ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 2, 2013

BSDES UNDER FILTRATION-CONSISTENT NONLINEAR EXPECTATIONS AND THE CORRESPONDING DECOMPOSITION THEOREM FOR $\mathcal{E}$-SUPERMARTINGALES IN L^p

ZHAOJUN ZONG AND FENG HU

ABSTRACT. In this paper, we introduce the notion of $\mathcal{F}_t$-consistent expectation defined on $\mathcal{L}(\Omega, \mathcal{F}, P)$ and prove an existence and uniqueness theorem for solutions and a comparison theorem of BSDE under $\mathcal{E}^\mu$-dominated $\mathcal{F}$-expectations. Furthermore, as an application of this comparison theorem, we obtain the decomposition theorem for $\mathcal{E}$-supermartingales.

1. Introduction. By [**7**], we know that there exists a unique adapted and square integrable solution to a backward stochastic differential equation (BSDE for short) of the type

$$(1) \qquad y_t = \xi + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s - \int_t^T z_s\,\mathrm{d}W_s, \quad 0 \le t \le T,$$

provided the function g is Lipschitz in both variables y and z, and ξ and $(g(t,0,0))_{t\in[0,T]}$ are square integrable. The function g is said to be the generator of BSDE (1). We denote the unique adapted and square integrable solution of BSDE (1) by $(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)})_{t\in[0,T]}$. When g also satisfies $g(\cdot, y, 0) = 0$ for any $y \in R$, then $y_0^{(T,g,\xi)}$, denoted by $\mathcal{E}_g[\xi]$, is called the g-expectation of ξ; $y_t^{(T,g,\xi)}$, denoted by $\mathcal{E}_g[\xi|\mathcal{F}_t]$, is called the conditional g-expectation of ξ (see [**8**]).

The g-expectation is a kind of nonlinear expectation, which can be considered to be a nonlinear extension of the well-known Girsanov

2010 AMS *Mathematics subject classification.* Primary 60H10, 60G48.
Keywords and phrases. $\mathcal{F}_t$-consistent nonlinear expectation ($\mathcal{F}$-expectation), backward stochastic differential equation (BSDE), generalized g-expectation, comparison theorem.
This work was partly supported by the National Natural Science Foundation of China (No. 11171179), the Doctoral Program Foundation of Ministry of Education of China (No. 20123705120005), the Postdoctoral Science Foundation of China (No. 2012M521301) and the Natural Science Foundation of Shandong Province of China (No. ZR2012AQ009)
Received by the editors on July 8, 2010, and in revised form on August 26, 2010.
DOI:10.1216/RMJ-2013-43-2-677

transformations. The original motivation for studying g-expectations comes from the theory of expected utility (see [**2**]), which is fundamentally important in economics. This theory is seriously challenged by the famous Allais and Ellsberg paradoxes (see [**6**]). The notion of non-additive probability, or capacity, is then introduced to axiomatize preferences which do not satisfy the von Neumann-Morgenstern axioms. Nonlinear expectations are another useful notion in this setting. Since the notion of g-expectation was introduced, many researchers have studied the properties of g-expectation. In 2002, Coquet et al. [**3**] introduced the notion of $\mathcal{F}$-expectation defined on L^2 and obtained the existence and uniqueness theorem for solutions of BSDE under $\mathcal{E}^{\mu}$-dominated $\mathcal{F}$-expectations, comparison theorems and decomposition theorems for $\mathcal{E}$-supermartingales. Furthermore, by using the decomposition theorem for $\mathcal{E}$-supermartingales, they were able to prove the important result that any $\mathcal{E}^{\mu}$-dominated $\mathcal{F}$-expectation can be represented as a g-expectation.

Obviously, we know that, if a random variable ξ is in $L^1(\Omega, \mathcal{F}, P)$, the classical mathematical expectation of ξ is meaningful. But, in the past several years, researchers have studied g-expectations which are confined to $L^2(\Omega, \mathcal{F}, P)$. Recently, [**4, 5**] have given the extensions of g-expectations, i.e., the definitional domain of g-expectation is extended to $\mathcal{L}(\Omega, \mathcal{F}, P)$, which are called generalized g-expectations and investigated with their related properties.

In this paper, inspired by [**4, 5**], we study $\mathcal{F}_t$-consistent expectations defined on $\mathcal{L}(\Omega, \mathcal{F}, P)$ and prove an existence and uniqueness theorem for solutions and a BSDE comparison theorem under $\mathcal{E}^{\mu}$-dominated $\mathcal{F}$-expectations in $\mathcal{L}$. Furthermore, as an application of this comparison theorem and the monotonic limit theorem of BSDE in [**5**], we obtain a decomposition theorem for $\mathcal{E}$-supermartingales. These results nontrivially generalize the corresponding results of [**3**].

This paper is organized as follows. In Section 2, we present the notions of generalized g-expectation and $\mathcal{F}_t$-consistent nonlinear expectations defined on $\mathcal{L}$ and give their related properties that are useful in this paper. In Section 3, we give our main results such as the existence and uniqueness theorem for solutions of BSDE under $\mathcal{E}^{\mu}$-dominated $\mathcal{F}$-expectations, comparison theorems and decomposition theorems for $\mathcal{E}$-supermartingales, including the proofs.

2. Generalized g-expectations and $\mathcal{F}_t$-consistent nonlinear expectation.

2.1. Notations and assumptions. For a given $T \in (0, \infty)$, let $(W_t)_{t\in[0,T]}$ be a d-dimensional standard Brownian motion defined on a completed probability space $(\Omega, \mathcal{F}, P)$, and let $(\mathcal{F}_t)_{t\in[0,T]}$ be the natural filtration generated by Brownian motion $(W_t)_{t\in[0,T]}$, that is,

$$\mathcal{F}_t = \sigma\{W_s; s \le t\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the set of all P-null subsets.

