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EXPONENTIAL SYNCHRONIZATION OF DELAYED REACTION-DIFFUSION NEURAL NETWORKS WITH GENERAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we study delayed reaction-diffusion neural networks with general boundary conditions including Dirichlet and Neumann boundary conditions. By using some inequality techniques and constructing suitable Lyapunov functionals, some sufficient conditions are given to ensure the exponential synchronization of the drive-response neural networks. Finally, an example is given to verify the theoretical analysis.

1. Introduction. In recent years, different types of neural networks with or without delays [**2, 4–6, 10, 14, 15, 16, 18–20, 23, 26–30**], have been widely investigated due to their applicability in solving some problems for image processing, signal processing and pattern recognition. Since chaos synchronization was introduced by Pecora and Carroll in the 1990s [**12, 13**] by proposing the drive-response concept, its potential applications are found in many different areas including secure communication, chaos generators, chemical reactions, biological systems, information science, etc. As artificial neural networks can exhibit some complicated dynamics and even chaotic behaviors, the synchronization of coupled chaos systems and chaos neural networks has received considerable attention [**1, 7–9, 11, 17, 21, 22, 24, 25**].

Because of the finite processing speed of information, it is sometimes necessary to take account of time delays in the modeling of biological or artificial neural networks. Time delays may lead to bifurcation, oscillation, divergence or instability, which may be harmful to a system; thus, it is extremely important to study the synchronization of delayed

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neural networks [**1, 7–9, 11, 17, 21, 22, 24, 25**]. Although the models of delayed feedback with discrete delays are a good approximation in simple circuits, neural networks usually have a spatial nature due to the presence of a number of parallel pathways of variety of axon sizes and length. A distribution of conduction velocities along these pathways will lead to a distribution of propagation delays. Therefore, the models with time-varying delays and continuous distributed delays are more appropriate to the synchronization of neural networks [**9, 17, 22, 24**].

Just as diffusion effects cannot be avoided in the research of biological and artificial neural networks, they must be considered in the research of synchronization of chaotic neural networks because electrons are moving in the asymmetric electromagnetic field and the activations vary in space as well as in time. References [**9, 17, 22, 24**] have considered the synchronization of neural networks with diffusion terms, which are expressed by partial differential equations. In these papers, the boundary conditions are all assumed to be Dirichlet boundary conditions with the unknown functions being zeros, or to be that the derivatives of unknown functions with respect to spatial variables are zeros, which are special cases of Neumann boundary conditions.

To the best of our knowledge, exponential synchronization has not been considered for delayed reaction-diffusion neural networks with general boundary conditions. In this paper, by using some inequality techniques and the Lyapunov functional, exponential synchronization will be investigated for delayed reaction-diffusion neural networks with general boundary conditions. Some sufficient conditions are presented, which only rely upon the coefficients in the drive networks and the suitably designed controller gain matrix in response networks. It is easy to verify these conditions because they are expressed by algebraic inequalities. These results generalize the corresponding ones in [**9, 17, 22, 24**].

This paper is organized as follows. In Section 2, model description and main results are presented. In Section 3, by constructing a suitable Lyapunov functional, some sufficient conditions are obtained to ensure exponential synchronization of the drive and response neural networks. In Section 4, an example is given to verify the theoretical analysis. Section 5 contains the conclusions of the paper.

2. Model description and main results. Consider the following delayed reaction-diffusion neural networks with general boundary conditions:

$$
(1)\qquad \begin{cases} \dfrac{\partial u_i(x,t)}{\partial t} = \displaystyle\sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_i \frac{\partial u_i(x,t)}{\partial x_k}\right) - a_i u_i(x,t) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{n} c_{ij} f_j(u_j(x,t)) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{n} \omega_{ij} f_j(u_j(x,t-\tilde{t}_{ij}(t))) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{n} b_{ij} \int_{-\infty}^{0} \kappa_{ij}(t-s) f_j(u_j(x,s))\,ds + I_i, \\ u_i(x,t) = \varphi_i(x,t), \quad -\infty < t \le 0,\ x \in \Omega, \\ \sigma_{i1} \dfrac{\partial u_i(x,t)}{\partial n} + \sigma_{i2} u_i(x,t) = 0, \\ \qquad (x \in \partial\Omega, \sigma_{i1} \ge 0, \sigma_{i2} \ge 0 \text{ and } \sigma_{i1}^2 + \sigma_{i2}^2 \ne 0), \end{cases}
$$

where $i = 1, \dots, n$. n is the number of neurons in the networks. $u(x,t) = (u_1(x,t), \dots, u_n(x,t))^T$, $u_i(x,t)$ denotes the state of the ith neural unit at time t and in space $x \in \Omega$. Ω is a bounded open domain in $\mathbf{R}^m$ with smooth boundary $\partial\Omega$ and $\operatorname{mes}\Omega > 0$ denotes the measure of Ω. The smooth function $D_i = D_i(x,t,u) \ge 0$ corresponds to the transmission diffusion coefficient along the ith unit. $a_i > 0$ is a constant and represents the rate with which the ith neuron will reset its potential to the resting state in isolation when it is disconnected from networks and external inputs. c_{ij}, ω_{ij} and b_{ij} are constants and represent the weights of neuron interconnection without delays and with delays, respectively. $\tilde{t}_{ij}(t)$ corresponds to transmission delays along the axon of the jth neuron from the ith neuron at time t and satisfies

$$0 \le \tilde{t}_{ij}(t) \le \tau, \qquad \dot{\tilde{t}}_{ij}(t) < \rho < 1,$$

where $\tau > 0$ and $\rho < 1$ are constants. $f_i(\cdot)$ shows how the ith neuron reacts to the input. $\kappa_{ij}(\cdot)$, $(i,j = 1, \dots, n)$ are delay kernels. $I = (I_1, \dots, I_n)^T$, I_i is the input from outside the system. $\varphi(x,t) = (\varphi_1(x,t), \dots, \varphi_n(x,t))^T$, for any given $i = 1, \dots, n$, $\varphi_i(x,t)$ is a given smooth function defined on $\Omega \times (-\infty, 0]$ with the norm

$$\|\varphi\|_2 = \sqrt{\sum_{i=1}^{n} \int_{\Omega} |\varphi_i(x,\cdot)|_{\infty}^2 dx},$$

$$\text{where } |\varphi_i(x,\cdot)|_{\infty} = \sup_{-\infty < s \leq 0} |\varphi_i(x,s)|.$$

