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EXISTENCE-UNIQUENESS OF THE SOLUTION FOR NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The main aim of this paper is to develop some basic theories of neutral stochastic functional differential equations (NSFDEs). Firstly, we establish a local existence-uniqueness theorem under the local Lipschitz condition for the right side and a lower Lipschitz condition for the left side of the equation. Then continuation theorems and global existence theorems for NSFDEs are obtained. Some classical results, such as the Picard local existence-uniqueness theorem, continuation theorems and the Wintner global existence theorems for deterministic differential equations, are extended to the NSFDEs. Two examples are given to illustrate the efficiency of our results.

1. Introduction. Many physical phenomena can be modeled by stochastic dynamical systems whose evolution on time is governed by random forces as well as intrinsic dependence of the state on a finite part of its past history. Such models may be identified as SFDEs. Neutral stochastic functional differential equations (NSFDEs) not only depend on past and present values but also involve derivatives with delays. In recent years, investigation of NSFDEs has attracted the considerable attention of researchers, and many qualitative properties of solutions to NSFDEs have been obtained [**1, 7, 8**].

Mao [**8**] considered the following NSFDE of Itô-type with finite delay

$$(1) \qquad \mathrm{d}[x(t) - G(x_t)] = f(t, x_t)\,\mathrm{d}t + g(t, x_t)\,\mathrm{d}\omega(t), \quad t_0 \le t < T,$$

with the initial condition

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$$x_{t_0} = \xi, \tag{2}$$

where $x_t(s) = x(t+s), s \in [-\tau, 0]$, $\tau > 0$, T is a constant, or $T = \infty$.

Under G satisfying uniform Lipschitz condition with Lipschitz coefficient $\kappa < 1$, Mao obtained the existence-uniqueness of solutions of the initial value problem (1) with (2) if f and g satisfy linear growth condition and f, g satisfy uniform Lipschitz condition [**8**, Theorem 2.2, page 202] or quasi-local Lipschitz condition [**8**, Theorem 2.5, page 207]. Wei and Wang [**11**] gave the existence-uniqueness of solutions for SFDEs with infinite delay under uniform Lipschitz condition and weakened linear growth condition. Bao and Cao [**1**] extend the above results to the NSFDE (1) and (2) with infinite delay. However, the linear growth condition is strict, and many equations do not obey it such as the examples in Section 6. Xu et al. [**13**, Lemma 3.1] obtained the existence-uniqueness of solutions for SFDEs if f and g only satisfy uniform Lipschitz condition. On the other hand, Xu et al. [**15**] presented the Picard local existence-uniqueness theorem, continuation theorems and the Wintner global existence theorems for SFDEs under local Lipschitz condition.

Motivated by the above discussions, a natural question then is that of asking whether the Picard theorem holds for general NSFDEs? That is, is the local Lipschitz condition alone sufficient to prove a local existence-uniqueness theorem for NSFDEs? Our first objective is to give a positive answer of the above question under a lower Lipschitz condition for the operator D. A local existence-uniqueness theorem is given for NSFDEs by the contraction mapping principle. Under the lower Lipschitz condition of D, our new theorems need only the local Lipschitz condition without the linear growth condition; therefore, the theorems cover a wider class of nonlinear NSFDEs. Our second objective is to give some continuation and global existence theorems for NSFDEs [**15**], which extend the related results in [**1, 8, 11**].

This paper is organized as follows. In Section 2, we introduce some notations and definitions. Section 3 is devoted to obtaining the local existence-uniqueness of solutions of NSFDEs. In Section 4, we will establish some continuation theorems for NSFDEs. In Section 5, we shall give the global existence theorems for NSFDEs. Two examples are given in Section 6 to illustrate the efficiency of the obtained results.

2. Preliminaries. In this section, we introduce some notations and recall some basic definitions.

$C(X,Y)$ denotes the space of continuous mappings from the topological space X to the topological space Y. Especially, let $BC \triangleq BC([-\tau,0],R^n)$ be the space of all bounded continuous R^n-value functions ϕ defined on $[-\tau,0]$ with the norm $\|\phi\| = \sup_{-\tau\le s\le 0}|\phi(s)|$, where $|\cdot|$ is any norm in R^n and τ is a fixed number or $\tau=\infty$. When $\tau=\infty$ we mean, of course, that $BC \triangleq BC((-\infty,0],R^n)$.

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge t_0},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge t_0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets in $\mathcal{F}$). $\omega(t) = (\omega_1(t),\dots,\omega_m(t))^T$ is an m-dimensional Brownian motion defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge t_0},P)$.

For Banach space $L^p(\Omega,R^n)$, $p>0$, we define the norm

$$|x|_\Omega \triangleq (\mathbf{E}|x|^p)^{1/p}, \quad p>0.$$

We also employ $|\cdot|_\Omega$ to denote the norm of Banach space $L^p(\Omega,R^{n\times m})$, $p>0$.

We shall adopt the usual manner (see [**5, 8, 15**]) and let $L^p(\Omega,C(J,R^n))$, $p>0$, be the space of ($\mathcal{F}$, Borel C)-measurable maps $\Omega\to C(J,R^n)$ which are L^p in the Bochner sense. Give $L^p(\Omega,C(J,R^n))$ the norm

$$\|\xi\|_{\Omega J} = \left[\int_\Omega \sup_{t\in J}|\xi(t,\omega)|^p \mathrm{d}P(\omega)\right]^{1/p} = [\mathbf{E}\sup_{t\in J}|\xi(t,\omega)|^p]^{1/p}, \quad p>0,$$

where $J\subset R$. Especially, when $J=[-\tau,0]$,

$$L^p(\Omega,C(J,R^n)) = L^p(\Omega,C).$$

For convenience, we denote the norm of $\xi\in L^p(\Omega,C)$ by

$$\|\xi\|_\Omega \triangleq \|\xi\|_{\Omega[-\tau,0]} = [\mathbf{E}\|\xi\|^p]^{1/p}, \quad p>0.$$

Let $L^p_D(\Omega,C([t_0-\tau,a],R^n))$ be the space of all processes $x(t)\in L^p(\Omega,C([t_0-\tau,a],R^n))$ such that $x(t)$ is $\mathcal{F}_{t_0}$-measurable for all $t\in[t_0-\tau,t_0]$ and $x(t)$ is $\mathcal{F}_t$-measurable for all $t\in[t_0,a]$.

Consider the following NSFDE of Itô-type

$$\mathrm{d}D(t,x_t) = f(t,x_t)\,\mathrm{d}t + g(t,x_t)\,\mathrm{d}\omega(t), \quad t_0 \le t < T, \tag{3}$$

with the initial condition

$$x_{t_0} = \xi, \tag{4}$$

where $x_t(s) = x(t+s)$ and $s \in [-\tau, 0]$ can be regarded as a BC-valued stochastic process. Since we only consider function $x(t)$ bounded on $[t_0-\tau, t]$, then $\|x_t\| = \sup_{-\tau\le s\le 0}|x(t+s)|$ always exists as a finite number almost surely whenever τ is a fixed number or $\tau = \infty$. For convenience, we always define $|||x_t||| = \sup_{-\infty<s\le 0}|x(t+s)|$; then $\|x_t\| \le |||x_t|||$. T is a constant, or $T = \infty$.