For simplicity, we consider the $d = 1$ case, but our method is easily extended to higher dimensions. We also make the natural choice of $\mathcal{F} = \mathcal{F}_T$. Let

$L^p(\Omega, \mathcal{F}, P) = \{\xi : \xi$ is an $\mathcal{F}$-measurable random variable such that $E|\xi|^p < \infty\}$;

$\mathcal{L}(\Omega, \mathcal{F}, P) = \cup_{p>1} L^p(\Omega, \mathcal{F}, P)$;

$\mathcal{S}^p(0, T; P; R) = \{V : (V_t)_{t\in[0,T]}$ is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process with $E[\sup_{0\le t\le T} |V_t|^p] < \infty\}$;

$\mathcal{S}(0, T; P; R) = \cup_{p>1} \mathcal{S}^p(0, T; P; R)$;

$L^p(0, T; P; R) = \{V : (V_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process with $E[(\int_0^T |V_s|^2 \mathrm{d}s)^{p/2}] < \infty\}$;

$\mathcal{L}(0, T; P; R) = \cup_{p>1} L^p(0, T; P; R)$;

$M^p(0, T; P; R) = \{V : (V_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process with $E[\int_0^T |V_s|^p \mathrm{d}s] < \infty\}$;

$\mathcal{M}(0, T; P; R) = \cup_{p>1} M^p(0, T; P; R)$.

Suppose a function g:

$$g(\omega, t, y, z) : \Omega \times [0, T] \times R \times R \longmapsto R \tag{2}$$

satisfies the following conditions:

(A.1) There exists a Lipschitz constant $\mu > 0$ such that, for all $y_1, y_2 \in R$, $z_1, z_2 \in R$,

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \le \mu(|y_1 - y_2| + |z_1 - z_2|), \quad \text{for all } t \in [0, T];$$

(A.2) $g(\cdot, y, 0) = 0$, for any $y \in R$.

2.2. Generalized g-expectations.

Lemma 2.1 (see [**1**]). *Suppose g satisfies* (A.1) *and* (A.2)*. Then, for any $\xi \in L^p(\Omega, \mathcal{F}, P)$, $1 < p < 2$, the* BSDE

$$(3) \qquad y_t = \xi + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s - \int_t^T z_s \mathrm{d}W_s, \quad t \in [0,T]$$

has a unique pair of adapted processes $(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)})_{t\in[0,T]} \in \mathcal{S}^p(0,T; P; R) \times L^p(0,T;P;R)$.

Remark 2.1. From Lemma 2.1, we have: suppose g satisfies (A.1) and (A.2). Then, for each given $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, BSDE (3) has a unique pair of adapted processes $(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)})_{t\in[0,T]} \in \mathcal{S}(0,T;P;R) \times \mathcal{L}(0,T;P;R)$.

Now, we introduce the notions of generalized g-expectation and generalized conditional g-expectation. For more details, see [**5**].

Definition 2.1 (Generalized g-expectation). Suppose that g satisfies (A.1) and (A.2). The generalized g-expectation $\mathcal{E}_g[\cdot] : \mathcal{L}(\Omega, \mathcal{F}, P) \longmapsto R$ is defined by

$$\mathcal{E}_g[\xi] = y_0^{(T,g,\xi)}.$$

Definition 2.2 (Generalized conditional g-expectation). The generalized conditional g-expectation of ξ with respect to $\mathcal{F}_t$ is defined by

$$\mathcal{E}_g[\xi|\mathcal{F}_t] = y_t^{(T,g,\xi)}.$$

Generalized and generalized conditional g-expectations are in general nonlinear. However they meet the following basic properties of usual expectations (see [**5**]).

Proposition 2.1. *Suppose $\xi, \xi_1, \xi_2 \in \mathcal{L}(\Omega, \mathcal{F}, P)$. Then:*

(i) *If ξ is $\mathcal{F}_t$-measurable, then $\mathcal{E}_g[\xi|\mathcal{F}_t] = \xi$;*

(ii) *For all stopping times τ and $\sigma \le T$, $\mathcal{E}_g[\mathcal{E}_g[\xi|\mathcal{F}_\tau]|\mathcal{F}_\sigma] = \mathcal{E}_g[\xi|\mathcal{F}_{\tau\wedge\sigma}]$, where $\tau \wedge \sigma$ denotes* $\min(\tau, \sigma)$;

(iii) *If* $\xi_1 \geq \xi_2$ *almost everywhere, then* $\mathcal{E}_g[\xi_1|\mathcal{F}_t] \geq \mathcal{E}_g[\xi_2|\mathcal{F}_t]$; *if, moreover,* $P(\xi_1 > \xi_2) > 0$, *then* $P(\mathcal{E}_g[\xi_1|\mathcal{F}_t] > \mathcal{E}_g[\xi_2|\mathcal{F}_t]) > 0$;

(iv) *For each* $B \in \mathcal{F}_t$, $\mathcal{E}_g[1_B\xi|\mathcal{F}_t] = 1_B\mathcal{E}_g[\xi|\mathcal{F}_t]$;

(v) $\mathcal{E}_g[\xi|\mathcal{F}_t]$ *is the unique random variable* η *in* $\mathcal{L}(\Omega, \mathcal{F}_t, P)$ *such that* $\mathcal{E}_g[1_A\xi] = \mathcal{E}_g[1_A\eta]$, *for all* $A \in \mathcal{F}_t$;

(vi) *For any* $(\varsigma, \eta) \in \mathcal{L}(\Omega, \mathcal{F}, P) \times \mathcal{L}(\Omega, \mathcal{F}_t, P)$, $\mathcal{E}_g[\varsigma + \eta|\mathcal{F}_t] = \mathcal{E}_g[\varsigma|\mathcal{F}_t] + \eta$ *if and only if* g *does not depend on* y.

The next proposition will be useful in this paper.

Proposition 2.2 (see [**5**]). *Suppose* g *satisfies* (A.1) *and* (A.2).

(i) *For any* $\xi, \eta \in L^p(\Omega, \mathcal{F}, P)$ $(1 < p < 2)$, *then*

$$|\mathcal{E}_g[\xi|\mathcal{F}_t] - \mathcal{E}_g[\eta|\mathcal{F}_t]| \leq e^{1/2(p-1)\mu^2 T + \mu T}(E[|\xi - \eta|^p|\mathcal{F}_t])^{1/p}, \quad \text{a.s.};$$

(ii) *For any* $\xi \in L^p(\Omega, \mathcal{F}, P)$ $(1 < p < 2)$, *then*

$$|\mathcal{E}_g[\xi|\mathcal{F}_t]| \leq \mathrm{e}^{1/2(p-1)\mu^2 T}(E[|\xi|^p|\mathcal{F}_t])^{1/p}, \quad a.s.$$

Definition 2.3 (Generalized g-martingale). A process $(X_t)_{t\in[0,T]}$ satisfying that, for each t, $X_t \in \mathcal{L}(\Omega, \mathcal{F}_t, P)$ is called a generalized g-martingale (respectively, generalized g-supermartingale, generalized g-submartingale) if, for any t, s satisfying $t \leq s \leq T$,

$$\mathcal{E}_g[X_s|\mathcal{F}_t] = X_t \quad (\text{respectively} \leq X_t, \geq X_t), \quad \text{a.s..}$$

The decomposition theorem of generalized g-supermartingale obtained by [**4**] will play an important role in this paper.

