} 1 + \frac{1}{\varphi(k)} \sum_{\chi \neq \chi_0} \overline{\chi}(b) \sum_{\substack{u \leq x \\ u \in A}} \chi(u), \tag{2.5}$$

where χ_0 is the principal character modulo k. For the main contribution in (2.5), consider

(2.6)

$$\sum_{\substack{u \leq x \\ u \in A \\ (u,k)=1}} 1 = \sum_{\substack{u \leq x \\ u \in A}} \sum_{\substack{d|u \\ d|k}} \mu(d) = \sum_{\substack{d|k \\ d \in A}} \mu(d) \sum_{\substack{u \leq x \\ u \in A \\ d|u}} 1$$

$$= \sum_{\substack{d|k \\ d \in A}} \mu(d) \sum_{\substack{q \leq x/d \\ q \in A}} 1 = \sum_{\substack{d|k \\ d \in A}} \mu(d) \left(\frac{\alpha x}{d} + O\left(\frac{x^{1-\varepsilon}}{d^{1-\varepsilon}}\right)\right).$$

Since A is divisor closed and $k \in A$, the conditions $d \mid k$ and $d \in A$ reduce to $d \mid k$, and the rightmost term of (2.6) simplifies to

$$\alpha x \sum_{d|k} \frac{\mu(d)}{d} + O_k(x^{1-\varepsilon}) = \frac{\varphi(k)\alpha x}{k} + O_k(x^{1-\varepsilon}). \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\frac{1}{\varphi(k)} \sum_{\substack{u \leq x \\ u \in A \\ (u,k)=1}} 1 = \frac{\alpha x}{k} + O_k(x^{1-\varepsilon}). \tag{2.8}$$

For the error term in (2.5), note that

$$\left|\frac{1}{\varphi(k)}\sum_{\chi\neq\chi_0}\overline{\chi}(b)\sum_{\substack{u\leq x\\u\in A}}\chi(u)\right|\leq\max_{\chi\neq\chi_0}\left|\sum_{\substack{u\leq x\\u\in A}}\chi(u)\right|. \tag{2.9}$$

We may now use the fact that

$$\chi_A(u)=\sum_{d^s|u}\nu\left(\frac{u}{d^s}\right)h(d)$$

to obtain

$$\begin{aligned}\sum_{\substack{u\leq x\\u\in A}}\chi(u)&=\sum_{u\leq x}\chi(u)\chi_A(u)\\&=\sum_{u\leq x}\chi(u)\sum_{d^s|u}\nu\left(\frac{u}{d^s}\right)h(d)\\&=\sum_{d\leq x^{1/s}}h(d)\chi(d^s)\sum_{q\leq x/d^s}\nu(q)\chi(q)\end{aligned} \tag{2.10}$$

for any non-principal χ. Recall that ν is supported on U, and put

$$P=\prod_{\substack{p\leq x\\p\notin U}}p,$$

so that $\omega(P)\leq C\log x$ by our assumption on U, where $0<C<1/\log 2(1-(1/s))$ and $\omega(P)$ is the number of distinct prime factors of P. Using this, one can deduce

$$\begin{aligned}\sum_{q\leq x}\nu(q)\chi(q)&=\sum_{\substack{q\leq x\\(q,P)=1}}\chi(q)=\sum_{q\leq x}\chi(q)\sum_{\substack{d|q\\d|P}}\mu(d)\\&=\sum_{d|P}\mu(d)\chi(d)\sum_{j\leq x/d}\chi(j)=O_k\left(2^{\omega(P)}\right)=O_k\left(x^{C\log 2}\right),\end{aligned} \tag{2.11}$$

where we used the fact that

$$\sum_{j\leq x/d}\chi(j)=O_k(1)$$

since χ is non-principal (or it is $O(\sqrt{k}\log k)$ by the Polya-Vinogradov estimate). As a consequence of (2.10) and (2.11),

$$\max_{\chi\neq\chi_0}\left|\sum_{\substack{u\leq x\\ u\in A}}\chi(u)\right| = O_k\left(x^{(1/s)+C\log 2}\right) \tag{2.12}$$

follows. Combining (2.5), (2.8), (2.9) and (2.12), one sees that

$$\sum_{\substack{u\leq x\\ u\in A\\ u\equiv b \pmod{k}}} 1 = \frac{\alpha x}{k} + O_k(x^{\delta}), \tag{2.13}$$

where $\delta = \max((1/s) + C\log 2, 1-\varepsilon) < 1$. Next consider

$$\sum_{\substack{u\leq x\\ u\in A\\ u\equiv b \pmod{k}\\ (u,n)=1}} 1 = \sum_{d|n}\mu(d)\sum_{\substack{m\leq x/d\\ m\in A\\ dm\equiv b \pmod{k}}} 1. \tag{2.14}$$