Chaos dynamics is extremely sensitive to initial conditions. Even infinitesimal changes in initial conditions will lead to an asymptotic divergence of orbits. In order to observe the synchronization behavior in this class of neural networks, we give two neural networks in which the drive system with state variable denoted by $u_i(x,t)$ drives the response system having identical dynamical equations with the state variable denoted by $\widetilde{u}_i(x,t)$. However, the initial condition of the system derived is different from that of the response system. Therefore, response neural networks can be described as in the following model.

$$(2)\qquad \begin{cases} \dfrac{\partial \widetilde{u}_i(x,t)}{\partial t} = \displaystyle\sum_{k=1}^{m} \frac{\partial}{\partial x_k}(D_i \frac{\partial \widetilde{u}_i(x,t)}{\partial x_k}) - a_i \widetilde{u}_i(x,t) \\ \qquad + \displaystyle\sum_{j=1}^{n} c_{ij} f_j(\widetilde{u}_j(x,t)) \\ \qquad + \displaystyle\sum_{j=1}^{n} \omega_{ij} f_j(\widetilde{u}_j(x, t - \tilde{t}_{ij}(t))) \\ \qquad + \displaystyle\sum_{j=1}^{n} b_{ij} \int_{-\infty}^{0} \kappa_{ij}(t-s) f_j(\widetilde{u}_j(x,s))\, ds \\ \qquad + I_i + v_i(x,t), \\ \widetilde{u}_i(x,t) = \psi_i(x,t), \quad -\infty < t \leq 0,\ x \in \Omega, \\ \sigma_{i1} \dfrac{\partial \widetilde{u}_i(x,t)}{\partial n} + \sigma_{i2} \widetilde{u}_i(x,t) = 0, \\ \qquad (x \in \partial\Omega, \sigma_{i1} \geq 0, \sigma_{i2} \geq 0 \text{ and } \sigma_{i1}^2 + \sigma_{i2}^2 \neq 0), \end{cases}$$

where $i = 1, \ldots, n$. $v = (v_1, \ldots, v_n)^T$ is the control input vector, and v_i stands for the external control input that will be appropriately designed for a certain control objective. In this paper, the control input vector $v = (v_1, \dots, v_n)^T$ is assumed to take as the following form

$$(3) \quad (v_1, \dots, v_n)^T = M(u_1(x,t) - \widetilde{u}_1(x,t), \dots, u_n(x,t) - \widetilde{u}_n(x,t))^T,$$

where $M = (M_{ij})_{n\times n}$ is the controller gain matrix and will be appropriately chosen for synchronization in both drive and response systems. $\psi = (\psi_1, \dots, \psi_n)^T$, $\psi_i(x,s) (i = 1, \dots, n)$ are bounded smooth functions defined on $\Omega \times (-\infty, 0]$ with the norm

$$\|\psi\|_2 = \sqrt{\sum_{i=1}^{n} \int_\Omega |\psi_i(x,\cdot)|_\infty^2 \, dx},$$

where

$$|\psi_i(x,\cdot)|_\infty = \sup_{-\infty < s \le 0} |\psi_i(x,s)|.$$

In order to deal with the case in which $\sigma_{i1} = 0$, $\sigma_{i2} \neq 0$ $(i = 1, \dots, n)$, we introduce the following lemma [**3**].

Lemma 1. *Let Ω be a bounded open domain in $\mathbf{R}^m$ with smooth boundary $\partial\Omega$. If $u = u(x)$ defined on Ω is a smooth function with $u|_{\partial\Omega} = 0$, then the following inequality holds*:
(4)

$$\int_\Omega u^2(x)\, dx \le \left(\frac{|\Omega|}{\omega_m}\right)^{1/m} \int_\Omega \left(\frac{\partial u}{\partial x}\right)^2 dx \triangleq \left(\frac{|\Omega|}{\omega_m}\right)^{1/m} \int_\Omega \sum_{k=1}^{m} \left(\frac{\partial u}{\partial x_k}\right)^2 dx,$$

where $|\Omega|$ denotes the volume of Ω and ω_m denotes the surface area of unit ball in $\mathbf{R}^m$.

Throughout the paper, we always assume that system (1) has a smooth solution $u(x,t)$ with the norm

$$\|u(\cdot,t)\|_2 = \sqrt{\sum_{i=1}^{n} \int_\Omega |u_i(x,t)|^2 dx}, \quad \text{for all } t \in [0, +\infty).$$

Definition 1. The drive system (1) and the response system (2) are said to be exponentially synchronized if there exist positive constants $\gamma > 0$ and $\varepsilon > 0$ such that

$$\|u(\cdot,t) - \widetilde{u}(\cdot,t)\|_2 \leq \gamma\|\varphi - \psi\|_2 \exp^{-\varepsilon t}, \quad \text{for all } t \geq 0,$$

where the constant ε is said to be the exponential synchronization rate.

Definition 2. Let $V : \mathbf{R} \to \mathbf{R}$ be a continuous function. The upper right-hand Dini derivative D^+V is defined by

$$D^+V = \limsup_{h\to+0} \frac{V(t+h) - V(t)}{h}.$$

In this paper, we always assume that:

$(H1)$ There exist constants $B_i (i = 1, \ldots, n)$ such that

$$D_i(x,t,u) \geq B_i \geq 0,$$
$$\text{for all } x \in \Omega, \text{ for all } t \in [0,+\infty), \text{ for all } u \in \mathbf{R}^n, \ i = 1, \ldots, n.$$

$(H2)$ There exist positive constants $\beta_i (i = 1, \ldots, n)$ such that

$$|f_i(x_i) - f_i(y_i)| \leq \beta_i |x_i - y_i|, \quad \text{for all } x_i, y_i \in \mathbf{R}, \ i = 1, \ldots, n.$$

$(H3)$ The delay kernels $\kappa_{ij} (i, j = 1, \ldots, n)$ satisfy the following assumptions:

(5)
$$\kappa_{ij} : [0,+\infty) \to [0,+\infty) \text{ is a bounded and continuous function.}$$
(6)
$$\int_0^{+\infty} \kappa_{ij}(s)\, ds = 1.$$
(7)
$$\text{There exists a constant } \sigma > 0 \text{ such that } \int_0^{+\infty} \kappa_{ij}(s) e^{2\sigma s} ds < +\infty.$$