Proposition 2.3 (Decomposition theorem of generalized g-supermartingale). *Assume that* g *satisfies* (A.1), (A.2) *and* g *is independent of* y. *Let* (X_t) *be a right-continuous generalized* g*-supermartingale on* $[0, T]$ *with* $E[\sup_{0\leq t\leq T}|X_t|^p] < \infty$ $(1 < p < 2)$. *Then* (X_t) *has the following decomposition*

$$X_t = M_t - A_t, \quad \textit{for all } t \in [0, T].$$

Here (M_t) is a generalized g-martingale and (A_t) is an RCLL *increasing process (i.e., a process (A_t) is said to be* RCLL *if it a.s. has sample paths which are right continuous with left limit; a process (A_t) is said to be increasing if its paths $A : t \mapsto A_t(\omega)$ are a.s. nondecreasing) with $A_0 = 0$ and $E(A_T)^p < \infty$. More specifically, (X_t) is the unique solution of the* BSDE

$$X_t = X_T + \int_t^T g(s, Z_s)\,\mathrm{d}s + (A_T - A_t) - \int_t^T Z_s \mathrm{d}W_s, \quad t \in [0, T].$$

For notational simplification, we will henceforth write $\mathcal{E}^{\mu}[\cdot|\mathcal{F}_t] \equiv \mathcal{E}_g[\cdot|\mathcal{F}_t]$ for $g := \mu|z|$ and $\mathcal{E}^{-\mu}[\cdot|\mathcal{F}_t] \equiv \mathcal{E}_g[\cdot|\mathcal{F}_t]$ for $g := -\mu|z|$.

2.3. $\mathcal{F}_t$-consistent nonlinear expectations.

Definition 2.4. A nonlinear expectation defined on $\mathcal{L}(\Omega, \mathcal{F}, P)$ is a functional:

$$\mathcal{E}[\cdot] : \mathcal{L}(\Omega, \mathcal{F}, P) \longmapsto R$$

satisfying the following properties:

(i) Strict monotonicity:

$$\text{if} \quad X_1 \geq X_2 \text{ a.s.}, \mathcal{E}[X_1] \geq \mathcal{E}[X_2];$$

$$\text{if} \quad X_1 \geq X_2 \text{ a.s.}, \mathcal{E}[X_1] = \mathcal{E}[X_2] \Longleftrightarrow X_1 = X_2, \text{a.s.};$$

(ii) Preservation of constants:

$$\mathcal{E}[c] = c, \quad \text{for each constant } c.$$

Definition 2.5. A nonlinear expectation is called an $\mathcal{F}_t$-consistent nonlinear expectation ($\mathcal{F}$-expectation) defined on $\mathcal{L}(\Omega, \mathcal{F}, P)$ if, for each $Y \in \mathcal{L}(\Omega, \mathcal{F}, P)$ and $t \in [0, T]$, there exists a random variable $\eta \in \mathcal{L}(\Omega, \mathcal{F}_t, P)$, such that

$$\mathcal{E}[Y 1_A] = \mathcal{E}[\eta 1_A], \quad \text{for all } A \in \mathcal{F}_t.$$

The $\mathcal{F}$-expectation has the following property:

Proposition 2.4. *For each $Y \in \mathcal{L}(\Omega, \mathcal{F}, P)$, η is the unique random variable in $\mathcal{L}(\Omega, \mathcal{F}_t, P)$ such that*

$$\mathcal{E}[Y1_A] = \mathcal{E}[\eta 1_A], \quad \textit{for all } A \in \mathcal{F}_t. \tag{4}$$

The proof is very similar to that of Lemma 3.1 in [**3**]. We omit it.

From Proposition 2.4, such an η is uniquely defined. We also denote it by $\eta = \mathcal{E}[Y|\mathcal{F}_t]$. $\mathcal{E}[Y|\mathcal{F}_t]$ is called the conditional $\mathcal{F}$-expectation of Y under $\mathcal{F}_t$. It is characterized by

$$\mathcal{E}[Y1_A] = \mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t]1_A], \quad \text{for all } A \in \mathcal{F}_t. \tag{5}$$

Remark 2.2. Let g satisfy (A.1) and (A.2); then the related generalized g-expectation $\mathcal{E}_g[\cdot]$ is an $\mathcal{F}$-expectation.

Definition 2.6 ($\mathcal{E}^\mu$-domination). Given $\mu > 0$, we say that an $\mathcal{F}$-expectation $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$ if

$$\mathcal{E}[X+Y] - \mathcal{E}[X] \le \mathcal{E}^\mu[Y], \quad \text{for all } X, Y \in \mathcal{L}(\Omega, \mathcal{F}, P). \tag{6}$$

Remark 2.3. Suppose that g satisfies (A.1), (A.2) and g does not depend upon y; then the associated generalized g-expectation is dominated by $\mathcal{E}^\mu$, where μ is the Lipschitz constant in (A.1).

Lemma 2.2. *If $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$ for some $\mu > 0$, then*

$$\mathcal{E}^{-\mu}[Y] \le \mathcal{E}[X+Y] - \mathcal{E}[X] \le \mathcal{E}^\mu[Y], \quad \textit{for all } X, Y \in \mathcal{L}(\Omega, \mathcal{F}, P).$$

Proof. A simple consequence of

$$\mathcal{E}^{-\mu}[Y|\mathcal{F}_t] = -\mathcal{E}^\mu[-Y|\mathcal{F}_t].$$

The proof of Lemma 2.2 is complete. ◻

From now on, we will deal with the $\mathcal{F}$-expectation $\mathcal{E}[\cdot]$ also satisfying the following condition:

$$\mathcal{E}[X+Y|\mathcal{F}_t] = \mathcal{E}[X|\mathcal{F}_t] + Y, \tag{7}$$
$$\text{for all } X \in \mathcal{L}(\Omega, \mathcal{F}, P) \text{ and for all } Y \in \mathcal{L}(\Omega, \mathcal{F}_t, P).$$

Proposition 2.5. *Let $\mathcal{E}[\cdot]$, $\mathcal{E}_1[\cdot]$ and $\mathcal{E}_2[\cdot]$ be $\mathcal{F}$-expectations satisfying* (6) *and* (7)*. Then*