Since $u = dm$ and $(u,k) = 1$, we see that $(d,k) = 1$. Using (2.13) and (2.14), we have

(2.15)

$$\begin{aligned}\sum_{d|n}\mu(d)\sum_{\substack{m\leq x/d\\ m\in A\\ m\equiv b\overline{d} \pmod{k}}} 1 &= \sum_{d|n}\mu(d)\left(\frac{\alpha x}{kd} + O_k\left(\frac{x^{\delta}}{d^{\delta}}\right)\right)\\ &= \frac{\varphi(n)\alpha x}{nk} + O_{k,n}(x^{\delta}).\end{aligned}$$

Note that, if $u \in A$ and $(u,n) = 1$, then $un \in A$ since $n \in A$ and A is multiplicative closed. Moreover using (2.4), it is easy to see that

$$\frac{\sigma(un)}{un} \geq \frac{\sigma(n)}{n} \geq Mt. \tag{2.16}$$

If $un \leq x$, then $u \leq x/n$, and using (2.15), we have

$$\sum_{\substack{u\leq x/n\\ u\in A\\ u\equiv b\overline{n} \pmod{k}\\ (u,n)=1}} 1 \sim \frac{\varphi(n)\alpha x}{n^2 k}, \tag{2.17}$$

so that the set

$$\left\{v \in A : v \equiv b \pmod k,\ \frac{\sigma(v)}{v} \geq Mt\right\}$$

contains a set of density $\varphi(n)\alpha/n^2k$ by (2.17). Note that $\varphi(n)/n^2 \gg 1/n \log\log n$ and, to estimate $n = n(t)$, we use the prime number theorem with de la Vallée Poussin error term to get

(2.18)
$$\log n = \sum_{\substack{p\leq y\\ p\in A\\ (p,k)=1}} \log p \leq \sum_{p\leq y} \log p = y\left(1 + O\left(e^{-c\sqrt{\log y}}\right)\right) = e^t(1+o(1))$$

for some constant $c > 0$. Therefore, replacing t by t/M and absorbing both α/k and $\log\log n = O(t)$ into the exponential, one can deduce from (2.17) and (2.18) that the set

$$\left\{v \in A : v \equiv b \pmod k,\ \frac{\sigma(v)}{v} \geq t\right\}$$

contains a set whose density is at least

$$\exp\left\{-e^{Et}(1+o(1))\right\}$$

for any constant

$$E = \frac{1}{M} > \frac{\pi^2 e^{-\gamma}}{6} \prod_{\substack{p\notin A \text{ or}\\ p\mid k}} \left(1 + \frac{1}{p}\right).$$

This completes the proof of Theorem 1. □

3. Proof of Theorem 2. Let $k \in A$ be fixed, and let A be the set of l-free numbers not divisible by a thin subset of primes. Note that, using Rankin's method, we have

$$\sum_{\substack{n\leq x\\ n\in A\\ n\equiv b \pmod k\\ n\geq t\varphi(n)}} 1 \leq \sum_{\substack{n\leq x\\ n\in A\\ n\equiv b \pmod k}} \left(\frac{n}{t\varphi(n)}\right)^m \tag{3.1}$$

for every integer $m \geq 0$. Again, by the orthogonality of Dirichlet characters modulo k, the right side of (3.1) becomes

(3.2)
$$\sum_{\substack{n\leq x\\ n\in A\\ n\equiv b \pmod k}} \left(\frac{n}{t\varphi(n)}\right)^m = \frac{1}{\varphi(k)} \sum_{\substack{n\leq x\\ n\in A\\ (n,k)=1}} \left(\frac{n}{t\varphi(n)}\right)^m + \frac{1}{\varphi(k)} \sum_{\chi\neq\chi_0} \overline{\chi}(b) \sum_{\substack{n\leq x\\ n\in A}} \chi(n)\left(\frac{n}{t\varphi(n)}\right)^m.$$