Throughout this paper, we always set $p \ge 2$ and suppose $\xi \in L^p(\Omega, BC)$ in (4) is an $\mathcal{F}_{t_0}$-measurable process and, for (3), let

$$\begin{aligned} f &: [t_0, T) \times L^p(\Omega, BC) \longrightarrow L^p(\Omega, R^n), \\ g &: [t_0, T) \times L^p(\Omega, BC) \longrightarrow L^p(\Omega, R^{n\times m}), \\ D &: [t_0, T) \times L^p(\Omega, BC) \longrightarrow L^p(\Omega, R^n). \end{aligned}$$

Definition 1. An R^n-valued stochastic process $x(t)$ defined on $t_0 - \tau \le t < T$ is called a solution of (3) with initial data (4) if it has the following properties:

(i) $x(t) \in L^p_D(\Omega, C([t_0-\tau, T), R^n))$;

(ii) $\{f(t,x_t)\} \in L^p_D(\Omega, C([t_0, T), R^n))$ and $\{g(t,x_t)\} \in L^p_D(\Omega, C([t_0, T), R^{n\times m}))$;

(iii) $x_{t_0} = \xi$ and, for each $t_0 \le t < T$, $D(t,x_t) = D(t_0,\xi) + \int_{t_0}^t f(s,x_s)\,\mathrm{d}s + \int_{t_0}^t g(s,x_s)\,\mathrm{d}\omega(s)$ a.s.

Solution $x(t)$ of (3) and (4) is said to be unique if any other solution $\overline{x}(t)$ is indistinguishable from it, that is,

$$P\{x(t) = \overline{x}(t) \quad \text{for all } t \in [t_0-\tau, T)\} = 1.$$

Definition 2. Let $x(t)$ on J_1 and $\overline{x}(t)$ on J_2 both be solutions of (3) and (4). If $J_1 \subset J_2$, $J_1 \ne J_2$ and $P\{x(t) = \overline{x}(t) \text{ for all } t \in J_1\} = 1$, we say $\overline{x}(t)$ is a continuation of $x(t)$, or $x(t)$ can be continued to J_2. A solution $x(t)$ is non-continuable if it has no continuation. The existing interval of non-continuable solution $x(t)$ is called the maximum existing interval of $x(t)$.

Definition 3. It is said that a sample path $x(t,\omega)$ explodes in $[t_0-\tau,T)$ if, for any integer $k>0$, there exists a time $s\in[t_0-\tau,T)$ such that $|x(s,\omega)|\geq k$. And the solution $x(t,\omega)$ of (3) and (4) explodes in $[t_0-\tau,T)$ if there exists a measurable subset $S\subset\Omega$ with $P(S)>0$ such that the sample path $x(t,\omega)$ explodes in $[t_0-\tau,T)$ for almost all $\omega\in S$.

Definition 4. The solution $x(t)$ of (3) and (4) for $t\in[t_0,\beta)$ is said to be bounded if there exists a constant $\alpha=\alpha(t_0,\xi)>0$ such that

$$|x(t)|_\Omega<\alpha,\quad \text{for all } t\in[t_0,\beta).$$

Definition 5. The functional $F:[t_0,T)\times L^p(\Omega,BC)\to L^p(\Omega,R^n)$ is said to be quasi-bounded if, for any constant $\beta\in(t_0,T)$ and $\alpha>0$, there exists a positive constant M such that

$$|F(t,\phi)|_\Omega\leq M,$$

provided that

$$t\in[t_0,\beta]\quad\text{and}\quad\|\phi\|_\Omega\leq\alpha.$$

Definition 6. The functional $F:[t_0,T)\times L^p(\Omega,BC)\to L^p(\Omega,R^n)$ is said to satisfy the local Lipschitz condition at point $(\overline{t}_0,\xi)$ if there exist positive constants b, r and K such that

$$|F(t,\phi)-F(t,\psi)|_\Omega\leq K\|\phi-\psi\|_\Omega,$$

for all $t\in[\overline{t}_0-b,\overline{t}_0+b]\cap[t_0,T)$ and $\phi,\psi\in S(\xi,r)$. Moreover, F is said to satisfy the local Lipschitz condition in the region $[t_0,T)\times L^p(\Omega,BC)$ if F satisfies the local Lipschitz condition for any point $(\overline{t},\overline{\xi})\in[t_0,T)\times L^p(\Omega,BC)$.

Definition 7. The functional $D(t,x_t):(-\infty,T)\times L^p(\Omega,BC)\to L^p(\Omega,R^n)$ is said to satisfy the lower Lipschitz condition in $J\times L^p(\Omega,BC)$, $J\subseteq(-\infty,T)$ if there exists a positive constant K such that

$$\sup_{-\infty<r\le 0}|D(u+r,x_{u+r})-D(v+r,y_{v+r})|\ge K\|x_u-y_v\|,$$

for all $u,v\in J$ and $x_u,y_v\in L^p(\Omega,BC)$.

3. Local existence and uniqueness.

Lemma 3.1. Let $p\ge 2$. For any $t_0\in R$ and $\beta>0$, we let g be a process in $L^p([t_0,t_0+\beta];R^{n\times m})$ such that $\mathbf{E}\int_{t_0}^{t_0+\beta}|g(s)|^p\mathrm{d}s<\infty$, then

$$\mathbf{E}\bigg|\int_{t_0}^{t_0+\beta}g(s)\,\mathrm{d}\omega(s)\bigg|^p\le C_p\beta^{(p-2)/2}\mathbf{E}\int_{t_0}^{t_0+\beta}|g(s)|^p\mathrm{d}s,$$

where

$$C_p=\left(\frac{p(p-1)}{2}\right)^{p/2}.$$

Proof. The proof is similar to that of Theorem 7.1 [**8**, page 39], except letting

$$x(t)=\int_{t_0}^{t}|g(s)|^p\mathrm{d}\omega(s),\quad\forall\ t\in[t_0,t_0+\beta],$$

and noting $x(t_0)=0$. So it is omitted. $\square$

Lemma 3.2. *Under the same assumptions as Lemma* 3.1*, then*

$$\mathbf{E}\sup_{t_0\le t\le t_0+\beta}\bigg|\int_{t_0}^{t}g(s)\,\mathrm{d}\omega(s)\bigg|^p\le c_p\beta^{(p-2)/2}\mathbf{E}\int_{t_0}^{t_0+\beta}|g(s)|^p\mathrm{d}s,$$

where

$$c_p=\left(\frac{p^3}{2(p-1)}\right)^{p/2}.$$

Proof. The proof is similar to that of Theorem 7.2 [**8**, page 40], so it is omitted. □

Theorem 3.1 (Local existence-uniqueness theorem). *Assume that*:

(i) *f, g are continuous in $[t_0, T) \times L^p(\Omega, BC)$ and satisfy the local Lipschitz condition at (t_0, ξ).*

(ii) *D is continuous and satisfies the lower Lipschitz condition in $[t_0, T) \times L^p(\Omega, BC)$ if we set $x(t) = \xi(-\tau)$ for $t \le t_0 - \tau$ when the delay is finite and $D(t, x_t) = D(t_0, \xi)$ for $t \le t_0$.*

Then there is an $h > 0$ such that the initial problem (3) *and* (4) *has a unique solution on $[t_0, t_0 + h]$*

Proof. Since f and g satisfy the local Lipschitz condition at (t_0, ξ), there exist positive constants b_1, r_1 and K_1 such that

$$|f(t,\phi) - f(t,\psi)|_\Omega \le K_1 \|\phi - \psi\|_\Omega, \tag{5}$$

$$|g(t,\phi) - g(t,\psi)|_\Omega \le K_1 \|\phi - \psi\|_\Omega, \tag{6}$$

for all $t \in [t_0, t_0 + b_1]$, and $\phi, \psi \in S(\xi, r_1)$.