(i) *If $\mathcal{E}_1[X] \leq \mathcal{E}_2[X]$, for all $X \in \mathcal{L}(\Omega, \mathcal{F}, P)$, then $\mathcal{E}_1[X|\mathcal{F}_t] \leq \mathcal{E}_2[X|\mathcal{F}_t]$, a.s., for all $t \in [0, T]$. In particular, $\mathcal{E}^{-\mu}[X|\mathcal{F}_t] \leq \mathcal{E}[X|\mathcal{F}_t] \leq \mathcal{E}^{\mu}[X|\mathcal{F}_t]$, a.s., for all $t \in [0, T]$.*

(ii) *For a given $\zeta \in \mathcal{L}(\Omega, \mathcal{F}, P)$, the mapping $\mathcal{E}_\zeta[\cdot] = \mathcal{E}[\cdot + \zeta] - \mathcal{E}[\zeta] : \mathcal{L}(\Omega, \mathcal{F}, P) \mapsto R$ is also an $\mathcal{F}$-expectation satisfying* (6) *and* (7)*. Its conditional expectation under $\mathcal{F}_t$ is $\mathcal{E}_\zeta[\cdot|\mathcal{F}_t] = \mathcal{E}[\cdot + \zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t]$, a.s. In particular, $\mathcal{E}^{-\mu}[Y|\mathcal{F}_t] \leq \mathcal{E}[X+Y|\mathcal{F}_t] - \mathcal{E}[X|\mathcal{F}_t] \leq \mathcal{E}^{\mu}[Y|\mathcal{F}_t]$, a.s., for all $X, Y \in \mathcal{L}(\Omega, \mathcal{F}, P)$.*

The proof of Proposition 2.5 is very similar to those of Lemmas 4.3, 4.4 and 4.5 in [**3**]. We omit it. ◻

The following lemma will be very useful later.

Lemma 2.3. *If $\mathcal{E}[\cdot]$ meets* (6) *and* (7)*, there exists a constant $C_p > 0$ such that, for all $Y \in L^p(\Omega, \mathcal{F}, P)$ $(1 < p < 2)$ and for each $t \leq T$,*

$$|\mathcal{E}[X+Y|\mathcal{F}_t] - \mathcal{E}[X|\mathcal{F}_t]| \leq C_p(E[|Y|^p|\mathcal{F}_t])^{1/p}, \quad a.s. \tag{8}$$

Proof. By Propositions 2.2 and 2.5, we have

$$\begin{aligned}|\mathcal{E}[X+Y|\mathcal{F}_t] - \mathcal{E}[X|\mathcal{F}_t]| &\leq |\mathcal{E}^{\mu}[Y|\mathcal{F}_t]| \vee |\mathcal{E}^{-\mu}[Y|\mathcal{F}_t]| \\ &\leq \mathrm{e}^{1/2(p-1)\mu^2 T}(E[|Y|^p|\mathcal{F}_t])^{1/p}, \quad \text{a.s.,}\end{aligned}$$

where $|\mathcal{E}^{\mu}[Y|\mathcal{F}_t]| \vee |\mathcal{E}^{-\mu}[Y|\mathcal{F}_t]|$ denotes $\max(|\mathcal{E}^{\mu}[Y|\mathcal{F}_t]|, |\mathcal{E}^{-\mu}[Y|\mathcal{F}_t]|)$. The proof of Lemma 2.3 is complete. ◻

From now on, we will always assume that $\mathcal{E}[\cdot]$ is an $\mathcal{F}$-expectation satisfying (6) and (7).

3. Main results and proofs.

3.1. BSDE under $\mathcal{F}$-expectations. Let a function $f:\Omega\times[0,T]\times R\mapsto R$ satisfying:

(B.1) There exists a Lipschitz constant $K>0$ such that for all y_1, $y_2\in R$,

$$|f(t,y_1)-f(t,y_2)|\le K|y_1-y_2|,\quad \text{for all } t\in[0,T];$$

(B.2) $f(\cdot,0)\in\mathcal{M}(0,T;P;R)$.

For a given terminal data $X\in\mathcal{L}(\Omega,\mathcal{F},P)$, we consider the following type of equation:

$$Y_t=\mathcal{E}\Big[X+\int_t^T f(s,Y_s)\,\mathrm{d}s|\mathcal{F}_t\Big]. \tag{9}$$

Theorem 3.1. *We assume* (B.1) *and* (B.2). *Then there exists a unique process* $Y(\cdot)\in\mathcal{M}(0,T;P;R)$ *which solves* (9).

Proof. For a given terminal data $X\in\mathcal{L}(\Omega,\mathcal{F},P)$ and each $y(\cdot)\in\mathcal{M}(0,T;P;R)$, we define a mapping $\pi.(y(\cdot))$ by

$$\pi_t(y(\cdot))=\mathcal{E}\Big[X+\int_t^T f(s,y(s))\,\mathrm{d}s|\mathcal{F}_t\Big].$$

Obviously, $\pi.(y(\cdot))\in\mathcal{M}(0,T;P;R)$.

Without loss of generality, suppose $y_1(\cdot)$, $y_2(\cdot)\in M^p(0,T;P;R)$ $(1<p<2)$. By Lemma 2.3, we have

$$\begin{aligned}&|\pi_t(y_1(\cdot))-\pi_t(y_2(\cdot))|\\&\quad\le C_p\bigg(E\bigg[\bigg|\int_t^T[f(s,y_1(s))-f(s,y_2(s))]\,\mathrm{d}s\bigg|^p|\mathcal{F}_t\bigg]\bigg)^{1/p},\quad \text{a.s.}\end{aligned}\tag{10}$$

By Hölder's inequality and (B.1),

$$\begin{aligned}\Big|\int_t^T [f(s,y_1(s)) - f(s,y_2(s))]\,\mathrm{d}s\Big|^p \\ &\le T^{p-1}\int_t^T |f(s,y_1(s)) - f(s,y_2(s))|^p \mathrm{d}s \\ &\le T^{p-1}K^p \int_t^T |y_1(s)-y_2(s)|^p \mathrm{d}s, \quad \text{a.s.}\end{aligned}$$

This with (10) yields

$$\begin{aligned}E[|\pi_t(y_1(\cdot)) - \pi_t(y_2(\cdot))|^p] \\ &\le T^{p-1}(C_pK)^p E\Big[\int_t^T |y_1(s)-y_2(s)|^p \mathrm{d}s\Big].\end{aligned}$$