To study the main contribution in (3.2), consider

$$\sum_{\substack{n\leq x\\ n\in A\\ (n,k)=1}} \left(\frac{n}{\varphi(n)}\right)^m = \sum_{\substack{n\leq x\\ (n,k)=1}} \chi_A(n)\left(\frac{n}{\varphi(n)}\right)^m. \tag{3.3}$$

Since χ_A is multiplicative, the partial sums on the right side of (3.3) can be determined with the help of its Dirichlet series and Euler product

(3.4)
$$\begin{aligned}\sum_{\substack{n\geq 1\\ (n,k)=1}} \frac{\chi_A(n)(n/\varphi(n))^m}{n^s} &= \prod_{\substack{p\in A\\ (p,k)=1}} \left(1+\frac{(p/p-1)^m}{p^s}+\cdots+\frac{(p/p-1)^m}{p^{(l-1)s}}\right)\\ &= \prod_{\substack{p\in A\\ (p,k)=1}} \left(1+\frac{(p/p-1)^m(1-(1/p^{(l-1)s}))}{p^s-1}\right),\end{aligned}$$

which is valid for all complex numbers s with $\mathrm{Re}\,(s) > 1$. To complete the product on the right side of (3.4), note that it can be written as

$$\prod_{p}\left(1+\frac{(p/p-1)^m(1-(1/p^{(l-1)s}))}{p^s-1}\right) \prod_{\substack{p\notin A \text{ or}\\ p\mid k}} \left(1+\frac{(p/p-1)^m(1-(1/p^{(l-1)s}))}{p^s-1}\right)^{-1}. \tag{3.5}$$

Since the primes $p \notin A$ form a thin subset and satisfy

$$\sum_{\substack{p\leq x\\ p\notin A}} 1 = O(\log x),$$

we see that

$$\sum_{p\notin A}\frac{1}{p^{\eta_0}}$$

converges for any $1/2 \le \eta_0 < 1$. Fix such an η_0 and consider the half plane of complex numbers s with $\mathrm{Re}\,(s) > \eta_0$. Since

$$\left|\left(\frac{p}{p-1}\right)^m\right| \le 2^m \quad \text{and} \quad \left|1-\frac{1}{p^{(l-1)s}}\right| \le 2,$$

we have

$$\begin{aligned}\sum_{p\notin A}\frac{|(p/p-1)^m(1-(1/p^{(l-1)s}))|}{|p^s-1|} &\le \sum_{p\notin A}\frac{2^{m+1}}{|p^s-1|}\\ &\ll_m \sum_{p\notin A}\frac{1}{p^{\mathrm{Re}\,(s)}-1} \ll_m \sum_{p\notin A}\frac{1}{p^{\eta_0}} < \infty.\end{aligned}$$

Observe that

$$\begin{aligned}&\prod_{\substack{p\notin A \text{ or}\\ p\mid k}}\left|1+\frac{(p/p-1)^m(1-(1/p^{(l-1)s}))}{p^s-1}\right|\\ &= \prod_{\substack{p\notin A \text{ or}\\ p\mid k\\ |(p/p-1)^m(1-(1/p^{(l-1)s}))|\ge|p^s-1|}}\left|1+\frac{(p/p-1)^m(1-(1/p^{(l-1)s}))}{p^s-1}\right|\\ &\quad\times \prod_{\substack{p\notin A \text{ or}\\ p\mid k\\ |(p/p-1)^m(1-(1/p^{(l-1)s}))|<|p^s-1|}}\left|1+\frac{(p/p-1)^m(1-(1/p^{(l-1)s}))}{p^s-1}\right|.\end{aligned}$$