From the lower Lipschitz condition, for all $u, v \in [t_0, T)$ and $x_u, y_v \in L^p(\Omega, BC)$, there exists a positive constant K_0 such that

$$\sup_{-\infty < r \le 0} |D(u + r, x_{u+r}) - D(v + r, y_{v+r})| \ge K_0 \|x_u - y_v\|. \tag{7}$$

It implies that there exists a constant $K > 0$ such that

$$\mathbf{E} \sup_{-\infty < r \le 0} |D(u + r, x_{u+r}) - D(v + r, y_{v+r})|^p \ge K \|x_u - y_v\|_\Omega^p. \tag{8}$$

For $\xi \in L^p(\Omega, BC)$ and $r_1 > 0$, we will find an $h > 0$ and $M^* > 0$, and consider the complete metric space

$$\begin{aligned} S_h = \Big\{ x \in L^p(\Omega, BC([t_0 - \tau, t_0 + h], R^n)) :& \\ x_t \in S(\xi, r_1) \quad &\text{for } t \in [t_0, t_0 + h]; \\ |x(t_2) - x(t_1)|_\Omega \le M^* |t_2 - t_1|^{1/2}& \\ &\text{for } t_1, t_2 \in [t_0, t_0 + h];\ x_{t_0} = \xi \Big\}. \end{aligned} \tag{9}$$

The metric ρ of S_h is defined by

$$\rho(x,y)=\Big[\mathbf{E}\sup_{t_0\le t\le t_0+h}|x(t)-y(t)|^p\Big]^{1/p}.$$

Since f and g are continuous, $|f(t,\xi)|_\Omega$ and $|g(t,\xi)|_\Omega$ are continuous on t for the above given $\xi\in L^p(\Omega,BC)$. So, there exists a positive constant M such that

$$|f(t,\xi)|_\Omega\le M,\qquad |g(t,\xi)|_\Omega\le M,\quad \text{for all } t\in[t_0,t_0+1]. \tag{10}$$

Firstly, we choose $M^*=[4^{p-1}(K_1^pr_1^p+M^p)(2+c_p+C_p)/K]^{1/p}$ and $0<h\le\min\{1,b_1\}$ satisfying that

$$\frac{1}{K}\Big[4^{p-1}(K_1^pr_1^p+M^p)(h^p+c_ph^{p/2})\Big]\le r_1^p.$$

Later, h will be further restricted.

Let Γ be the following transformation:

$$\begin{cases}\Gamma(x(t)):D(t,\Gamma x_t)=D(t_0,\xi)+\int_{t_0}^t f(s,x_s)\,\mathrm{d}s\\ \qquad +\int_{t_0}^t g(s,x_s)\,\mathrm{d}\omega(s), & t\in[t_0,t_0+h],\\ \Gamma(x(t))=\xi(t-t_0), & t\in[t_0-\tau,t_0],\end{cases}\qquad x\in S_h. \tag{11}$$

Next, we prove

$$\Gamma:S_h\longrightarrow S_h. \tag{12}$$

From (8) and (11), for any $t\in[t_0,t_0+h]$, we have

$$\begin{aligned}K\|\Gamma x_t-\xi\|_\Omega^p&=K\|\Gamma x_t-x_{t_0}\|_\Omega^p\\ &\le\mathbf{E}\sup_{-\infty<r\le0}|D(t+r,\Gamma x_{t+r})-D(t_0+r,x_{t_0+r})|^p\\ &\le\mathbf{E}\sup_{t_0-t\le r\le0}|D(t+r,\Gamma x_{t+r})-D(t_0,\xi)|^p\\ &\le 2^{p-1}\mathbf{E}\sup_{t_0-t\le r\le0}\left|\int_{t_0}^{t+r}f(s,x_s)\,\mathrm{d}s\right|^p+2^{p-1}\mathbf{E}\\ &\quad\times\sup_{t_0-t\le r\le0}\left|\int_{t_0}^{t+r}g(s,x_s)\,\mathrm{d}\omega(s)\right|^p.\end{aligned} \tag{13}$$

By Hölder's inequality, Jonson's inequality and the local Lipschitz condition, we obtain

$$
\begin{aligned}
(14)\quad & \mathbf{E} \sup_{t_0 - t \le r \le 0} \left| \int_{t_0}^{t+r} f(s, x_s) \mathrm{d}s \right|^p \\
&\le (t - t_0)^{p-1} \mathbf{E} \int_{t_0}^{t} |f(s, x_s)|^p \mathrm{d}s \\
&\le h^{p-1} 2^{p-1} \int_{t_0}^{t} \mathbf{E} \Big[|f(s, x_s) - f(s, \xi)|^p + |f(s, \xi)|^p \Big] \, \mathrm{d}s \\
&\le h^{p-1} 2^{p-1} \int_{t_0}^{t} \Big[K_1^p \|x_s - \xi\|_\Omega^p + M^p \Big] \, \mathrm{d}s \\
&\le 2^{p-1} h^p (K_1^p r_1^p + M^p).
\end{aligned}
$$

By Lemma 3.2, we obtain

$$
\begin{aligned}
(15)\quad & \mathbf{E} \sup_{t_0 - t \le r \le 0} \left| \int_{t_0}^{t+r} g(s, x_s) \, \mathrm{d}\omega(s) \right|^p \\
&\le c_p (t - t_0)^{(p-2)/2} \mathbf{E} \int_{t_0}^{t} |g(s, x_s)|^p \mathrm{d}s \\
&\le c_p h^{(p-2)/2} 2^{p-1} \int_{t_0}^{t} \mathbf{E} \Big[|g(s, x_s) - g(s, \xi)|^p + |g(s, \xi)|^p \Big] \, \mathrm{d}s \\
&\le c_p h^{(p-2)/2} 2^{p-1} \int_{t_0}^{t} \Big[K_1^p \|x_s - \xi\|_\Omega^p + M^p \Big] \, \mathrm{d}s \\
&\le 2^{p-1} h^{p/2} c_p (K_1^p r_1^p + M^p).
\end{aligned}
$$

Thus, it follows from (13)–(15) that

$$
\begin{aligned}
\|\Gamma x_t - \xi\|_\Omega^p &\le \frac{1}{K} \Big[4^{p-1} (K_1^p r_1^p + M^p)(h^p + c_p h^{p/2}) \Big] \\
&\le r_1^p.
\end{aligned}
\tag{16}
$$