Let $C = T^{p-1}(C_pK)^p$. By multiplying both sides of the above inequality by e^{2Ct} and then integrating both sides on $[0,T]$ with respect to t, it follows that

$$\begin{aligned}E\Big[\int_0^T \mathrm{e}^{2Ct}|\pi_t(y_1(\cdot)) - \pi_t(y_2(\cdot))|^p \mathrm{d}t\Big] \\ &\le CE\Big[\int_0^T \mathrm{e}^{2Ct}\int_t^T |y_1(s)-y_2(s)|^p \mathrm{d}s\,\mathrm{d}t\Big] \\ &= CE\Big[\int_0^T |y_1(s)-y_2(s)|^p \int_0^s \mathrm{e}^{2Ct}\mathrm{d}t\,\mathrm{d}s\Big] \\ &= (2C)^{-1}CE\Big[\int_0^T (\mathrm{e}^{2Ct}-1)|y_1(t)-y_2(t)|^p \mathrm{d}t\Big] \\ &\le \frac{1}{2}E\Big[\int_0^T \mathrm{e}^{2Ct}|y_1(t)-y_2(t)|^p \mathrm{d}t\Big].\end{aligned}$$

We observe that, for any constant $\alpha \in R$, the following two norms are equivalent in $M^p(0,T;P;R)$:

$$\Big(E\Big[\int_0^T |Y_t|^p \mathrm{d}t\Big]\Big)^{1/p} \sim \Big(E\Big[\int_0^T |Y_t|^p \mathrm{e}^{\alpha t}\mathrm{d}t\Big]\Big)^{1/p}.$$

From this, we can obtain that $\pi.(y(\cdot))$ is a contraction mapping on $M^p(0,T;P;R)$. It follows that this mapping has a unique fixed point $Y(\cdot)$:

$$Y_t = \mathcal{E}\Big[X + \int_t^T f(s, Y_s)\,\mathrm{d}s|\mathcal{F}_t\Big].$$

So the proof of Theorem 3.1 is complete. □

Theorem 3.2 (Comparison theorem). *Let Y be the solution of* (9) *and Y' the solution of*

$$Y'_t = \mathcal{E}\Big[X' + \int_t^T [f(s, Y'_s) + \phi(s)]\,\mathrm{d}s|\mathcal{F}_t\Big],$$

where $X' \in \mathcal{L}(\Omega, \mathcal{F}, P)$ and $\phi \in \mathcal{M}(0,T;P;R)$. If

$$X \geq X', \quad a.s.,\ \phi(t) \geq 0, \mathrm{d}P \times \mathrm{d}t - a.e., \tag{11}$$

then we have

$$Y_t \geq Y'_t, \quad a.s. \tag{12}$$

Equation (12) *becomes an equality if and only if* (11) *become equalities.*

Proof. The main approach of the following proof derives from Coquet et al. [**3**]. In the case of $\phi(t) = 0$, $\mathrm{d}P \times \mathrm{d}t$–almost everywhere, the proof is very similar to that of Theorem 6.2 in [**3**], so we omit it.

In order to prove the general case $\phi(t) \geq 0$, $\mathrm{d}P \times \mathrm{d}t$–almost everywhere, we define for $n = 1, 2, 3, \ldots$, Y^n to be the solution of

$$Y_t^n = \mathcal{E}\bigg[\bigg(X' + \int_{iT/n}^T \phi(s)\,\mathrm{d}s\bigg) + \int_t^T f(s, Y_s^n)\,\mathrm{d}s\bigg|\mathcal{F}_t\bigg],$$

for $t \in [t_i^n, t_{i+1}^n)$, $t_i^n = iT/n$, $i = 0, 1, 2, \ldots, n-1$.

This equation can be written, piece by piece, as

$$Y_t^n = \mathcal{E}\bigg[\bigg(Y^n_{t^n_{i+1}} + \int_{t_i^n}^{t_{i+1}^n} \phi(s)\,\mathrm{d}s\bigg) + \int_t^{t_{i+1}^n} f(s, Y_s^n)\,\mathrm{d}s|\mathcal{F}_t\bigg],$$
$$t \in [t_i^n, t_{i+1}^n),\ Y_T^n = Y^n_{t_n^n} = X'.$$

From the first part of the proof, we have for $i = n-1$, $Y_t^n \geq Y_t$, $t \in [t_{n-1}^n, T)$. In particular, $Y_{t_{n-1}^n}^n \geq Y_{t_{n-1}^n}$. An obvious iteration of this algorithm gives $Y_t^n \geq Y_t$, $t \in [t_i^n, t_{i+1}^n)$, $i = 0, \dots, n-2$. Thus, $Y_t^n \geq Y_t$, $t \in [0,T]$.

In order to prove that $Y_t' \geq Y_t$, it suffices to show the convergence of the sequence (Y^n) to Y'. Without loss of generality, we assume that there exists a $1 < p < 2$ such that Y^n, Y' and $\phi \in M^p(0,T;P;R)$. By Lemma 2.3, for fixed $t \in [t_i^n, t_{i+1}^n)$,

$$E[|Y_t^n - Y_t'|^p] \leq (C_p)^p E\left[\left(\int_{iT/n}^t |\phi(s)|\,\mathrm{d}s + K\int_t^T |Y_s^n - Y_s'|\,\mathrm{d}s\right)^p\right].$$

Using Hölder's inequality, we have, for each $t \in [0,T]$,

$$E[|Y_t^n - Y_t'|^p] \leq (2C_p)^p\left(\frac{T}{n}\right)^{p-1} E\left[\int_0^T |\phi(s)|^p \mathrm{d}s\right] + (2KC_p)^p T^{p-1} E\left[\int_t^T |Y_s^n - Y_s'|^p \mathrm{d}s\right].$$

Gronwall's lemma applied to the above inequality shows that

$$E[|Y_t^n - Y_t'|^p] \longrightarrow 0, \quad \text{as } n \to \infty.$$

Thus, $Y_t' \geq Y_t$.