Note that the condition

$$\left|\left(\frac{p}{p-1}\right)^m\left(1-\frac{1}{p^{(l-1)s}}\right)\right| \ge |p^s-1|$$

can hold for only finitely many primes p, since

$$\lim_{p\to\infty}\left(\frac{p}{p-1}\right)^m\left(1-\frac{1}{p^{(l-1)s}}\right) = 1$$

and

$$\begin{aligned}|p^s-1| &\geq |p^s|-1 \geq \sqrt{p}-1\\ &\geq \sqrt{5}-1>1\end{aligned}$$

when $p \geq 5$. Taking into account the finitely many prime divisors of k as well, we see that

$$\begin{aligned}&\prod_{\substack{p\notin A \text{ or}\\ p|k}} \left|1+\frac{(p/p-1)^m(1-(1/p^{(l-1)s}))}{p^s-1}\right|\\ &\gg_{\eta_0,k,m,A}\\ &\times \prod_{\substack{p\notin A \text{ or}\\ p|k\\ |(p/p-1)^m(1-(1/p^{(l-1)s}))|<|p^s-1|}} \left(1-\frac{|(p/p-1)^m(1-(1/p^{(l-1)s}))|}{|p^s-1|}\right)\\ &\geq \frac{1}{C(\eta_0,k,m,A)}>0,\end{aligned}$$

where $C(\eta_0,k,m,A)>0$ is a constant depending only upon η_0,k,m,A. Consequently, one obtains that

$$\left|\prod_{\substack{p\notin A \text{ or}\\ p|k}} \left(1+\frac{(p/p-1)^m(1-(1/p^{(l-1)s}))}{p^s-1}\right)^{-1}\right| \leq C(\eta_0,k,m,A) \tag{3.6}$$

for all complex numbers s with $\mathrm{Re}\,(s)>\eta_0$ and some fixed $1/2 \leq \eta_0 < 1$. We remark that the full force of the assumption on $p \notin A$ will be exploited in the estimation of the error term in (3.2). We may now write, for $\mathrm{Re}\,(s)>1$,

$$\prod_p \left(1+\frac{(p/p-1)^m(1-(1/p^{(l-1)s}))}{p^s-1}\right)=\zeta(s)F(s), \tag{3.7}$$

where

$$F(s)=\prod_p \left(1-\frac{1}{p^s}\right)\left(1+\frac{(p/p-1)^m(1-(1/p^{(l-1)s}))}{p^s-1}\right) \tag{3.8}$$

in (3.7). It turns out that $F(s)$ is analytic more to the left of the half plane $\mathrm{Re}\,(s) > 1$. To make this precise, note that, if we expand the product

$$\left(1 + \frac{p^m}{(p-1)^m} \sum_{j=1}^{l-1} \frac{1}{p^{js}}\right)\left(1 - \frac{1}{p^s}\right),$$

then the terms

$$\left|\frac{p^m}{(p-1)^m p^s} - \frac{1}{p^s}\right| = O\left(\frac{1}{p^{1+\mathrm{Re}\,(s)}}\right)$$

and

$$\left|\frac{p^m}{(p-1)^m p^{ls}}\right| = O\left(\frac{1}{p^{l\mathrm{Re}\,(s)}}\right)$$

dictate a half plane of analyticity for $F(s)$ in (3.8). Therefore, $F(s)$ is analytic for $\mathrm{Re}\,(s) > 1/l$ (note that $1/l \le 1/2$). Now we may use a truncated form of Perron's formula (see [**6**]). Precisely, let $f(n)$ be an arithmetic function with the corresponding Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

having finite abscissa of absolute convergence σ_a. Then, for any $c > \max(0, \sigma_a)$, non-integer value of $x > 1$ and $T > 0$, the formula

$$\sum_{n \le x} f(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} F(s)\, ds + R(x, T)$$

holds, where

$$|R(x,T)| \le \frac{x^c}{T} \sum_{n=1}^{\infty} \frac{|f(n)|}{n^c\, |\log(x/n)|}.$$