From (8) and (11), for all $t_0 \le t_1 < t_2 \le t_0 + h$, we have

$$
\begin{aligned}
(17)\quad & K|\Gamma x(t_2) - \Gamma x(t_1)|_\Omega^p \\
& \le K\|\Gamma x_{t_2} - \Gamma x_{t_1}\|_\Omega^p \\
& \le \mathbf{E} \sup_{-\infty < r \le 0} |D(t_2 + r, \Gamma x_{t_2+r}) - D(t_1 + r, \Gamma x_{t_1+r})|^p \\
& \le \mathbf{E} \sup_{t_0 - t_2 \le r \le t_0 - t_1} |D(t_2 + r, \Gamma x_{t_2+r}) - D(t_0, \xi)|^p \\
& \quad + \mathbf{E} \sup_{t_0 - t_1 \le r \le 0} |D(t_2 + r, \Gamma x_{t_2+r}) - D(t_1 + r, \Gamma x_{t_1+r})|^p \\
& \le \mathbf{E} \sup_{t_0 - t_2 \le r \le t_0 - t_1} \left| \int_{t_0}^{t_2+r} f(s, x_s)\,\mathrm{d}s + \int_{t_0}^{t_2+r} g(s, x_s)\mathrm{d}\omega(s) \right|^p \\
& \quad + \mathbf{E} \sup_{t_0 - t_1 \le r \le 0} \left| \int_{t_1+r}^{t_2+r} f(s, x_s)\,\mathrm{d}s + \int_{t_1+r}^{t_2+r} g(s, x_s)\,\mathrm{d}\omega(s) \right|^p \\
& \le 2^{p-1}\mathbf{E} \sup_{t_0 - t_2 \le r \le t_0 - t_1} \Bigg[\left| \int_{t_0}^{t_2+r} f(s, x_s)\,\mathrm{d}s \right|^p \\
& \qquad\qquad + \left| \int_{t_0}^{t_2+r} g(s, x_s)\,\mathrm{d}\omega(s) \right|^p \Bigg] \\
& \quad + 2^{p-1}\mathbf{E} \sup_{t_0 - t_1 \le r \le 0} \Bigg[\left| \int_{t_1+r}^{t_2+r} f(s, x_s)\,\mathrm{d}s \right|^p \\
& \qquad\qquad + \left| \int_{t_1+r}^{t_2+r} g(s, x_s)\,\mathrm{d}\omega(s) \right|^p \Bigg] \\
& \le 2^{p-1} \Bigg[(t_2 - t_1)^{p-1}\mathbf{E} \int_{t_0}^{t_0+t_2-t_1} |f(s, x_s)|^p \mathrm{d}s \\
& \qquad + c_p (t_2 - t_1)^{(p-2)/2}\mathbf{E} \int_{t_0}^{t_0+t_2-t_1} |g(s, x_s)|^p \mathrm{d}s \\
& \qquad + (t_2 - t_1)^{p-1} \sup_{t_0 - t_1 \le r \le 0} \mathbf{E} \int_{t_1+r}^{t_2+r} |f(s, x_s)|^p \mathrm{d}s \\
& \qquad + C_p (t_2 - t_1)^{(p-2)/2} \sup_{t_0 - t_1 \le r \le 0} \mathbf{E} \int_{t_1+r}^{t_2+r} |g(s, x_s)|^p \mathrm{d}s \Bigg].
\end{aligned}
$$

Using local Lipschitz conditions like (14) and (15), we can get

(18)
$$\begin{aligned}|\Gamma x(t_2)-\Gamma x(t_1)|_\Omega^p &\le \frac{4^{p-1}(K_1^p r_1^p+M^p)}{K}\\ &\quad\times[2(t_2-t_1)^p+c_p(t_2-t_1)^{p/2}+C_p(t_2-t_1)^{p/2}]\\ &\le \frac{4^{p-1}(K_1^p r_1^p+M^p)(2+c_p+C_p)}{K}(t_2-t_1)^{p/2}\\ &=[M^*(t_2-t_1)^{1/2}]^p.\end{aligned}$$

So, (12) holds.

On the other hand, from (8) and (11), for all $u,v\in S_h$, we obtain

(19)
$$\begin{aligned}K\rho^p(\Gamma u,\Gamma v) &= K\mathbf{E}\sup_{t_0\le t\le t_0+h}|\Gamma u(t)-\Gamma v(t)|^p\\ &= K\sup_{t_0\le t\le t_0+h}\|\Gamma u_t-\Gamma v_t\|_\Omega^p\\ &\le \mathbf{E}\sup_{-\infty<r\le 0}|D(t_0+h+r,\Gamma u_{t_0+h+r})\\ &\qquad\qquad - D(t_0+h+r,\Gamma v_{t_0+h+r})|^p\\ &\le \mathbf{E}\sup_{-h\le r\le 0}|D(t_0+h+r,\Gamma u_{t_0+h+r})\\ &\qquad\qquad - D(t_0+h+r,\Gamma v_{t_0+h+r})|^p\\ &= \mathbf{E}\sup_{-h\le r\le 0}\Bigg|\int_{t_0}^{t_0+h+r} f(s,u_s)-f(s,v_s)\,\mathrm{d}s\\ &\qquad\qquad + \int_{t_0}^{t_0+h+r} g(s,u_s)-g(s,v_s)\,\mathrm{d}\omega(s)\Bigg|^p\\ &\le 2^{p-1}\mathbf{E}\sup_{-h\le r\le 0}\left|\int_{t_0}^{t_0+h+r} f(s,u_s)-f(s,v_s)\,\mathrm{d}s\right|^p\\ &\quad + 2^{p-1}\mathbf{E}\sup_{-h\le r\le 0}\left|\int_{t_0}^{t_0+h+r} g(s,u_s)-g(s,v_s)\,\mathrm{d}\omega(s)\right|^p\\ &\le 2^{p-1}h^{p-1}\int_{t_0}^{t_0+h}\mathbf{E}|f(s,u_s)-f(s,v_s)|^p\mathrm{d}s\\ &\quad + 2^{p-1}c_p h^{(p-2)/2}\int_{t_0}^{t_0+h}\mathbf{E}|g(s,u_s)-g(s,v_s)|^p\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\le 2^{p-1}(h^{p-1}+c_p h^{(p/2)-1})\int_{t_0}^{t_0+h} K_1^p \|u_s - v_s\|_\Omega^p \mathrm{d}s \\
&\le 2^{p-1}K_1^p(h^p + c_p h^{p/2})\mathbf{E}\sup_{t_0\le t\le t_0+h}\|u_t - v_t\|^p \\
&= 2^{p-1}K_1^p(h^p + c_p h^{p/2})\mathbf{E}\sup_{t_0-\tau\le t\le t_0+h}|u(t) - v(t)|^p.
\end{aligned}
$$

For a given $\theta \in (0,1)$, we may choose h such that

$$2^{p-1}K_1^p(h^p + c_p h^{p/2}) \le \theta K;$$

then, we can get

$$K\rho^p(\Gamma u, \Gamma v) \le \theta K \mathbf{E}\sup_{t_0\le t\le t_0+h}|u(t)-v(t)|^p = \theta K \rho^p(u,v). \tag{20}$$

Thus, $\Gamma : S_h \to S_h$ is a contraction mapping in $L^p(\Omega, BC([t_0-\tau, t_0+h], R^n))$. Then $\Gamma : S_h \to S_h$ has a unique fixed point x in $L^p(\Omega, BC([t_0-\tau, t_0+h], R^n))$ which, as is easy to see, is a solution of (3) with initial data (4). The proof is completed. □

Remark 3.1. Condition (ii) of Theorem 3.1 is general and reachable because a similar assumption is required in many works on the qualitative analysis of NSFDEs. For example, in [**1, 7, 8**], $D(t,x_t) = x(t) - G(x_t)$ and $G(x_t)$ satisfies the uniform Lipschitz condition with Lipschitz coefficient $\kappa < 1$; then $D(t,x_t)$ satisfies condition (ii) of Theorem 3.1.

In fact, since G satisfies the uniform Lipschitz condition, for all $\phi, \psi \in BC([-\tau,0], R^n)$

$$|G(\phi) - G(\psi)| \le \kappa\|\phi - \psi\|, \quad \kappa \in (0,1). \tag{21}$$

Then, for all $u, v \in [t_0, T)$ and $x_u, y_v \in L^p(\Omega, BC)$, we have

$$
\begin{aligned}
&\sup_{-\infty<r\le 0}|D(u+r, x_{u+r}) - D(v+r, y_{v+r})| \\
&\ge \sup_{-\infty<r\le 0}|x(u+r) - y(v+r)| - \sup_{-\infty<r\le 0}|G(x_{u+r}) - G(y_{v+r})| \\
&\ge |\|x_u - y_v\|| - \kappa \sup_{-\infty<r\le 0}\|x_{u+r} - y_{v+r}\| \\
&\ge (1-\kappa)\|x_u - y_v\|.
\end{aligned} \tag{22}
$$

Remark 3.2. Theorem 3.1 is also a generalization of the Picard local existence-uniqueness theorem for NFDEs given by Driver in [**3**].

4. Continuation theorems. In this section, we will establish some continuation theorems for the NSFDE (3) and (4). We will give the conditions under which the trajectory (t, x) of a noncontinuable solution of $[t_0, \beta)$ approaches the boundary of $[t_0, T) \times L^p(\Omega, BC)$ as $t \to \beta^-$.