$(H4)$ For any given $i = 1, \ldots, n$, there exist constants $\delta > 0$, p_{ij}, p^*_{ij}, r_{ij}, r^*_{ij} $(j = 1, \ldots, n)$ such that

$$\begin{aligned} -2a_i - \frac{2B_i}{\delta} + \sum_{j=1}^{n} [(\beta_j |c_{ij}|)^{2p_{ij}} + (\beta_i |c_{ji}|)^{2-2p_{ji}}] \\ + \sum_{j=1}^{n} [(\beta_j |\omega_{ij}|)^{2p^*_{ij}} + \frac{1}{1-\rho} (\beta_i |\omega_{ji}|)^{2-2p^*_{ji}}] \\ + \sum_{j=1}^{n} [(\beta_j |b_{ij}|)^{2r_{ij}} + (\beta_i |b_{ji}|)^{2-2r_{ji}}] \\ + \sum_{j=1}^{n} [|M_{ij}|^{2r^*_{ij}} + |M_{ji}|^{2-2r^*_{ji}}] < 0, \end{aligned}$$

where $\rho < 1$ is defined by the time-varying delays.

Remark 1. In $(H4)$, positive constant δ depends upon the volume of the domain Ω and the surface area of the unit ball in $\mathbf{R}^m$.

Remark 2. $(H4)$ includes the constants B_i defined according to the diffusion coefficients D_i, which implies that the diffusion coefficients will affect the exponential synchronization of reaction-diffusion neural networks.

$(H5)$ For any given $i = 1, \ldots, n$, there exist constants p_{ij}, p^*_{ij}, r_{ij}, r^*_{ij} $(j = 1, \ldots, n)$ such that

$$\begin{aligned} -2a_i + \sum_{j=1}^{n} [(\beta_j |c_{ij}|)^{2p_{ij}} + (\beta_i |c_{ji}|)^{2-2p_{ji}}] \\ + \sum_{j=1}^{n} [(\beta_j |\omega_{ij}|)^{2p^*_{ij}} + \frac{1}{1-\rho} (\beta_i |\omega_{ji}|)^{2-2p^*_{ji}}] \\ + \sum_{j=1}^{n} [(\beta_j |b_{ij}|)^{2r_{ij}} + (\beta_i |b_{ji}|)^{2-2r_{ji}}] \\ + \sum_{j=1}^{n} [|M_{ij}|^{2r^*_{ij}} + |M_{ji}|^{2-2r^*_{ji}}] < 0. \end{aligned}$$

Remark 3. Assumption $(H5)$ is stronger than Assumption $(H4)$, that is, if $(H5)$ holds, then $(H4)$ must hold.

The main result is the following theorem:

Theorem 1. (1) *If* $\sigma_{i1} = 0$ $(i = 1, \dots, n)$ *and* $(H1)$–$(H4)$ *hold, then the drive-response neural networks* (1) *and* (2) *are exponential synchronization.*

(2) *If* $\sigma_{i1} \neq 0$ $(i = 1, \dots, n)$ *and* $(H2)$–$(H3)$, $(H5)$ *hold, then the drive-response neural networks* (1) *and* (2) *are exponential synchronization.*

Remark 4. Let $p_{ij} = p^*_{ij} = r_{ij} = r^*_{ij} = 1/2$. Then $(H4)$ can be changed into the following inequality:

$$(H4)_1 \quad -2a_i - \frac{2B_i}{\delta} + \sum_{j=1}^{n} \Big[\beta_j |c_{ij}| + \beta_i |c_{ji}| + \beta_j |\omega_{ij}| + \frac{1}{1-\rho} (\beta_i |\omega_{ji}|) + \beta_j |b_{ij}| + \beta_i |b_{ji}| + |M_{ij}| + |M_{ji}| \Big] < 0.$$

Remark 5. Let $p_{ij} = p^*_{ij} = r_{ij} = r^*_{ij} = 1/2$. Then $(H5)$ can be changed into the following inequality:

$$(H5)_1 \quad -2a_i + \sum_{j=1}^{n} \Big[\beta_j |c_{ij}| + \beta_i |c_{ji}| + \beta_j |\omega_{ij}| + \frac{1}{1-\rho} (\beta_i |\omega_{ji}|) + \beta_j |b_{ij}| + \beta_i |b_{ji}| + |M_{ij}| + |M_{ji}| \Big] < 0.$$

Then, by Theorem 1, we have the following corollary.

Corollary 1. (1) *If* $\sigma_{i1} = 0$ $(i = 1, \dots, n)$ *and* $(H1)$–$(H3)$ *hold, and, further, if* $(H4)_1$ *holds, then the drive-response neural networks* (1) *and* (2) *are exponential synchronization.*

(2) *If* $\sigma_{i1} \neq 0$ $(i = 1, \ldots, n)$ *and* $(H2)$–$(H3)$ *hold, and, further, if* $(H5)_1$ *holds, then the drive-response neural networks* (1) *and* (2) *are exponential synchronization.*

3. Proof of the main result. Let us define the synchronization error signal $\varepsilon_i(x,t) \triangleq u_i(x,t) - \widetilde{u}_i(x,t)$, where $u_i(x,t)$ and $\widetilde{u}_i(x,t)$ are the ith state variables of the drive and response neural networks, respectively. Therefore, the dynamics error between (1) and (2) can be expressed as follows:

(8)

$$\begin{cases} \dfrac{\partial \varepsilon_i(x,t)}{\partial t} = \displaystyle\sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_i \frac{\partial \varepsilon_i(x,t)}{\partial x_k}\right) - a_i \varepsilon_i(x,t) \\ \qquad + \displaystyle\sum_{j=1}^{n} c_{ij}(f_j(u_j(x,t)) - f_j(\widetilde{u}_j(x,t))) \\ \qquad + \displaystyle\sum_{j=1}^{n} \omega_{ij}(f_j(u_j(x,t-\tilde{t}_{ij}(t))) - f_j(\widetilde{u}_j(x,t-\tilde{t}_{ij}(t)))) \\ \qquad + \displaystyle\sum_{j=1}^{n} b_{ij} \int_{-\infty}^{0} \kappa_{ij}(t-s)(f_j(u_j(x,s)) - f_j(u_j(x,s)))\, ds - v_i, \\ \varepsilon_i(x,t) = \varphi_i(x,t) - \psi_i(x,t), \quad -\infty < t \leq 0,\ x \in \Omega, \\ \sigma_{i1} \dfrac{\partial \varepsilon_i(x,t)}{\partial n} + \sigma_{i2} \varepsilon_i(x,t) = 0, \\ \qquad (x \in \partial\Omega, \sigma_{i1} \geq 0, \sigma_{i2} \geq 0 \text{ and } \sigma_{i1}^2 + \sigma_{i2}^2 \neq 0), \end{cases}$$

where $i = 1, \ldots, n$ and the control input vector $v(x,t) = (v_1(x,t), \ldots, v_n(x,t))^T$ takes the form defined in (3).