In particular, if $|f(n)| \ll_\lambda n^\lambda$ for any $\lambda > 0$ and x is chosen as the average of two consecutive integers, then one further obtains

$$|R(x,T)| = O_\lambda\left(\frac{x^{c+\lambda}}{T}\right)$$

for any $\lambda > 0$. Since

$$\left(\frac{n}{\varphi(n)}\right)^m \ll_{\lambda,m} n^\lambda$$

for any $\lambda > 0$, one arrives at the formula

$$\sum_{\substack{n\le x\\ n\in A\\ (n,k)=1}} \left(\frac{n}{\varphi(n)}\right)^m = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\zeta(s)F(s)P(s)\,ds + O_{\lambda,k,m,A}\left(\frac{x^{c+\lambda}}{T}\right), \tag{3.9}$$

for any $\lambda > 0$, where $c > 1$, T are to be chosen later and

$$P(s) = \prod_{\substack{p\notin A \text{ or}\\ p|k}} \left(1 + \frac{(p/p-1)^m(1-(1/p^{(l-1)s}))}{p^s-1}\right)^{-1}. \tag{3.10}$$

Let C_1 be the line segment connecting $c-iT$ and $c+iT$. Also, let C_2, C_3 and C_4 be the line segments connecting $c+iT$ to η_0+iT, η_0+iT to η_0-iT and η_0-iT to $c-iT$, respectively. Let $C^\times$ be the rectangular contour oriented in the positive direction whose sides are the C_j's. Applying Cauchy's residue theorem to $C^\times$, one obtains:

$$\frac{1}{2\pi i}\left\{\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right\}\frac{x^s}{s}\,\zeta(s)F(s)P(s)\,ds = F(1)P(1)x. \tag{3.11}$$

Using well-known estimates on the Riemann zeta function (see [**14**]) in the form

$$|\zeta(s)| \ll_{\eta_0} |t|^{1-\eta_0} \ll \sqrt{|t|},$$

since $\eta_0 \ge 1/2$, which is uniform for $|t| \ge 1$, and then combining this with (3.6), (3.10) and the fact that $F(s), P(s)$ are both uniformly bounded in terms of η_0, k, m, A for $\mathrm{Re}\,(s) \ge \eta_0$, one derives the estimates

$$\left|\int_{C_2}, \int_{C_4}\right| \ll \frac{1}{\sqrt{T}}\int_{\eta_0}^{c} x^\rho\,d\rho \ll \frac{x^c}{\sqrt{T}\log x}. \tag{3.12}$$

To estimate the line integral over C_3, one can first refer to symmetry and then partition the interval $[0,T]$ as $[0,1]\cup[1,T]$. It follows that

$$\Big|\int_{C_3}\Big| \ll x^{\eta_0}\int_0^1 \frac{|\zeta(\eta_0+it)|}{|\eta_0+it|}|F(\eta_0+it)||P(\eta_0+it)|\,dt$$
$$+\,x^{\eta_0}\int_1^T \frac{|\zeta(\eta_0+it)|}{|\eta_0+it|}|F(\eta_0+it)||P(\eta_0+it)|\,dt.$$

Clearly, the integral over $[0,1]$ is bounded by a uniform constant in terms of η_0,k,m,A. Also, by the uniform estimate on the Riemann zeta function given above for $|t|\geq 1$, we see that

$$\Big|\int_1^T\Big| \ll x^{\eta_0}\int_1^T \frac{\sqrt{t}}{|\eta_0+it|}\,dt \ll x^{\eta_0}\sqrt{T}.$$

Gathering these estimates finally gives

$$\Big|\int_{C_3}\Big| \ll x^{\eta_0}+x^{\eta_0}\sqrt{T}\ll x^{\eta_0}\sqrt{T} \tag{3.13}$$

for $T\geq 1$. In (3.12) and (3.13), the implicit constants all depend upon η_0,k,m,A. Let $T=x^{\xi}$ with $\xi>0$ such that $\eta_0+(\xi/2)<1$. Then we may choose $c>1$ to satisfy $c<1+(\xi/2)$. As a consequence of (3.9), (3.11), (3.12) and (3.13), one can deduce the asymptotic formula:

$$\frac{1}{\varphi(k)}\sum_{\substack{n\leq x\\ n\in A\\ (n,k)=1}}\left(\frac{n}{\varphi(n)}\right)^m = \frac{1}{\varphi(k)}F(1)P(1)x+O_{\eta_0,k,m,A}(x^{\mu}) \tag{3.14}$$

for some $0<\mu<1$. To rewrite the constants on the right side of (3.14), observe that