Theorem 4.1. *Let condition* (ii) *of Theorem* 3.1 *hold. Assume that f and g are continuous and satisfy the local Lipschitz condition in $[t_0, T) \times L^p(\Omega, BC)$. Then the following conclusions are true.*

(I) *The initial value problem* (3) *has a unique non-continuable solution $x(t)$, whose maximum existing interval is assumed to be $[t_0 - \tau, \beta)$.*

(II) *For any compact set $D_1 \subset [t_0, T) \times L^p(\Omega, BC)$,*

$$(t, x_t) \notin D_1, \quad \textit{for some } t \in [t_0, \beta). \tag{23}$$

Proof. From Theorem 3.1, the initial value problem (3) has a unique solution $x(t) \in L^p(\Omega, BC([t_0 - \tau, t_1], R^n))$. Note that $x_{t_1} \in L^p(\Omega, BC)$ and f, g and D satisfy the local Lipschitz condition and lower Lipschitz condition in $[t_0, T) \times L^p(\Omega, BC)$. Thus, applying Theorem 3.1 to (3) with the initial condition (t_1, x_{t_1}), the solution $x(t)$ of (3) can be continued to $[t_0 - \tau, t_1 + \delta_1]$, where δ_1 is a positive constant satisfying $t_1 + \delta_1 < T$. Furthermore, $x(t)$ is the unique solution of (3) on $[t_0 - \tau, t_1 + \delta_1]$. Repeat the above procedure, and define

$$\beta = \sup\{s \in \mathbf{R} : x(t) \text{ can be continued to } [t_0 - \tau, s]\}.$$

Then $\beta \in (t_0, T]$, $x(t)$ is the unique non-continuable solution of the initial value problem (3) and its maximum existing interval is $[t_0 - \tau, \beta)$. Obviously, its maximum existing interval must not be $[t_0 - \tau, \beta]$ by the same continuation method of $x(t)$ at $t = t_1$.

Then, the proof of (I) is completed.

In (II), the case $\beta = T$ is trivial. So we suppose $\beta < T$. If the conclusion of (II) is not true, there are a compact set $D_1 \subset$

$[t_0, T) \times L^p(\Omega, BC)$, a sequence of real numbers $t_k \to \beta^-$ as $k \to +\infty$, and a $\psi \in L^p(\Omega, BC)$ such that

$$(t_k, x_{t_k}) \in D_1, \quad (b, \psi) \in D_1, \qquad (t_k, x_{t_k}) \longrightarrow (\beta, \psi), \text{ as } k \to +\infty \text{ a.s.}$$

Thus, for any $\varepsilon > 0$,

$$\lim_{k \to +\infty} \mathbf{E} \sup_{\theta \in [-\tau, -\varepsilon]} |x_{t_k}(\theta) - \psi(\theta)|^p = 0.$$

Since $x_t(\theta) = x(t + \theta), -\tau \le \theta \le 0$, and $\tau > 0$, this implies

$$\mathbf{E}|x(\beta + \theta) - \psi(\theta)|^p = 0, \quad -\tau \le \theta < 0.$$

Hence $\lim_{t \to \beta^-} x(t)$ exists. So, $x(t)$ can be continued to $[t_0 - \tau, \beta]$. This contradicts the fact that the maximum existing interval of $x(t)$ is $[t_0 - \tau, \beta)$. So, (II) is true.

Then, the proof of Theorem 4.1 is completed. $\square$

Theorem 4.2. *In addition to the conditions in Theorem* 4.1*, if* f *and* g *are quasi-bounded, then for any closed bounded set* $A \subset [t_0, T) \times L^p(\Omega, BC)$*, the non-continuable solution* $x(t)$ *of the initial value problem* (3) *on* $[t_0 - \tau, \beta)$ *satisfies that*:

$$(t, x_t) \notin A, \quad \textit{for some } t \in [t_0, \beta). \tag{24}$$

Proof. The case $\beta = T$ is trivial. So we suppose $\beta < T$. If the conclusion of the theorem is not true, there must exist a closed bounded set $A \subset [t_0, T) \times L^p(\Omega, BC)$ such that

$$(t, x_t) \in A, \quad \text{for all } t \in [t_0, \beta). \tag{25}$$

From the boundedness of A, there exists a constant $\alpha_1 > \|\xi\|_\Omega$ such that

$$\|x_t\|_\Omega < \alpha_1, \quad \text{for all } t \in [t_0, \beta). \tag{26}$$

By the quasi-boundedness of f and g, there is a positive constant μ such that

$$|f(t, x_t)|_\Omega \le \mu, \qquad |g(t, x_t)|_\Omega \le \mu, \quad \text{for all } t \in [t_0, \beta). \tag{27}$$

By the uniqueness of the local solution of initial value problem (3) on $[t_0 - \tau, \beta)$, we have

$$D(t, x_t) = D(t_1, x_{t_1}) + \int_{t_1}^{t} f(s, x_s)\,\mathrm{d}s + \int_{t_1}^{t} g(s, x_s)\,\mathrm{d}\omega(s), \tag{28}$$

where $t_0 \le t_1 < t < \beta$. So, by the same way in (17), we obtain for all $t_0 \le t_1 < t_2 < \beta$,

$$\begin{aligned}
(29)\quad & K|x(t_2) - x(t_1)|_\Omega^p \\
&\le 2^{p-1}\bigg[(t_2 - t_1)^{p-1}\int_{t_0}^{t_0+t_2-t_1} \mathbf{E}|f(s, x_s)|^p \mathrm{d}s \\
&\qquad + c_p(t_2 - t_1)^{(p-2)/2}\int_{t_0}^{t_0+t_2-t_1} \mathbf{E}|g(s, x_s)|^p \mathrm{d}s \\
&\qquad + (t_2 - t_1)^{p-1}\sup_{t_0 - t_1 \le r \le 0}\int_{t_1+r}^{t_2+r} \mathbf{E}|f(s, x_s)|^p \mathrm{d}s \\
&\qquad + C_p(t_2 - t_1)^{(p-2)/2}\sup_{t_0 - t_1 \le r \le 0}\int_{t_1+r}^{t_2+r} \mathbf{E}|g(s, x_s)|^p \mathrm{d}s\bigg] \\
&\le 2^p \mu^p (t_2 - t_1)^p + 2^{p-1}(c_p + C_p)\mu^p (t_2 - t_1)^{p/2}.
\end{aligned}$$

This implies that $\{(t, x_t) : t_0 \le t < \beta\}$ belongs to a compact set in $[t_0, T) \times L^p(\Omega, BC)$. This contradicts Theorem 4.1, and we have proved the theorem. $\square$

5. Global existence theorems. In this section, we will give some global existence theorems for the NSFDE (3) and (4).

Theorem 5.1. *With the same conditions as in Theorem* 4.2*, the following conclusions are true.*

(I) *If solution $x(t)$ of the initial value problem* (3) *is bounded, $x(t)$ exists on $[t_0 - \tau, T)$.*

(II) *If $\beta < T$, the solution $x(t)$ of the initial value problem* (3) *explodes in $[t_0 - \tau, \beta)$.*

Proof. If conclusion (I) of the theorem is not true, we must have $\beta < T$. From the boundedness of the solution of (3), there exists a constant α_1 such that

$$\|x_t\|_\Omega < \alpha_1, \quad \text{for all } t \in [t_0, \beta).$$

Then the non-continuable solution $x(t)$ of the initial value problem (3) satisfies that

$$(t, x_t) \in A, \quad \text{for all } t \in [t_0, \beta),$$

where $A = [t_0, \beta] \times \{x_t : \|x_t\|_\Omega \leq \alpha_1\}$ is a closed bounded set in $[t_0, T) \times L^p(\Omega, BC)$. This contradicts Theorem 4.2, and conclusion (I) is true. Conclusion (II) obviously holds by conclusion (I). Thus, the proof is completed. $\square$

Remark 5.1. In the BC space, a result similar to Theorem 5.1 is given by Xu et al. [**14**]. Therefore, the following results still hold under the local Lipschitz condition in BC if the drift and diffusion coefficients are square-integrable with respect to t when the state variable equals zero.