Finally, we investigate the possible equality in (12). From $Y_t' = Y_t$, a.s., then

$$\mathcal{E}\left[X + \int_0^T f(s,Y_s)\,\mathrm{d}s\right] = \mathcal{E}\left[X' + \int_0^T [f(s,Y_s) + \phi(s)]\,\mathrm{d}s\right].$$

Since $X' \geq X$, a.s., $\phi(t) \geq 0$, $\mathrm{d}P \times \mathrm{d}t -$ a.e., it follows from the strict monotonicity of $\mathcal{E}[\cdot]$ that $X' = X$, a.s., $\phi(t) = 0$, $\mathrm{d}P \times \mathrm{d}t -$ a.e. So the proof of Theorem 3.2 is complete.

3.2. $\mathcal{F}_t$-consistent martingales and decomposition theorem for $\mathcal{E}$-supermartingales.

Definition 3.1. A process $(X_t)_{t\in[0,T]} \in \mathcal{M}(0,T;P;R)$ is called an $\mathcal{E}$-martingale (respectively $\mathcal{E}$-supermartingale, $\mathcal{E}$-submartingale), if for each $0 \leq s \leq t \leq T$,

$$X_s = \mathcal{E}[X_t|\mathcal{F}_s], \quad (\text{resp.} \geq \mathcal{E}[X_t|\mathcal{F}_s], \leq \mathcal{E}[X_t|\mathcal{F}_s]) \text{ a.s.}$$

Remark 3.1. An $\mathcal{E}^{\mu}$-supermartingale $(X_t)_{t\in[0,T]}$ is both an $\mathcal{E}$-supermartingale and $\mathcal{E}^{-\mu}$-supermartingale. An $\mathcal{E}^{-\mu}$-submartingale $(X_t)_{t\in[0,T]}$ is both an $\mathcal{E}$-submartingale and $\mathcal{E}^{\mu}$-submartingale. An $\mathcal{E}$-martingale $(X_t)_{t\in[0,T]}$ is an $\mathcal{E}^{-\mu}$-supermartingale and an $\mathcal{E}^{\mu}$-submartingale.

The next lemma shows that every $\mathcal{E}$-martingale admits RCLL paths.

Lemma 3.1. *For each $X \in \mathcal{L}(\Omega, \mathcal{F}, P)$, the process $(\mathcal{E}[X|\mathcal{F}_t])_{t\in[0,T]}$ admits a unique modification with a.s.* RCLL *paths.*

The proof is very similar to that of Lemma 5.2 in [**3**]. We omit the proof. In the following, we do not distinguish $\mathcal{E}$-martingales and their RCLL modifications.

Lemma 3.1 has an immediate consequence as follows:

Lemma 3.2. *Let $\mathcal{E}[\cdot]$ be an $\mathcal{F}$-expectation satisfying* (6) *and* (7). *Then, for each $X \in \mathcal{L}(\Omega, \mathcal{F}, P)$ and $g \in \mathcal{M}(0, T; P; R)$, the process $(\mathcal{E}[X + \int_t^T g(s)\,\mathrm{d}s|\mathcal{F}_t])_{t\in[0,T]}$ is* RCLL *a.s.*

The next lemma shows that every $\mathcal{E}$-martingale admits continuous paths.

Lemma 3.3. *For each $X \in \mathcal{L}(\Omega, \mathcal{F}, P)$, let $y_t = \mathcal{E}[X|\mathcal{F}_t]$. Then there exists a pair $(g(t), z_t) \in \mathcal{L}(0, T; P; R \times R)$ with*

$$|g(t)| \le \mu |z_t| \tag{13}$$

such that

$$y_t = X + \int_t^T g(s)\,\mathrm{d}s - \int_t^T z_s\,\mathrm{d}W_s. \tag{14}$$

Furthermore, take $X' \in \mathcal{L}(\Omega, \mathcal{F}, P)$, put $y'_t = \mathcal{E}[X' \mid \mathcal{F}_t]$ and let $(g'(t), z'_t) \in \mathcal{L}(0, T; P; R \times R)$ be the corresponding pair. Then we have

$$|g(t) - g'(t)| \le \mu |z_t - z'_t|. \tag{15}$$

Proof. For each $X \in \mathcal{L}(\Omega, \mathcal{F}, P)$, there exists a $1 < p < 2$ such that $X \in L^p(\Omega, \mathcal{F}, P)$. Since $y_t = \mathcal{E}[X|\mathcal{F}_t]$, $t \in [0,T]$ is an RCLL $\mathcal{E}$-martingale, it is an RCLL $\mathcal{E}^{\mu}$-submartingale and $\mathcal{E}^{-\mu}$-supermartingale. By the domination $\mathcal{E}^{-\mu}[X|\mathcal{F}_t] \leq \mathcal{E}[X|\mathcal{F}_t] \leq \mathcal{E}^{\mu}[X|\mathcal{F}_t]$, we have $E[\sup_{0\leq t\leq T} |y_t|^p] < \infty$. Thus, by Proposition 2.3, and in a similar manner to Lemma 5.3 in [**3**], we can complete the rest proof. So we omit it. □

Remark 3.2. From Lemma 3.3, the result of Lemma 3.2 can be improved to: for each $X \in \mathcal{L}(\Omega, \mathcal{F}, P)$ and $g \in \mathcal{M}(0,T;P;R)$, the process $(\mathcal{E}[X + \int_t^T g(s)\mathrm{d}s|\mathcal{F}_t])_{t\in[0,T]}$ is continuous a.s.

Our next result generalizes the decomposition theorem for generalized g-supermartingales proved in [**4**] to right continuous $\mathcal{E}$-supermartingales. The proof mainly uses arguments from [**4**].

Theorem 3.3 (Decomposition theorem for $\mathcal{E}$-supermartingales). *Let $\mathcal{E}[\cdot]$ be an $\mathcal{F}$-expectation satisfying* (6), (7) *and let $Y \in \mathcal{S}(0,T;P;R)$ be a right-continuous $\mathcal{E}$-supermartingale. Then there exists an $A \in \mathcal{S}(0,T;P;R)$ such that A is* RCLL *and increasing with $A(0) = 0$ and such that $Y + A$ is an $\mathcal{E}$-martingale.*

Proof. For $n \geq 1$, we define y^n to be the solution of the following BSDE:

$$y_t^n = \mathcal{E}\Big[Y_T + \int_t^T n(Y_s - y_s^n)\,\mathrm{d}s|\mathcal{F}_t\Big].$$

We then have the following:

Lemma 3.4. *We have, for each t and $n \geq 1$,*

$$Y_t \geq y_t^n, \quad a.s.$$

In a similar manner as Lemma 6.2 in [**3**], we can prove Lemma 3.4. So we omit it. □

Lemma 3.4 with Theorem 3.2 implies that y_n monotonically converges to some $Y^0 \leq Y$. Indeed, writing $\phi(t) = Y_t - y_t^{n+1} \geq 0$ shows that (y^n) is an increasing sequence of functions.