$$\begin{aligned}\frac{1}{\varphi(k)}&F(1)P(1) \\ &=\frac{1}{k}\prod_{(p,k)=1}\left(1-\frac{1}{p}\right)\prod_{\substack{p\in A\\ (p,k)=1}}\left(1+\frac{(p/p-1)^m(1-(1/p^{l-1}))}{p-1}\right)\\ &=\frac{C_A}{k}\prod_{\substack{p\in A\\ (p,k)=1}}\left(1-\frac{1}{p}\right)\left(1+\frac{(p/p-1)^m(1-(1/p^{l-1}))}{p-1}\right)\\ &=\frac{C_A}{k}\prod_{\substack{p\in A\\ (p,k)=1}}\left(1+\frac{(1-(1/p))^{-m}(1-(1/p^{l-1}))-1}{p}\right).\end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15), one obtains that

$$\begin{aligned}(3.16)\quad \frac{1}{\varphi(k)} \sum_{\substack{n\le x\\ n\in A\\ (n,k)=1}} \left(\frac{n}{\varphi(n)}\right)^m &= \frac{C_A}{k} \prod_{\substack{p\in A\\ (p,k)=1}} \left(1+\frac{(1-(1/p))^{-m}(1-(1/p^{l-1}))-1}{p}\right)x\\ &\quad + O_{\eta_0,k,m,A}(x^{\mu})\end{aligned}$$

for some $0<\mu<1$. To estimate the error term in (3.2), note that

$$\chi_A(n)=\sum_{d^l\mid n}\nu\left(\frac{n}{d^l}\right)\mu_U(d), \tag{3.17}$$

where ν is the characteristic function of the set of integers all of whose prime factors are in U, and μ_U is the Möbius function supported on U. Using (3.17) and denoting by $[v_1,\dots,v_j,d^l]$, the least common multiple of $v_1,\dots,v_j,d^l$, we have

$$\begin{aligned}(3.18)\quad \sum_{\substack{n\le x\\ n\in A}}\chi(n)\left(\frac{n}{\varphi(n)}\right)^m &= \sum_{n\le x}\chi(n)\left(\frac{n}{\varphi(n)}\right)^m \sum_{d^l\mid n}\nu\left(\frac{n}{d^l}\right)\mu_U(d)\\ &= \sum_{d\le x^{1/l}}\mu_U(d)\sum_{\substack{n\le x\\ d^l\mid n}}\chi(n)\nu\left(\frac{n}{d^l}\right)\left(\prod_{j=1}^m\left\{\sum_{v_j\mid n}\frac{\mu^2(v_j)}{\varphi(v_j)}\right\}\right)\\ &= \sum_{d\le x^{1/l}}\mu_U(d)\sum_{\substack{v_j\le x\\ 1\le j\le m}}\frac{\mu^2(v_j)}{\varphi(v_j)}\chi\left([v_1,\dots,v_j,d^l]\right)\\ &\qquad\times\nu\left(\frac{[v_1,\dots,v_j,d^l]}{d^l}\right)\sum_{q\le x/[v_1,\dots,v_j,d^l]}\nu(q)\chi(q)\\ &= O_k\left(x^{B\log 2+(1/l)}(\log x)^m\right)=o(x)\end{aligned}$$

by estimating the sum over q similarly as in (2.11), where we further used the fact that ν is completely multiplicative. Collecting (3.1), (3.2), (3.16) and (3.18) together gives

(3.19)
$$\limsup_{x\to\infty}\Bigg(\frac{1}{x}\sum_{\substack{n\le x\\ n\in A\\ n\equiv b \pmod k\\ n\ge t\varphi(n)}} 1\Bigg) \le \lim_{x\to\infty}\frac{1}{x}\sum_{\substack{n\le x\\ n\in A\\ n\equiv b \pmod k}}\left(\frac{n}{t\varphi(n)}\right)^m = \frac{R(m)}{t^m}$$

for every integer $m \ge 0$, where there is no further need to distinguish between integer and non-integer values of x on the left side of (3.19), and

$$R(m) := \frac{C_A}{k}\prod_{\substack{p\in A\\ (p,k)=1}}\left(1+\frac{(1-(1/p))^{-m}(1-(1/p^{l-1}))-1}{p}\right). \tag{3.20}$$