In the following, we shall extend the main results in [**1, 11**]. To this end, we shall introduce some notations and lemmas. For the vector functions $x(t) = (x_1(t), \ldots, x_m(t))^T \in C(R, R^m)$, we denote

$$\overline{x}(t) = (\overline{x}_1(t), \ldots, \overline{x}_m(t))^T,$$
$$\overline{x}_i(t) = \sup_{-\tau \leq s \leq 0} x_i(t+s), \quad i = 1, 2, \ldots, m,$$

and define the Dini upper right derivative as follows:

$$D^+x(t) = (D^+x_1(t), \ldots, D^+x_m(t))^T,$$
$$D^+x_i(t) = \limsup_{h \to 0^+} \frac{x_i(t+h) - x_i(t)}{h}, \quad i = 1, 2, \ldots, m.$$

Lemma 5.1 (Lemma 8.2 in [**9**, page 72]). *Let $h \in C(R \times R, R)$. Assume that $x(t)$ and $y(t)$ are continuous. Furthermore, $x(t)$ is a solution of*

$$D^+x(t) \leq h(t, x(t)), \quad t \geq t_0,$$

and $y(t)$ is the maximal solution of

$$\dot{y}(t) = h(t, y(t)), \quad t \geq t_0.$$

Then, for all $t \geq t_0$,

$$x(t) \leq y(t),$$

provided that $x(t_0) \leq y(t_0)$.

Theorem 5.2. *Let the conditions of Theorem* 4.2 *hold. Assume that there exists a function* $F(t,u) \in C([t_0,T) \times R_+, R_+)$ *such that, for all* $t \in [t_0,T)$ *and* $\phi \in L^p(\Omega, BC)$,

$$|f(t,\phi)|^p + |g(t,\phi)|^p \leq F(t, \|\phi\|^p), \tag{30}$$

where $F(t,u)$ *is monotone nondecreasing and concave with respect to* $u \in R_+$ *for each fixed* $t \in [t_0,T)$. *If, for any* $\gamma > 0$ *and arbitrary given initial condition, the solution of the scalar differential equation*

$$u'(t) = \gamma F(t,u), \tag{31}$$

exists on $[t_0,T)$. *Then any solution of* (3) *and* (4) *also exists on* $[t_0-\tau,T)$.

Proof. From Theorem 4.1, the NSFDE (3) with the initial condition (4) has a solution $x(t) = x(t;t_0,\xi)$ with maximum existing interval $[t_0-\tau,\beta_1)$. Now, we only need to prove $\beta_1 = T$. If $\beta_1 < T$, by Theorem 5.1 there exists a measurable subset $S \subset \Omega$ with $P(S) > 0$ such that $x(t)$ explodes in $[t_0-\tau,\beta_1)$ for all $\omega \in S$. For any sufficiently large integer n, we define the stopping times

$$\tau_n = \beta_1 \wedge \inf\{t \in [t_0,\beta_1) : |x(t)| \geq n\},$$

where, as usual, we set $\inf \varnothing = \infty$. Clearly, the τ_n's are increasing. So they have the limit $\beta_1 = \lim_{n\to\infty} \tau_n$.

Since D satisfies the lower Lipschitz condition in $[t_0,T) \times L^p(\Omega, BC)$, for any $t \in [t_0,T)$, there exists a positive constant K_0 such that

$$\begin{aligned}\sup_{-\infty<r\leq 0} |D(t+r,x_{t+r}) - D(t_0+r,x_{t_0+r})| &\geq K_0\|x_t - x_{t_0}\| \\ &\geq K_0(\|x_t\| - \|\xi\|).\end{aligned} \tag{32}$$

Let $K = K_0^p$; then we can get

$$K\mathbf{E}\sup_{-\tau\le r\le 0}|x(t+r)|^p \le 2^{p-1}K\mathbf{E}\|\xi\|^p + 2^{p-1}\mathbf{E}\sup_{-\infty<r\le 0}|D(t+r,x_{t+r})-D(t_0,\xi)|^p. \tag{33}$$

Since

$$\begin{aligned} D(t\wedge\tau_n, x_{t\wedge\tau_n}) &= D(t_0,\xi) + \int_{t_0}^{t\wedge\tau_n} f(s,x_s)\,\mathrm{d}s \\ &\quad + \int_{t_0}^{t\wedge\tau_n} g(s,x_s)\,\mathrm{d}\omega(s), \quad t\in[t_0,\beta_1). \end{aligned} \tag{34}$$

By the assumptions, we obtain that

$$\begin{aligned} K\mathbf{E}I_S|x(t\wedge\tau_n)|^p &\le K\mathbf{E}I_S\sup_{-\tau\le r\le 0}|x(t\wedge\tau_n+r)|^p \\ &\le 2^{p-1}K\mathbf{E}I_S\|\xi\|^p + 2^{p-1}\mathbf{E}I_S \\ &\quad\times\sup_{-\infty<r\le 0}|D(t\wedge\tau_n+r,x_{t\wedge\tau_n+r})-D(t_0,\xi)|^p \\ &\le 2^{p-1}K\mathbf{E}I_S\|\xi\|^p + 2^{p-1}\mathbf{E}I_S \\ &\quad\times\sup_{t_0-t\wedge\tau_n\le r\le 0}\Big|\int_{t_0}^{t\wedge\tau_n+r} f(s,x_s)\,\mathrm{d}s \\ &\qquad + \int_{t_0}^{t\wedge\tau_n+r} g(s,x_s)\,\mathrm{d}\omega(s)\Big|^p \\ &\le 2^{p-1}K\mathbf{E}I_S\|\xi\|^p + 4^{p-1}\mathbf{E}I_S \\ &\quad\times\sup_{t_0-t\wedge\tau_n\le r\le 0}\Big|\int_{t_0}^{t\wedge\tau_n+r} f(s,x_s)\,\mathrm{d}s\Big|^p \\ &\quad + 4^{p-1}\mathbf{E}I_S\sup_{t_0-t\wedge\tau_n\le r\le 0}\Big|\int_{t_0}^{t\wedge\tau_n+r} g(s,x_s)\,\mathrm{d}\omega(s)\Big|^p \\ &\le 2^{p-1}K\mathbf{E}I_S\|\xi\|^p \\ &\quad + 4^{p-1}(\beta_1-t_0)^{p-1}\mathbf{E}I_S\int_{t_0}^{t\wedge\tau_n}|f(s,x_s)|^p\mathrm{d}s \\ &\quad + 4^{p-1}c_p(\beta_1-t_0)^{(p-2)/2}\mathbf{E}I_S\int_{t_0}^{t\wedge\tau_n}|g(s,x_s)|^p\mathrm{d}s \end{aligned} \tag{35}$$

$$
\begin{aligned}
&\le 2^{p-1} K \mathbf{E} I_S \|\xi\|^p \\
&\quad + 4^{p-1}\Big[(\beta_1 - t_0)^{p-1} + c_p(\beta_1 - t_0)^{(p-2)/2}\Big]\mathbf{E} I_S \\
&\quad \times \int_{t_0}^{t\wedge\tau_n} F(s, \|x_s\|^p)\,\mathrm{d}s \\
&\le 2^{p-1} K \mathbf{E} I_S \|\xi\|^p \\
&\quad + 4^{p-1}\Big[(\beta_1 - t_0)^{p-1} + c_p(\beta_1 - t_0)^{(p-2)/2}\Big] \\
&\quad \times \int_{t_0}^{t\wedge\tau_n} F(s, \mathbf{E} I_S\|x_s\|^p)\,\mathrm{d}s.
\end{aligned}
$$

So, we can obtain

$$
\begin{aligned}
(36)\quad & \mathbf{E} I_S \sup_{-\tau\le r\le 0} |x(t\wedge\tau_n + r)|^p \\
&\qquad \le M_1 + M_2 \int_{t_0}^{t\wedge\tau_n} F(s, \mathbf{E} I_S \sup_{-\tau\le r\le 0} |x(s+r)|^p)\,\mathrm{d}s,
\end{aligned}
$$

where $M_1 = 2^{p-1}\mathbf{E} I_S\|\xi\|^p$ and $M_2 = (4^{p-1})/K[(\beta_1 - t_0)^{p-1} + c_p(\beta_1 - t_0)^{(p-2)/2}]$.