By (7) and assumptions $(H4)$ or $(H5)$, there exists a positive constant λ such that

(9)

$$W_i^1 \triangleq 2\lambda - 2a_i - \frac{2B_i}{\delta} + \sum_{j=1}^{n} [(\beta_j |c_{ij}|)^{2p_{ij}} + (\beta_i |c_{ji}|)^{2-2p_{ji}}] + \sum_{j=1}^{n} \left[(\beta_j |\omega_{ij}|)^{2p_{ij}^*} + \frac{e^{2\lambda\tau}}{1-\rho} (\beta_i |\omega_{ji}|)^{2-2p_{ji}^*}\right]$$

$$+\sum_{j=1}^{n}\left[(\beta_j|b_{ij}|)^{2r_{ij}}+(\beta_i|b_{ji}|)^{2-2r_{ji}}\int_0^{+\infty}\kappa_{ji}(s)e^{2\lambda s}ds\right]$$
$$+\sum_{j=1}^{n}[|M_{ij}|^{2r_{ij}^*}+|M_{ji}|^{2-2r_{ji}^*}]\le 0.$$

or

$$\begin{aligned} W_i^2 \triangleq\ & 2\lambda-2a_i \\ & +\sum_{j=1}^{n}[(\beta_j|c_{ij}|)^{2p_{ij}}+(\beta_i|c_{ji}|)^{2-2p_{ji}}] \\ & +\sum_{j=1}^{n}\left[(\beta_j|\omega_{ij}|)^{2p_{ij}^*}+\frac{e^{2\lambda\tau}}{1-\rho}(\beta_i|\omega_{ji}|)^{2-2p_{ji}^*}\right] \\ & +\sum_{j=1}^{n}\left[(\beta_j|b_{ij}|)^{2r_{ij}}+(\beta_i|b_{ji}|)^{2-2r_{ji}}\int_0^{+\infty}\kappa_{ji}(s)e^{2\lambda s}ds\right] \\ & +\sum_{j=1}^{n}[|M_{ij}|^{2r_{ij}^*}+|M_{ji}|^{2-2r_{ji}^*}]\le 0. \end{aligned} \tag{10}$$

Now we begin to prove Theorem 1.

Proof of Theorem 1. Taking the Laypunov functional as follows:

$$\begin{aligned} V(t)=\sum_{i=1}^{n}\int_\Omega\Bigg[& \varepsilon_i^2(x,t)e^{2\lambda t} \\ & +\sum_{j=1}^{n}\frac{1}{1-\rho}(\beta_j|\omega_{ij}|)^{2-2p_{ij}^*}\int_{t-\tilde{t}_{ij}(t)}^{t}\varepsilon_j^2(x,s)e^{2\lambda(s+\tau)}ds \\ & +\sum_{j=1}^{n}(\beta_j|b_{ij}|)^{2-2r_{ij}}\int_0^{+\infty}\kappa_{ij}(s) \\ & \qquad\times\int_{t-s}^{t}\varepsilon_j^2(x,z)e^{2\lambda(z+s)}dz\,ds\Bigg]dx. \end{aligned}$$

Calculating $D^+V(t)$ along the solution to system (8), we have

$$D^+V(t)=\sum_{i=1}^{n}\int_\Omega\left[2\lambda\varepsilon_i^2(x,t)e^{2\lambda t}+2\varepsilon_i(x,t)e^{2\lambda t}\frac{\partial\varepsilon_i(x,t)}{\partial t}\right.$$

$$
\begin{aligned}
&+ \sum_{j=1}^{n} (\beta_j |\omega_{ij}|)^{2-2p_{ij}^*} \left(\frac{e^{2\lambda(t+\tau)}}{1-\rho} \varepsilon_j^2(x,t) - \frac{e^{2\lambda t} e^{2\lambda(\tau - \tilde{t}_{ij}(t))}}{1-\rho} \right. \\
&\qquad\qquad \left. \times (1 - \dot{\tilde{t}}_{ij}(t)) \varepsilon_j^2(x, t - \tilde{t}_{ij}(t)) \right) \\
&+ \sum_{j=1}^{n} (\beta_j |b_{ij}|)^{2-2r_{ij}} \\
&\qquad \left. \times \int_0^{+\infty} \kappa_{ij}(s) (\varepsilon_j^2(x,t) e^{2\lambda(t+s)} - \varepsilon_j^2(x, t-s) e^{2\lambda t}) \, ds \right] dx \\
< &\sum_{i=1}^{n} \int_\Omega \left[2\lambda \varepsilon_i^2(x,t) e^{2\lambda t} + 2\varepsilon_i(x,t) e^{2\lambda t} \frac{\partial \varepsilon_i(x,t)}{\partial t} \right. \\
&+ \sum_{j=1}^{n} (\beta_j |\omega_{ij}|)^{2-2p_{ij}^*} \left(\frac{e^{2\lambda(t+\tau)}}{1-\rho} \varepsilon_j^2(x,t) - e^{2\lambda t} \varepsilon_j^2(x, t - \tilde{t}_{ij}(t)) \right) \\
&+ \sum_{j=1}^{n} (\beta_j |b_{ij}|)^{2-2r_{ij}} \int_0^{+\infty} \kappa_{ij}(s) (\varepsilon_j^2(x,t) e^{2\lambda(t+s)} \\
&\qquad\qquad \left. - \varepsilon_j^2(x, t-s) e^{2\lambda t}) \, ds \right] dx \\
= &\sum_{i=1}^{n} \int_\Omega \left\{ 2\lambda \varepsilon_i^2(x,t) e^{2\lambda t} + 2\varepsilon_i(x,t) e^{2\lambda t} \right. \\
&\qquad \times \left\{ \sum_{k=1}^{m} \frac{\partial}{\partial x_k} \left(D_i \frac{\partial \varepsilon_i(x,t)}{\partial x_k} \right) - a_i \varepsilon_i(x,t) \right. \\
&\qquad\quad + \sum_{j=1}^{n} c_{ij} (f_j(u_j(x,t)) - f_j(\widetilde{u}_j(x,t))) \\
&\qquad + \sum_{j=1}^{n} \left[\omega_{ij} (f_j(u_j(x, t - \tilde{t}_{ij}(t))) - f_j(\widetilde{u}_j(x, t - \tilde{t}_{ij}(t)))) \right. \\
&\quad \left. \left. + b_{ij} \int_{-\infty}^{0} \kappa_{ij}(t-s) (f_j(u_j(x,s)) - f_j(u_j(x,s))) \, ds \right] - v_i \right\} \\
&+ \sum_{j=1}^{n} (\beta_j |\omega_{ij}|)^{2-2p_{ij}^*} \left(\frac{e^{2\lambda(t+\tau)}}{1-\rho} \varepsilon_j^2(x,t) - e^{2\lambda t} \varepsilon_j^2(x, t - \tilde{t}_{ij}(t)) \right)
\end{aligned}
$$