Next we take $m = m(t) = [y\log y]$, where $y = e^{te^{-\gamma}}$. To estimate the product in (3.20), let us write

$$\prod_{\substack{p\in A\\ (p,k)=1}} = \prod_{\substack{p\le y\\ p\in A\\ (p,k)=1}} \cdot \prod_{\substack{p>y\\ p\in A\\ (p,k)=1}} = P_1\cdot P_2. \tag{3.21}$$

Using $\log(1+x)\le x$, one obtains

(3.22)
$$\begin{aligned}\log P_2 &= \sum_{\substack{p>y\\ p\in A\\ (p,k)=1}}\log\left(1+\frac{(1-(1/p))^{-m}(1-(1/p^{l-1}))-1}{p}\right)\\ &\le \sum_{p>y}\frac{(1-(1/p))^{-m}-1}{p}\\ &= \sum_{y<p\le m}\frac{(1-(1/p))^{-m}-1}{p} + \sum_{p>m}\frac{(1-(1/p))^{-m}-1}{p}.\end{aligned}$$

Observe that

$$\left(1-\frac{1}{p}\right)^{-m} = e^{-m\log(1-(1/p))} = e^{m((1/p)+O(1/p^2))}. \tag{3.23}$$

Using (3.23) and the fact that $e^x = 1 + O(x)$ when $|x|$ is bounded, one gets

$$\sum_{p>m} \frac{(1-(1/p))^{-m}-1}{p} \ll \sum_{p>m} \frac{m}{p^2} = O(1) \tag{3.24}$$

and

(3.25)

$$\begin{aligned}\sum_{y<p\le m} \frac{(1-(1/p))^{-m}-1}{p} &\le \sum_{y<p\le m} \frac{1}{p} e^{m((1/p)+O(1/p^2))} \\ &= \sum_{y<p\le m} \frac{1}{p} \exp\left(\frac{m}{p}+O(1)\right) \\ &\ll \sum_{y<p\le m} \frac{1}{p} e^{m/p} \ll \int_y^{y\log y} \frac{1}{t\log t} e^{m/t}\,dt \\ &\ll \frac{1}{\log y} \int_1^{\log y} \frac{e^u}{u}\,du \ll \frac{y}{(\log y)^2},\end{aligned}$$

where we used the change of variable $u = m/t$. Therefore, combining (3.22), (3.24) and (3.25), we have

$$\log P_2 \ll \frac{y}{\log^2 y}. \tag{3.26}$$

Next observe that

$$\log P_1 \le \sum_{\substack{p\le y\\ p\in A\\ (p,k)=1}} \log\left(1+\frac{(1-(1/p))^{-m}(1-(1/p^{l-1}))}{p}\right). \tag{3.27}$$

Using $\log(1+x) \leq \log x + (1/x)$, one gets from (3.27) that

(3.28)

$$\log P_1 \leq -\sum_{\substack{p\leq y\\ p\in A\\ (p,k)=1}} \log p + m \log \prod_{\substack{p\leq y\\ p\in A\\ (p,k)=1}} \left(1-\frac{1}{p}\right)^{-1} + \log \prod_{\substack{p\leq y\\ p\in A\\ (p,k)=1}} \left(1-\frac{1}{p^{l-1}}\right) + \sum_{\substack{p\leq y\\ p\in A\\ (p,k)=1}} p\left(1-\frac{1}{p}\right)^{m}\left(1-\frac{1}{p^{l-1}}\right)^{-1}.$$

Similarly as above, one can show that

(3.29)

$$\sum_{\substack{p\leq y\\ p\in A\\ (p,k)=1}} p\left(1-\frac{1}{p}\right)^{m}\left(1-\frac{1}{p^{l-1}}\right)^{-1} \leq \sum_{\substack{p\leq y\\ p\in A\\ (p,k)=1}} p\left(1-\frac{1}{p}\right)^{m-1} \ll \sum_{p\leq y} pe^{-m/p} \ll \frac{y}{\log^2 y}.$$

Moreover, using a strong form of Mertens' estimate due to Vinogradov [**17, 18**] (see also [**12**] and the recent work of Languasco and Zaccagnini [**11**] for even stronger forms), we have