Define $v(t) = \mathbf{E} I_S \sup_{-\tau\le r\le 0} |x(t+r)|^p$ and

$$
w(t) = M_1 + M_2 \int_{t_0}^{t} F(s, \mathbf{E} I_S \sup_{-\tau\le r\le 0} |x(s+r)|^p)\,\mathrm{d}s.
$$

Since $F(t,u)$ is monotone nondecreasing with respect u, (36) yields that $v(t) \le w(t)$ and

$$
(37)\qquad \begin{cases} w'(t) = M_2 F(t, v(t)) \le M_2 F(t, w(t)), \\ w(t_0) = M_1. \end{cases}
$$

For every $t_0 \le t$, Lemma 5.1 shows that

$$
v(t\wedge\tau_n) \le w(t\wedge\tau_n) \le u(t\wedge\tau_n), \quad t\in[t_0,\beta_1),\ \text{for each } n \gg 1,
$$

where $u(t)$ is the maximal solution of the equation

$$
(38)\qquad \begin{cases} u'(t) = M_2 F(t, u(t)), \\ u(t_0) = M_1. \end{cases}
$$

So,

$$\mathbf{E}I_S|x(\beta_1 \wedge \tau_n)|^p \le u(\beta_1 \wedge \tau_n), \quad \text{for each } n \gg 1. \tag{39}$$

Since the solution of (31) exists in $[t_0-\tau, T)$, we have $u(\beta_1 \wedge \tau_n) < \infty$. However, the left side of (39) approaches ∞ as $n \to \infty$, which is a contradiction. Consequently, the proof is completed. □

By Theorem 5.2 and Lemma 5 in [**15**], we can get the following generalization of the Wintner global existence-uniqueness theorems.

Corollary 5.1. *Suppose that all the conditions of Theorem* 5.2 *are satisfied except that conditions* (30) *and* (31) *are replaced by the following inequality*:

$$|f(t,\phi)|^p + |g(t,\phi)|^p \le a(t) + b(t)k(\|\phi\|^p), \tag{40}$$

where $a(t)$, $b(t) \in C([t_0,T), R_+)$ *and* $k(u) \in C(R_+, R_+)$ *is monotone nondecreasing, concave and satisfies*

$$\int_0^{+\infty} \frac{\mathrm{d}u}{1+k(u)} = +\infty. \tag{41}$$

Then any solution of (3) *and* (4) *exists on* $[t_0-\tau, T)$.

Remark 5.2. Corollary 5.2 is a generalization of Theorem 3.4 in [**1**]. In fact, conditions (i) and (iii) of Theorem 3.4 in [**1**] imply that D satisfies the lower-Lipschitz condition in $[t_0,T) \times L^p(\Omega, BC)$ (see Remark 3.1), and f and g satisfy the local Lipschitz condition in $[t_0,T) \times L^p(\Omega, BC)$. Taking $a(t) = b(t) = K(T)$ and $k(u) = u$, condition (40) becomes (ii) of Theorem 3.4 in [**1**], which can guarantee that (41) holds.

The following theorems on global existence of the solution of (3), (4) can be implied by the analogous methods in [**15**].

Let $C^{1,2}(R \times R^n, R)$ denote the family of all nonnegative functions $V(t,u)$ on $R \times R^n$ which are twice continuously differentiable in u and once in t. For each $V(t,u) \in C^{1,2}(R \times R^n, R)$, we define an Itô operator $\mathcal{L}V$, associated with the NSFDE (3), from $R \times R^n$ to R by

$$\mathcal{L}V(t,D) = V_t(t,D) + V_D(t,D)f(t,x_t) + \frac{1}{2}\text{trace}\,[g^T(t,x_t)V_{DD}g(t,x_t)],$$

$$V_t(t,D) = \frac{\partial V(t,D)}{\partial t},$$
$$V_D(t,D) = \left(\frac{\partial V(t,D)}{\partial D_1}, \dots, \frac{\partial V(t,D)}{\partial D_n}\right),$$
$$V_{DD}(t,D) = \left(\frac{\partial V^2(t,D)}{\partial D_i \partial D_j}\right)_{n\times n}.$$

Theorem 5.3. *Let the conditions of Theorem* 4.2 *hold. Suppose that there are functions* $V \in C^{1,2}([t_0-\tau,T)\times R^n, R_+^m)$ *and* $F \in C([t_0,T)\times R_+^m \times R_+^m, R_+^m)$ *such that*

(42) $$\max_{1\le i\le m}\left\{\lim_{|D|\to\infty}\left[\inf_{t_0-\tau\le t<T} V_i(t,D)\right]\right\} = \infty,$$

(43) $$\mathbf{E}\mathcal{L}V(t,D) \le F(t,\mathbf{E}V(t,D),\overline{\mathbf{E}V}(t,D)), \quad \text{for all } t\in[t_0,T),\ D\in R^n,$$

where $\mathcal{L}V = (\mathcal{L}V_1,\dots,\mathcal{L}V_m)^T$, $\mathbf{E}V = (\mathbf{E}V_1,\dots,\mathbf{E}V_m)^T$ *and* $R_+ = [0,\infty)$.

Assume, moreover, that F *is an* H_m*-function and, for an arbitrary given initial condition, the solution* $u(t)$ *of the delay differential equation*

(44) $$\dot u(t) = F(t,u(t),\overline{u}(t))$$

exists on $[t_0-\tau,T)$. *Then any solution of* (3) *and* (4) *also exists on* $[t_0-\tau,T)$.

Proof. The proof is similar to that of Theorem 3 in [**15**] except for establishing the relationship between x and D by the lower Lipschitz condition

(45) $$|||D_t - D_{t_0}||| \ge K_0\|x_t - x_{t_0}\|, \quad \text{for all } t\in[t_0,T),$$

where $D_t = D(t+s) = D(t+s,x_{t+s})$. From (45), we can get

(46) $$(\mathbf{E}|||D_t - D_{t_0}|||^p)^{1/p} \ge K_0\|x_t - x_{t_0}\|_\Omega \ge K_0\|x_t\|_\Omega - K_0\|x_{t_0}\|_\Omega, \quad \text{for all } t\in[t_0,T).$$

This implies that

(47)

$$K_0\|x_t\|_\Omega \le (\mathbf{E}|\|D_t\||^p)^{1/p} + (\mathbf{E}|\|D_{t_0}\||^p)^{1/p} + K_0\|x_{t_0}\|_\Omega$$
$$\le (\mathbf{E}|\|D_t\||^p)^{1/p} + |D(t_0,\xi)|_\Omega + K_0\|x_{t_0}\|_\Omega, \text{ for all } t\in[t_0,T).$$