$$+\sum_{j=1}^{n}(\beta_j|b_{ij}|)^{2-2r_{ij}}\int_0^{+\infty}\kappa_{ij}(s)(\varepsilon_j^2(x,t)e^{2\lambda(t+s)}$$
$$-\varepsilon_j^2(x,t-s)e^{2\lambda t})\,ds\Big\}\,dx$$
$$\leq\sum_{i=1}^{n}\int_\Omega\Big\{2\lambda\varepsilon_i^2(x,t)e^{2\lambda t}+2\varepsilon_i(x,t)e^{2\lambda t}\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\Big(D_i\frac{\partial\varepsilon_i(x,t)}{\partial x_k}\Big)$$
$$-2a_ie^{2\lambda t}\varepsilon_i^2(x,t)+2|\varepsilon_i(x,t)|e^{2\lambda t}$$
$$\times\sum_{j=1}^{n}\Big[\beta_j|c_{ij}||\varepsilon_j(x,t)|+\beta_j|\omega_{ij}||\varepsilon_j(x,t-\tilde{t}_{ij}(t))|$$
$$+\beta_j|b_{ij}|\int_0^{+\infty}\kappa_{ij}(s)|\varepsilon_j(x,t-s)|\,ds+|M_{ij}||\varepsilon_j(x,t)|\Big]$$
$$+\sum_{j=1}^{n}(\beta_j|\omega_{ij}|)^{2-2p_{ij}^*}\Big(\frac{e^{2\lambda(t+\tau)}}{1-\rho}\varepsilon_j^2(x,t)-e^{2\lambda t}\varepsilon_j^2(x,t-\tilde{t}_{ij}(t))\Big)$$
$$+\sum_{j=1}^{n}(\beta_j|b_{ij}|)^{2-2r_{ij}}\int_0^{+\infty}\kappa_{ij}(s)(\varepsilon_j^2(x,t)e^{2\lambda(t+s)}$$
$$-\varepsilon_j^2(x,t-s)e^{2\lambda t})\,ds\Big\}\,dx.$$

It is easy to see that

$$2\beta_j|c_{ij}||\varepsilon_i(x,t)||\varepsilon_j(x,t)|$$
$$\leq(\beta_j|c_{ij}|)^{2p_{ij}}\varepsilon_i^2(x,t)+(\beta_j|c_{ij}|)^{2-2p_{ij}}\varepsilon_j^2(x,t),$$
$$2\beta_j|\omega_{ij}||\varepsilon_i(x,t)||\varepsilon_j(x,t-\tilde{t}_{ij}(t))|$$
$$\leq(\beta_j|\omega_{ij}|)^{2p_{ij}^*}\varepsilon_i^2(x,t)+(\beta_j|\omega_{ij}|)^{2-2p_{ij}^*}\varepsilon_j^2(x,t-\tilde{t}_{ij}(t)),$$
$$2\beta_j|b_{ij}||\varepsilon_i(x,t)||\varepsilon_j(x,t-s)|$$
$$\leq(\beta_j|b_{ij}|)^{2r_{ij}}\varepsilon_i^2(x,t)+(\beta_j|b_{ij}|)^{2-2r_{ij}}\varepsilon_j^2(x,t-s),$$
$$2|M_{ij}\varepsilon_i(x,t)\varepsilon_j(x,t)|$$
$$\leq|M_{ij}|^{2r_{ij}^*}\varepsilon_i^2(x,t)+|M_{ij}|^{2-2r_{ij}^*}\varepsilon_j^2(x,t).$$

By using the Green formula, one has

(12)
$$\int_\Omega\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\Big(D_i\frac{\partial\varepsilon_i(x,t)}{\partial x_k}\Big)\varepsilon_i(x,t)\,dx$$

$$
\begin{aligned}
&= \int_\Omega \sum_{k=1}^m \frac{\partial}{\partial x_k}\left(D_i \frac{\partial \varepsilon_i(x,t)}{\partial x_k}\varepsilon_i(x,t)\right) dx \\
&\quad - \int_\Omega \sum_{k=1}^m \frac{\partial \varepsilon_i(x,t)}{\partial x_k} D_i \frac{\partial \varepsilon_i(x,t)}{\partial x_k}\, dx \\
&= \int_{\partial\Omega} D_i \varepsilon_i(x,t) \sum_{k=1}^m \frac{\partial \varepsilon_i(x,t)}{\partial x_k} \cos\widehat{(x_k,n)}\, ds \\
&\quad - \int_\Omega D_i \sum_{k=1}^m \left(\frac{\partial \varepsilon_i(x,t)}{\partial x_k}\right)^2 dx \\
&= \int_{\partial\Omega} D_i \varepsilon_i(x,t) \frac{\partial \varepsilon_i(x,t)}{\partial n}\, ds - \int_\Omega D_i \left(\frac{\partial \varepsilon_i(x,t)}{\partial x}\right)^2 dx.
\end{aligned}
$$