(3.30)

$$\begin{aligned}\log \prod_{\substack{p\leq y\\ p\in A\\ (p,k)=1}} \left(1-\frac{1}{p}\right)^{-1} &= \log \prod_{p\leq y} \left(1-\frac{1}{p}\right)^{-1} + \log \prod_{\substack{p\leq y\\ p\notin A \text{ or}\\ p\mid k}} \left(1-\frac{1}{p}\right)\\ &= \log\left(e^{\gamma}\log y + O\left(e^{-c_0\sqrt{\log y}}\right)\right)\\ &\quad - \log C_{A,k} + o(1)\\ &= \log t + O\left(\frac{e^{-c_0\sqrt{\log y}}}{\log y}\right)\\ &\quad - \log C_{A,k} + o(1)\end{aligned}$$

for some constants $c_0 > 0$ and $C_{A,k} \geq 1$. One has to look for a finer estimate for the $o(1)$ term in (3.30). To this end, note that

(3.31)
$$\sum_{\substack{p\leq y\\ p\notin A \text{ or}\\ p|k}} \log\left(1-\frac{1}{p}\right) = \sum_{\substack{p\notin A \text{ or}\\ p|k}} \log\left(1-\frac{1}{p}\right) + O_k\left(\sum_{\substack{p>y\\ p\notin A}} \left|\log\left(1-\frac{1}{p}\right)\right|\right).$$

Since
$$\sum_{\substack{p>y\\ p\notin A}} \left|\log\left(1-\frac{1}{p}\right)\right| \longrightarrow 0$$
as y tends to infinity, it follows that
$$-\log C_{A,k} = \sum_{\substack{p\notin A \text{ or}\\ p|k}} \log\left(1-\frac{1}{p}\right).$$

Clearly,

(3.32)
$$\sum_{\substack{p>y\\ p\notin A}} \left|\log\left(1-\frac{1}{p}\right)\right| \leq \sum_{\substack{p>y\\ p\notin A}} \frac{1}{p-1} \leq 2\sum_{\substack{p>y\\ p\notin A}} \frac{1}{p}$$

holds. Let p_n be the nth prime not in A. Since
$$\sum_{\substack{p\leq p_n\\ p\notin A}} 1 = \sum_{\substack{p\leq p_n\\ p\notin U}} 1 = n \leq B\log p_n,$$
$p_n \geq n^2$ holds for all n large enough. Therefore,

(3.33)
$$\sum_{\substack{p>y\\ p\notin A}} \frac{1}{p} \leq \sum_{n>\sqrt{y}} \frac{1}{n^2} + \sum_{n\leq\sqrt{y}} \frac{1}{y} \ll \frac{1}{\sqrt{y}}$$

follows. Gathering (3.30)–(3.33), we see that

(3.34)
$$m\log\prod_{\substack{p\le y\\ p\in A\\ (p,k)=1}}\left(1-\frac{1}{p}\right)^{-1}=m\log t+O\left(ye^{-c_0\sqrt{\log y}}\right)$$
$$-(\log C_{A,k})[y\log y]+O\left(\sqrt{y}\log y\right).$$

Combining (3.28), (3.29) and (3.34), we have

(3.35)
$$\log P_1\le m\log t-(\log C_{A,k})[y\log y]-\sum_{\substack{p\le y\\ p\in A\\ (p,k)=1}}\log p$$
$$+\log\prod_{\substack{p\le y\\ p\in A\\ (p,k)=1}}\left(1-\frac{1}{p^{l-1}}\right)+O\left(\frac{y}{\log^2 y}\right).$$

It follows from (3.19), (3.20), (3.21), (3.26) and (3.35) that

$$\limsup_{x\to\infty}\left(\frac{1}{x}\sum_{\substack{n\le x\\ n\in A\\ n\equiv b \pmod k\\ n\ge t\varphi(n)}}1\right)\le\frac{C_A}{k}\exp\left(\log P_1+\log P_2-m\log t\right)$$
$$\le\frac{C_A}{k}\prod_{\substack{p\le y\\ p\in A\\ (p,k)=1}}\left(1-\frac{1}{p^{l-1}}\right)$$
$$\times\exp\left(-(\log C_{A,k})[y\log y]-\sum_{\substack{p\le y\\ p\in A\\ (p,k)=1}}\log p+O\left(\frac{y}{\log^2 y}\right)\right).$$

This completes the proof of Theorem 2. $\square$

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