Observe then that $y_t^n + \int_0^t n(Y_s - y_s^n)\,\mathrm{d}s$ is an $\mathcal{E}$-martingale. By Lemma 3.3, there exists a $(g^n(t), z_t^n) \in \mathcal{L}(0,T;P;R\times R)$ with

$$|g^n(t)| \leq \mu |z_t^n|, \quad n = 1, 2, \dots, \tag{16}$$

such that

$$y_t^n + \int_0^t n(Y_s - y_s^n)\,\mathrm{d}s = y_T^n + \int_0^T n(Y_s - y_s^n)\,\mathrm{d}s + \int_t^T g^n(s)\,\mathrm{d}s - \int_t^T z_s^n \mathrm{d}W_s;$$

hence, as $y_T^n = Y_T$,

$$y_t^n = Y_T + \int_t^T [g^n(s) + n(Y_s - y_s^n)]\,\mathrm{d}s - \int_t^T z_s^n \mathrm{d}W_s. \tag{17}$$

Equation (15) also tells us that

$$|g^n(t) - g^m(t)| \leq \mu |z_t^n - z_t^m|, \quad n, m = 1, 2, \dots. \tag{18}$$

Let us denote, for each $n = 1, 2, \dots$, $A^n(t) = \int_0^t n(Y_s - y_s^n)\mathrm{d}s$; then A^n is a continuous increasing process such that $A^n(0) = 0$.

Without loss of generality, assume $Y \in \mathcal{S}^p(0,T;P;R)$ $(1 < p < 2)$. We have the following:

Lemma 3.5. *There exists a positive constant C which is independent of n such that*:

$$\begin{aligned} &(1) \quad E\left(\int_0^T |z_s^n|^2 \mathrm{d}s\right)^{p/2} \leq C, \\ &(2) \quad E(A^n(T))^p \leq C. \end{aligned} \tag{19}$$

Proof. From (16) and (17), we have

$$(A^n(T))^p = \Big|y_0^n - y_T^n - \int_0^T g^n(s)\,\mathrm{d}s + \int_0^T z_s^n \mathrm{d}W_s\Big|^p$$
$$\le 4^p\bigg(|y_0^n|^p + |y_T^n|^p + \Big(\int_0^T \mu|z_s^n|\,\mathrm{d}s\Big)^p + \Big|\int_0^T z_s^n \mathrm{d}W_s\Big|^p\bigg).$$

For the third term on the right side, applying Schwarz's inequality, we obtain

$$\Big(\int_0^T \mu|z_s^n|\,\mathrm{d}s\Big)^p \le m_p\Big(\int_0^T |z_s^n|^2\mathrm{d}s\Big)^{p/2},$$

where m_p is a positive constant independent of n.

For the last term, it follows by the BDG (Burkholder-Davis-Gundy) inequality that

$$E\Big|\int_0^T z_s^n \mathrm{d}W_s\Big|^p \le n_p E\Big(\int_0^T |z_s^n|^2\mathrm{d}s\Big)^{p/2},$$

where the positive constant n_p is independent of n. Observe that $|y_t^n|$ is dominated by $|y_t^1| + |Y_t|$. Thus, there exists a positive constant C' independent of n, such that

$$E\big[\sup_{0\le t\le T} |y_t^n|^p\big] \le C'.$$

It follows that there exist two positive constants C_1 and C_2, independent of n, such that

$$E(A^n(T))^p \le C_1 + C_2 E\Big(\int_0^T |z_s^n|^2\mathrm{d}s\Big)^{p/2}. \tag{20}$$

On the other hand, we apply Ito's formula to $|y_t^n|^2$:

$$\int_0^T |z_s^n|^2\mathrm{d}s \le |y_T^n|^2 + 2\int_0^T y_s^n g^n(s)\,\mathrm{d}s + 2\int_0^T y_s^n \mathrm{d}A^n(s) - 2\int_0^T y_s^n z_s^n \mathrm{d}W_s.$$

Then

$$\Big(\int_0^T |z_s^n|^2\mathrm{d}s\Big)^{p/2} \le d_p\bigg(|y_T^n|^p + \Big(\int_0^T |y_s^n g^n(s)|\,\mathrm{d}s\Big)^{p/2} + \Big(\int_0^T |y_s^n|\,\mathrm{d}A^n(s)\Big)^{p/2} + \Big|\int_0^T y_s^n z_s^n \mathrm{d}W_s\Big|^{p/2}\bigg), \tag{21}$$

where d_p is a positive constant independent of n. For the second term on the right side of (21), we get

$$E\Big(\int_0^T |y_s^n g^n(s)|\,\mathrm{d}s\Big)^{p/2} \le E\Big(\int_0^T |y_s^n||\mu||z_s^n|\,\mathrm{d}s\Big)^{p/2} \le e_p E\big[\sup_{0\le t\le T}|y_t^n|^p\big] + \frac{1}{4d_p}E\Big(\int_0^T |z_s^n|^2\mathrm{d}s\Big)^{p/2},$$

where e_p is a positive constant independent of n. For the third term,

$$\begin{aligned} E\Big(\int_0^T |y_s^n|\mathrm{d}A^n(s)\Big)^{p/2} &\le E\Big[\sup_{0\le t\le T}|y_t^n|^{p/2}(A^n(T))^{p/2}\Big] \\ &\le \Big(E\Big[\sup_{0\le t\le T}|y_t^n|^p\Big]\Big)^{1/2}(E(A^n(T))^p)^{1/2} \\ &\le \frac{1}{4C_2d_p}E(A^n(T))^p + C_2d_pE\Big[\sup_{0\le t\le T}|y_t^n|^p\Big]. \end{aligned}$$

For the last term, it follows from the BDG inequality that

$$\begin{aligned} E\Big|\int_0^T y_s^n z_s^n \mathrm{d}W_s\Big|^{p/2} &\le h_pE\Big(\int_0^T |y_s^n z_s^n|^2\mathrm{d}s\Big)^{p/4} \\ &\le h_pE\Big[\sup_{0\le t\le T}|y_t^n|^{p/2}\Big(\int_0^T |z_s^n|^2\mathrm{d}s\Big)^{p/4}\Big] \\ &\le \frac{1}{4d_p}E\Big(\int_0^T |z_s^n|^2\mathrm{d}s\Big)^{p/2} + d_ph_p^2E\Big[\sup_{0\le t\le T}|y_t^n|^p\Big], \end{aligned}$$

where h_p is a positive constant independent of n. Consequently,

$$E\Big(\int_0^T |z_s^n|^2\mathrm{d}s\Big)^{p/2} \le C_3 + \frac{1}{2C_2}E(A^n(T))^p, \tag{22}$$

where C_3 is a positive constant independent of n. This with (20) implies that (1) and (2) hold. The proof of Lemma 3.5 is complete.