And, noting the boundary condition, if $\sigma_{i1} = 0$ $(i = 1, \dots, n)$ and $(H1)$ hold, by $D_i \geq B_i \geq 0$ and Lemma 1, from (12), we have

$$
\begin{aligned}
\int_\Omega \sum_{k=1}^m \frac{\partial}{\partial x_k}\left(D_i \frac{\partial \varepsilon_i(x,t)}{\partial x_k}\right)\varepsilon_i(x,t)\, dx &= -\int_\Omega D_i \left(\frac{\partial \varepsilon_i(x,t)}{\partial x}\right)^2 dx \\
&\leq -\frac{B_i}{\delta}\int_\Omega \varepsilon_i^2(x,t)\, dx,
\end{aligned} \tag{13}
$$

where $\delta = (|\Omega|/\omega_m)^{1/m} > 0$. And, further noting the boundary condition, if $\sigma_{i1} \neq 0$ $(i = 1, \dots, n)$, by $D_i \geq 0$ and $\partial\varepsilon_i(x,t)/\partial n = -(\sigma_{i2}/\sigma_{i1})\varepsilon_i(x,t)$, from (12), we have

$$
\begin{aligned}
\int_\Omega \sum_{k=1}^m \frac{\partial}{\partial x_k}\left(D_i \frac{\partial \varepsilon_i(x,t)}{\partial x_k}\right)\varepsilon_i(x,t)\, dx &= -\int_{\partial\Omega} \frac{\sigma_{i2}}{\sigma_{i1}} D_i \varepsilon_i^2(x,t)\, ds \\
&\quad - \int_\Omega D_i \left(\frac{\partial \varepsilon_i(x,t)}{\partial x}\right)^2 dx \leq 0.
\end{aligned} \tag{14}
$$

Hence, if $\sigma_{i1} = 0$ $(i = 1, \dots, n)$ and $(H1)$ hold, it follows from (11) and (13) that

$$
D^+V(t) < \sum_{i=1}^n \int_\Omega \left\{ e^{2\lambda t}\varepsilon_i^2(x,t)\left[2\lambda - 2a_i - \frac{2B_i}{\delta} + \sum_{j=1}^n ((\beta_j|c_{ij}|)^{2p_{ij}} \right.\right. \tag{15}
$$

$$
\begin{aligned}
&+(\beta_i|c_{ji}|)^{2-2p_{ji}}+(\beta_j|\omega_{ij}|)^{2p^*_{ij}}+(\beta_j|b_{ij}|)^{2r_{ij}}+M_{ij}^{2r^*_{ij}}+M_{ji}^{2-2r^*_{ji}})\Big]\\
&\quad+\sum_{j=1}^{n} e^{2\lambda t}\Big[(\beta_j|\omega_{ij}|)^{2-2p^*_{ij}}\varepsilon_j^2(x,t-\tilde{t}_{ij}(t))\\
&\quad+(\beta_j|b_{ij}|)^{2-2r_{ij}}\int_0^{+\infty}\kappa_{ij}(s)\varepsilon_j^2(x,t-s)\,ds\Big]\\
&\quad+\sum_{j=1}^{n} e^{2\lambda t}(\beta_j|\omega_{ij}|)^{2-2p^*_{ij}}\left(\frac{e^{2\lambda\tau}}{1-\rho}\varepsilon_j^2(x,t)-\varepsilon_j^2(x,t-\tilde{t}_{ij}(t))\right)\\
&\quad+\sum_{j=1}^{n} e^{2\lambda t}(\beta_j|b_{ij}|)^{2-2r_{ij}}\int_0^{+\infty}\kappa_{ij}(s)(\varepsilon_j^2(x,t)e^{2\lambda s}\\
&\qquad\qquad\qquad\qquad-\varepsilon_j^2(x,t-s))\,ds\Big\}\,dx\\
&=e^{2\lambda t}\sum_{i=1}^{n}\int_\Omega \varepsilon_i^2(x,t)\Big[2\lambda-2a_i-\frac{2B_i}{\delta}\\
&+\sum_{j=1}^{n}\Big((\beta_j|c_{ij}|)^{2p_{ij}}+(\beta_i|c_{ji}|)^{2-2p_{ji}}+(\beta_j|\omega_{ij}|)^{2p^*_{ij}}+(\beta_j|b_{ij}|)^{2r_{ij}}\\
&\quad+M_{ij}^{2r^*_{ij}}+M_{ji}^{2-2r^*_{ji}}+\frac{e^{2\lambda\tau}}{1-\rho}(\beta_i|\omega_{ji}|)^{2-2p^*_{ji}}\\
&\qquad\qquad+(\beta_i|b_{ji}|)^{2-2r_{ji}}\int_0^{+\infty}\kappa_{ji}(s)e^{2\lambda s}ds\Big)\Big]\,dx\\
&=2e^{2\lambda t}\sum_{i=1}^{n}W_i^1\int_\Omega\varepsilon_i^2(x,t)\,dx\le 0,
\end{aligned}
$$

where W_i^1 is defined in (9).

In addition, if $\sigma_{i1}\neq 0$ $(i=1,\cdots,n)$, it follows from (11) and (14) that

(16)

$$
\begin{aligned}
D^+V(t)<&\sum_{i=1}^{n}\int_\Omega\Big\{e^{2\lambda t}\varepsilon_i^2(x,t)\Big[2\lambda-2a_i\\
&+\sum_{j=1}^{n}((\beta_j|c_{ij}|)^{2p_{ij}}+(\beta_i|c_{ji}|)^{2-2p_{ji}}+(\beta_j|\omega_{ij}|)^{2p^*_{ij}}+(\beta_j|b_{ij}|)^{2r_{ij}}
\end{aligned}
$$