With the help of Lemma 3.5, we can end the proof of Theorem 3.3.

Equation (19) (1) with (16) imply that $E(\int_0^T |g^n(s)|^2 \mathrm{d}s)^{p/2} \leq \mu^p C$. But equation (19) (2) implies that $y^n \uparrow Y$. From Theorem 3.1 in [**4**], it follows that we can write Y in the form

$$Y_t = Y_T + \int_t^T g(s)\,\mathrm{d}s + A(T) - A(t) - \int_t^T Z_s \mathrm{d}W_s,$$

for some $(g, Z) \in L^p(0, T; P; R \times R)$ and an increasing process A with $A(0) = 0$ and $E(A(T))^p < \infty$. From the same theorem we have that, moreover,

$$\begin{aligned} z_t^n &\longrightarrow Z_t, \quad \text{weakly in } L^p(0,T;P;R);\\ g^n(t) &\longrightarrow g(t), \quad \text{weakly in } L^p(0,T;P;R);\\ A^n(t) &\longrightarrow A(t), \quad \text{weakly in } L^p(\Omega, \mathcal{F}, P). \end{aligned}$$

Since $y_t^n = \mathcal{E}[Y_T + A^n(T) - A^n(t)|\mathcal{F}_t]$, by Lemma 2.3, it follows that $Y_t = \mathcal{E}[Y_T + A(T) - A(t)|\mathcal{F}_t]$. Thus, $Y_t + A(t) = \mathcal{E}[Y_T + A(T)|\mathcal{F}_t]$ is an $\mathcal{E}$-martingale. The proof of Theorem 3.3 is complete. ◻

At last, we will prove an important result: an $\mathcal{F}$-expectation can be identified as a generalized g-expectation, proving that (6) and (7) hold.

Theorem 3.4. *We assume that an $\mathcal{F}$-expectation $\mathcal{E}[\cdot]$ satisfies* (6) *and* (7) *for some $\mu > 0$. Then there exists a unique function $g = g(t, z) : \Omega \times [0, T] \times R$ satisfying* (A.1) *and* (A.2) *such that*

$$\mathcal{E}[X|\mathcal{F}_t] = \mathcal{E}_g[X|\mathcal{F}_t], \quad \textit{a.s., for all } X \in \mathcal{L}(\Omega, \mathcal{F}, P).$$

Proof. For any $X \in \mathcal{L}(\Omega, \mathcal{F}, P)$, let $X^n = (X \wedge n) \vee (-n)$, $n = 1, 2, \ldots$. Then, for each n, $X^n \in L^2(\Omega, \mathcal{F}, P)$. By Theorem 7.1 in [**3**], then there exists a unique function $g = g(t, z) : \Omega \times [0, T] \times R$ satisfying (A.1) and (A.2) such that $\mathcal{E}[X] = \mathcal{E}g[X]$, for all $X \in L^2(\Omega, \mathcal{F}, P)$. So $\mathcal{E}[X^n] = \mathcal{E}_g[X^n]$, $n = 1, 2, \ldots$. By Lemma 2.3, choosing $t = 0$, $\mathcal{E}[X^n] \to \mathcal{E}[X]$, as $n \to \infty$. On the other hand, by Proposition 2.2, choosing $t = 0$, $\mathcal{E}_g[X^n] \to \mathcal{E}_g[X]$, as $n \to \infty$. Thus, $\mathcal{E}[X] = \mathcal{E}_g[X]$, for all $X \in \mathcal{L}(\Omega, \mathcal{F}, P)$.

For all $A \in \mathcal{F}_t$,

$$\mathcal{E}[\mathcal{E}[X|\mathcal{F}_t]1_A]=\mathcal{E}[X1_A]=\mathcal{E}_g[X1_A]=\mathcal{E}_g[\mathcal{E}_g[X|\mathcal{F}_t]1_A]=\mathcal{E}[\mathcal{E}_g[X|\mathcal{F}_t]1_A].$$

By the uniqueness of the conditional $\mathcal{F}$-expectation of X under $\mathcal{F}_t$, we have

$$\mathcal{E}[X|\mathcal{F}_t] = \mathcal{E}_g[X|\mathcal{F}_t], \quad \text{a.s., for all } X \in \mathcal{L}(\Omega, \mathcal{F}, P).$$

So we have completed the proof of Theorem 3.4. □

Acknowledgments. We would like to thank the anonymous referee who gave us many constructive suggestions and comments on the previous version of this paper.

REFERENCES

1. P. Briand, B. Delyon, Y. Hu, E. Pardoux and L. Stoica, *L^p solutions of backward stochastic differential equations*, Stoch. Proc. Appl. **108** (2003), 109–129.

2. Z. Chen and L. Epstein, *Ambiguity, risk and asset returns in continuous time*, Econom. **70** (2002), 1403–1443.

3. F. Coquet, Y. Hu, J. Mémin and S. Peng, *Filtration-consistent nonlinear expectations and related g-expectations*, Probab. Theory Relat. Fields **123** (2002), 1–27.

4. F. Hu, *Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer's type in L^p*, Chinese J. Appl. Probab. Statist., submitted.

5. F. Hu and Z. Chen, *Generalized g-expectations and related properties*, Statist. Probab. Lett. **80** (2010), 191–195.

6. I. Levi, *The paradoxes of Allais and Ellsberg*, Econ. Philos. **2** (1986), 23–53.

7. E. Pardoux and S. Peng, *Adapted solution of a backward stochastic differential equation*, Systems Control Lett. **14** (1990), 55–61.

8. S. Peng, *Backward* SDE *and related g-expectation*, in *Backward stochastic differential equations*, N. El Karoui and L. Mazliak, eds., Pitman Res. Notes **364**, Longman, Harlow, 1997.

School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China
Email address: zongzj001@163.com

School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China and School of Mathematics, Shandong University, Jinan 250100, China
Email address: hufengqf@163.com