$$
\begin{aligned}
&+ M_{ij}^{2r_{ij}^*} + M_{ji}^{2-2r_{ji}^*}\Big)\Big] + \sum_{j=1}^{n} e^{2\lambda t}\Big[(\beta_j|\omega_{ij}|)^{2-2p_{ij}^*}\varepsilon_j^2(x,t-\tilde{t}_{ij}(t))\\
&\qquad\qquad + (\beta_j|b_{ij}|)^{2-2r_{ij}}\int_0^{+\infty}\kappa_{ij}(s)\varepsilon_j^2(x,t-s)\,ds\Big]\\
&+ \sum_{j=1}^{n} e^{2\lambda t}(\beta_j|\omega_{ij}|)^{2-2p_{ij}^*}\left(\frac{e^{2\lambda\tau}}{1-\rho}\varepsilon_j^2(x,t) - \varepsilon_j^2(x,t-\tilde{t}_{ij}(t))\right)\\
&+ \sum_{j=1}^{n} e^{2\lambda t}(\beta_j|b_{ij}|)^{2-2r_{ij}}\int_0^{+\infty}\kappa_{ij}(s)(\varepsilon_j^2(x,t)e^{2\lambda s}\\
&\qquad\qquad - \varepsilon_j^2(x,t-s))\,ds\Big\}\,dx\\
&= e^{2\lambda t}\sum_{i=1}^{n}\int_\Omega \varepsilon_i^2(x,t)\Big[2\lambda - 2a_i + \sum_{j=1}^{n}\Big((\beta_j|c_{ij}|)^{2p_{ij}} + (\beta_i|c_{ji}|)^{2-2p_{ji}}\\
&+ (\beta_j|\omega_{ij}|)^{2p_{ij}^*} + (\beta_j|b_{ij}|)^{2r_{ij}} + M_{ij}^{2r_{ij}^*} + M_{ji}^{2-2r_{ji}^*}\\
&+ \frac{e^{2\lambda\tau}}{1-\rho}(\beta_i|\omega_{ji}|)^{2-2p_{ji}^*} + (\beta_i|b_{ji}|)^{2-2r_{ji}}\int_0^{+\infty}\kappa_{ji}(s)e^{2\lambda s}ds\Big)\Big]\,dx\\
&= 2e^{2\lambda t}\sum_{i=1}^{n} W_i^2\int_\Omega \varepsilon_i^2(x,t)\,dx \le 0,
\end{aligned}
$$

where W_i^2 is defined in (10).

From (15) and (16), we know that, for the general boundary conditions, the following inequality always holds.

$$
\begin{aligned}
V(t) \le V(0) = \sum_{i=1}^{n}\int_\Omega \Big[&|\varphi_i(x,0) - \psi_i(x,0)|^2\\
&+ \sum_{j=1}^{n}\frac{(\beta_j|\omega_{ij}|)^{2-2p_{ij}^*}}{1-\rho}\int_{-\tilde{t}_{ij}(0)}^{0}|\varphi_j(x,s) - \psi_j(x,s)|^2 e^{2\lambda(s+\tau)}ds\\
&+ \sum_{j=1}^{n}(\beta_j|b_{ij}|)^{2-2r_{ij}}\int_0^{+\infty}\kappa_{ij}(s)\int_{-s}^{0}|\varphi_j(x,z)\\
&\qquad - \psi_j(x,z)|^2 e^{2\lambda(z+s)}dz\,ds\Big]\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \int_{\Omega} \bigg[|\varphi_i(x,0) - \psi_i(x,0)|^2 \\
&\quad + \sum_{j=1}^{n} \frac{(\beta_j |\omega_{ij}|)^{2-2p_{ij}^*}}{1-\rho} \\
&\times \int_0^{\tilde{t}_{ij}(0)} |\varphi_j(x, s-\tilde{t}_{ij}(0)) - \psi_j(x, s-\tilde{t}_{ij}(0))|^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times e^{2\lambda(s-\tilde{t}_{ij}(0)+\tau)} ds \\
&\qquad + \sum_{j=1}^{n} (\beta_j |b_{ij}|)^{2-2r_{ij}} \int_0^{+\infty} \kappa_{ij}(s) \\
&\qquad\quad \times \int_0^{s} |\varphi_j(x, z-s) - \psi_j(x, z-s)|^2 e^{2\lambda z} dz\, ds \bigg] dx.
\end{aligned}
$$

Let

$$
\begin{aligned}
(17) \quad \Upsilon = \sqrt{\max_i \bigg\{ 1 + \sum_{j=1}^{n} \bigg[\frac{e^{2\lambda\tau}}{2\lambda} \frac{(\beta_i |\omega_{ji}|)^{2-2p_{ij}^*}}{1-\rho}} \\
+ \sqrt{\frac{1}{2\lambda} (\beta_i |b_{ji}|)^{2-2r_{ji}} \int_0^{+\infty} \kappa_{ji}(s) e^{2\lambda s} ds} \bigg] \bigg\} > 1.
\end{aligned}
$$

For $V(t) \geq \|\varepsilon(\cdot,t)\|_2^2 e^{2\lambda t}$, the following inequality holds.

$$
\|\varepsilon(\cdot,t)\|_2^2 \leq \Upsilon^2 \|\varphi - \psi\|_2^2 e^{-2\lambda t},
$$

that is,

$$
\|u(\cdot,t) - \widetilde{u}(\cdot,t)\|_2 \leq \Upsilon \|\varphi - \psi\|_2 e^{-\lambda t},
$$

and, by Definition 1, we know that the drive-response neural networks are exponentially synchronized. So we are finished with the proof of Theorem 1. $\square$

Remark 6. If $\sigma_{i2} = 0$ and $\sigma_{i1} \neq 0$ $(i = 1, \dots, n)$, the result of Theorem 1 is that in [**17**]. Furthermore, suppose that $b_{ij} = 0$ $(i, j = 1, \dots, n)$; the result of Theorem 1 is that in [**9, 22**]. And, if $\sigma_{i1} = 0$ and $\sigma_{i2} \neq 0$ $(i = 1, \dots, n)$, the result of Theorem 1 is that in [**24**].

4. Example. In this section, let us discuss a numerical example as follows.

Example 1. Consider delayed reaction-diffusion neural networks with general boundary conditions:

$$
\begin{cases}
\dfrac{\partial u_i(x,t)}{\partial t} = \dfrac{\partial^2 u_i(x,t)}{\partial x^2} - a_i u_i(x,t) \\
\qquad\qquad - \displaystyle\sum_{j=1}^{2} c_{ij} f_j(u_j(x,t)) - \sum_{j=1}^{2} \omega_{ij} f_j(u_j(x,t-\tilde{t}_j)) \\
\qquad\qquad + \displaystyle\sum_{j=1}^{2} b_{ij} \int_{-\infty}^{0} \kappa_{ij}(t-s) f_j(u_j(x,s))\, ds + I_i, \quad i = 1,2, \\
u_i(x,t) = \varphi_i(x,t), \quad -\infty < t \le 0,\ x \in \Omega,\ i = 1,2, \\
\sigma_{i1} \dfrac{\partial u_i(x,t)}{\partial n} + \sigma_{i2} u_i(x,t) = 0, \\
\qquad (x \in \partial\Omega, \sigma_{i1} \ge 0,\ \sigma_{i2} \ge 0 \text{ and } \sigma_{i1}^2 + \sigma_{i2}^2 \ne 0),\ i = 1,2.
\end{cases}
\tag{18}
$$

The corresponding response neural networks can be expressed as follows.

$$
\begin{cases}
\dfrac{\partial \widetilde{u}_i(x,t)}{\partial t} = \dfrac{\partial^2 \widetilde{u}_i(x,t)}{\partial x^2} - a_i \widetilde{u}_i(x,t) \\
\qquad\qquad - \displaystyle\sum_{j=1}^{2} c_{ij} f_j(\widetilde{u}_j(x,t)) - \sum_{j=1}^{2} \omega_{ij} f_j(\widetilde{u}_j(x,t-\tilde{t}_j)) \\
\qquad\qquad + \displaystyle\sum_{j=1}^{2} b_{ij} \int_{-\infty}^{0} \kappa_{ij}(t-s) f_j(\widetilde{u}_j(x,s))\, ds + I_i + v_i,\ i = 1,2, \\
\widetilde{u}_i(x,t) = \psi_i(x,t), \quad -\infty < t \le 0,\ x \in \Omega,\ i = 1,2, \\
\sigma_{i1} \dfrac{\partial \widetilde{u}_i(x,t)}{\partial n} + \sigma_{i2} \widetilde{u}_i(x,t) = 0, \\
\qquad (x \in \partial\Omega, \sigma_{i1} \ge 0, \sigma_{i2} \ge 0 \text{ and } \sigma_{i1}^2 + \sigma_{i2}^2 \ne 0), i = 1,2.
\end{cases}
\tag{19}
$$

Let Ω be a bounded open domain with $|\Omega| \le 1$, and we can have $\delta = 1$ and $B_i = 1$ $(i = 1,2)$. Let $a_1 = 5.3$ and $a_2 = 5.25$; $f_i = \tanh u_i(x,t)(i = 1,2)$ satisfy (H2) with $\beta_1 = 1$, $\beta_2 = 1$; $c_{11} = 1/6$,

$c_{12} = 1/3$, $c_{21} = 1/6$, $c_{22} = 2/3$, $\omega_{11} = 1/2$, $\omega_{12} = 1/3$, $\omega_{21} = 2/3$, $\omega_{22} = 1/2$ and $b_{11} = 1$, $b_{12} = 1/3$,$b_{21} = 1/2$, $b_{22} = 1/3$. Let $\rho = 1/2$ and $\kappa_{ij} = e^{-t}$ $(i, j = 1, 2)$ which satisfy (5)–(7). Let $M_{11} = 1$, $M_{12} = 1/6$, $M_{21} = 1/6$, $M_{22} = 1/2$ and $I_1 = I_2 = 0$. By a simple calculation, we can choose $p_{ij} = p^*_{ij} = r_{ij} = r^*_{ij} = 1/2$ $(i, j = 1, 2)$ such that

$$
\begin{aligned}
&-2a_1 - \frac{2B_1}{\delta} + \sum_{j=1}^{n} \Big[\beta_j|c_{1j}| + \beta_1|c_{j1}| \\
&\quad + \beta_j|\omega_{1j}| + \frac{1}{1-\rho}(\beta_1|\omega_{j1}|) + \beta_j|b_{1j}| + \beta_1|b_{j1}| + |M_{1j}| + |M_{j1}|\Big] \\
&\le -2a_1 + \sum_{j=1}^{n} \Big[\beta_j|c_{1j}| + \beta_1|c_{j1}| + \beta_j|\omega_{1j}| \\
&\quad + \frac{1}{1-\rho}(\beta_1|\omega_{j1}|) + \beta_j|b_{1j}| + \beta_1|b_{j1}| + |M_{1j}| + |M_{j1}|\Big] \\
&= -2\cdot 5.3 + [1/6 + 1/3 + 1/6 + 1/6 + 1/2 + 1/3 + 2 \\
&\quad \cdot (1/2 + 2/3) + 1 + 1/3 + 1 + 1/2 + 1 + 1/6 + 1 + 1/6] \\
&= -10.6 + 9\frac{1}{6} < 0.
\end{aligned}
$$

and

$$
\begin{aligned}
&-2a_2 - \frac{2B_2}{\delta} + \sum_{j=1}^{n} \Big[\beta_j|c_{2j}| + \beta_2|c_{j2}| + \beta_j|\omega_{2j}| \\
&\quad + \frac{1}{1-\rho}(\beta_2|\omega_{j2}|) + \beta_j|b_{2j}| + \beta_2|b_{j2}| + |M_{2j}| + |M_{j2}|\Big] \\
&\le -2a_2 + \sum_{j=1}^{n} \Big[\beta_j|c_{2j}| + \beta_2|c_{j2}| + \beta_j|\omega_{2j}| \\
&\quad + \frac{1}{1-\rho}(\beta_2|\omega_{j2}|) + \beta_j|b_{2j}| + \beta_2|b_{j2}| + |M_{2j}| + |M_{j2}|\Big] \\
&= -2\cdot 5.25 + [1/6 + 2/3 + 1/3 + 2/3 + 2/3 + 1/2 + 2 \\
&\quad \cdot (1/3 + 1/2) + 1/2 + 1/3 + 1/3 + 1/3 + 1/6 + 1/2 + 1/6 + 1/2] \\
&= -10.5 + 7.5 < 0.
\end{aligned}
$$

These inequalities mean that $(H4)_1$ and $(H5)_1$ hold for this example. It follows from Corollary 1 that the drive-response neural networks (18) and (19) are exponentially synchronized.

5. Conclusions. In this paper, by using some inequality techniques and constructing a suitable Lyapunov functional, some sufficient conditions have been given to ensure the drive-response neural networks (1) and (2) to be exponentially synchronized. These conditions are easy to verify as they are expressed by algebraic inequalities. The methods used here can be used to deal with the exponential synchronization of general problems. Our new results for delayed reaction-diffusion neural networks with general boundary conditions generalize the corresponding ones in [**9, 17, 22, 24**].

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