If the conclusion is not true, that is, $x(t)$ explodes in the interval $[t_0-\tau,T)$, then D also explodes in $[t_0-\tau,T)$ by $|x(t)|_\Omega \le \|x_t\|_\Omega$ and (47), which contradicts the solution $u(t)$ of (44) existing on $[t_0-\tau,T)$. □

Theorem 5.4. *Let the conditions of Theorem* 4.2 *hold. Suppose that there are functions* $V \in C^{1,2}([t_0-\tau,T)\times R^n,R_+)$ *and* $F \in C([t_0,T)\times R_+,R_+)$ *such that*

$$\lim_{|D|\to\infty}[\inf\nolimits_{t_0-\tau\le t<T} V(t,D)] = \infty, \tag{48}$$
$$\mathbf{E}\mathcal{L}V(t,D) \le F(t,\overline{\mathbf{E}V}(t,D)), \quad \textit{for all } t\in[t_0,T),\ D\in R^n.$$

Assume, moreover, that for an arbitrary given initial condition, the maximal solution $u(t)$ *of the differential equation*

$$\dot u(t) = F(t,u(t)) \tag{49}$$

exists on $[t_0,T)$. *Then any solution of* (3) *and* (4) *also exists on* $[t_0-\tau,T)$.

6. Examples. To illustrate the efficiency of the results we obtained above, in this section we will give two examples to which the existing results cannot be applied.

Example 6.1. Consider the following NSFDE

(50)

$$d[x(t)+\alpha x(t-\tau)] = -x^2[x(t)+\alpha x(t-\tau)]\,\mathrm{d}t + a[x(t)+\alpha x(t-\tau)]\,\mathrm{d}\omega(t),$$

where a is a constant and $\alpha\in(0,1)$.

Clearly, here f does not obey the linear growth condition. But f and g satisfy the local Lipschitz condition in $[t_0,T)\times L^p(\Omega,BC)$ and D satisfies the lower Lipschitz condition in $[t_0,T)\times L^p(\Omega,BC)$. Choose

$V(t,u) = u^2$; then we can compute the operator $\mathcal{L}V$ associated with the NSFDE (50) as follows

$$\mathcal{L}V = -2x^2[x(t)+\alpha x(t-\tau)]^2 + a^2[x(t)+\alpha x(t-\tau)]^2 \le a^2V. \tag{51}$$

Since the solution of the linear equation $\dot{u}(t) = a^2u(t)$ is a global unique existence, from Theorem 5.4 we can conclude the solution of equation (50) is a global unique existence for any initial data $(t_0,\xi) \in R \times L^p(\Omega, BC)$.

Example 6.2. Consider the following NSFDE

$$\begin{aligned} d\Big[x(t) - 0.5\int_{-\infty}^{t} e^{\alpha(s-t)}x(s)\,\mathrm{d}s\Big] &= \Big[-x^3(t) - x(t) + 0.5\int_{-\infty}^{t} e^{\alpha(s-t)}x(s)\,\mathrm{d}s\Big]\,dt \\ &\quad + \Big[0.5\int_{-\infty}^{t} e^{\alpha(s-t)}x^2(s)\,\mathrm{d}s\Big]\,\mathrm{d}\omega(t). \end{aligned} \tag{52}$$

Clearly, f and g do not obey the linear growth condition. But f, g and D satisfy all the assumptions in Theorem 5.1. Let us set $V(t,D) = V_1 + V_2 + V_4$, where

$$V_1 = D^2(t,x_t) = \Big[x(t) - 0.5\int_{-\infty}^{t} e^{\alpha(s-t)}x(s)\,\mathrm{d}s\Big]^2,$$

$$V_p = \int_{-\infty}^{t}\int_{-\infty}^{s-t} e^{\alpha u}|x(s)|^p\mathrm{d}u\,\mathrm{d}s, \quad p = 2, 4.$$

Computing the Itô operator $\mathcal{L}V$ associated with the NSFDE (52), we have

(53)

$$\begin{aligned} \mathcal{L}V_1 = 2\Big[&x - 0.5\int_{-\infty}^{t} e^{\alpha(s-t)}x(s)\,\mathrm{d}s\Big] \\ &\times\Big[-x^3 - x + 0.5\int_{-\infty}^{t} e^{\alpha(s-t)}x(s)\,\mathrm{d}s\Big] \\ &+ 0.25\Big[\int_{-\infty}^{t} e^{\alpha(s-t)}x^2(s)\,\mathrm{d}s\Big]^2 \end{aligned}$$

$$
\begin{aligned}
&= -2x^4 - 2x^2 + x^3 \int_{-\infty}^{t} e^{\alpha(s-t)} x(s)\,\mathrm{d}s \\
&\quad + 2x \int_{-\infty}^{t} e^{\alpha(s-t)} x(s)\,\mathrm{d}s - 0.5\left[\int_{-\infty}^{t} e^{\alpha(s-t)} x(s)\,\mathrm{d}s\right]^2 \\
&\quad + 0.25\left[\int_{-\infty}^{t} e^{\alpha(s-t)} x^2(s)\,\mathrm{d}s\right]^2 \\
&\le -1.25x^4 - x^2 \\
&\quad + 0.25\left[\int_{-\infty}^{t} e^{\alpha(s-t)} x(s)\,\mathrm{d}s\right]^4 + 0.5\left[\int_{-\infty}^{t} e^{\alpha(s-t)} x(s)\,\mathrm{d}s\right]^2 \\
&\quad + 0.25\left[\int_{-\infty}^{t} e^{\alpha(s-t)} x^2(s)\,\mathrm{d}s\right]^2.
\end{aligned}
$$

By Hölder's inequality, we have

$$
\begin{aligned}
\left[\int_{-\infty}^{t} e^{\alpha(s-t)} x(s)\,\mathrm{d}s\right]^2 &= \left[\int_{-\infty}^{t} e^{\alpha/2(s-t)} e^{\alpha/2(s-t)} x(s)\,\mathrm{d}s\right]^2 \\
&\le \int_{-\infty}^{t} e^{\alpha(s-t)}\mathrm{d}s \int_{-\infty}^{t} e^{\alpha(s-t)} x^2(s)\,\mathrm{d}s \\
&= \frac{1}{\alpha}\int_{-\infty}^{t} e^{\alpha(s-t)} x^2(s)\,\mathrm{d}s.
\end{aligned} \tag{54}
$$

In the same way, we also get

$$
\left[\int_{-\infty}^{t} e^{\alpha(s-t)} x(s)\,\mathrm{d}s\right]^4 \le \frac{1}{\alpha^3}\int_{-\infty}^{t} e^{\alpha(s-t)} x^4(s)\,\mathrm{d}s, \tag{55}
$$

$$
\left[\int_{-\infty}^{t} e^{\alpha(s-t)} x^2(s)\,\mathrm{d}s\right]^2 \le \frac{1}{\alpha}\int_{-\infty}^{t} e^{\alpha(s-t)} x^4(s)\,\mathrm{d}s. \tag{56}
$$

By the definition of V_p, we can get

$$
\begin{aligned}
V_p' &= \int_{-\infty}^{0} e^{\alpha s}\mathrm{d}s |x(t)|^p - \int_{-\infty}^{t} e^{\alpha(s-t)}|x(s)|^p \mathrm{d}s \\
&= \frac{1}{\alpha}|x(t)|^p - \int_{-\infty}^{t} e^{\alpha(s-t)}|x(s)|^p \mathrm{d}s.
\end{aligned} \tag{57}
$$

From (53)–(57), we can obtain

$$
\begin{aligned}
\mathcal{L}V &= \mathcal{L}V_1 + V_2' + V_4' \\
&\le -\left(1.25 - \frac{1}{\alpha}\right)x^4 - \left(1 - \frac{1}{\alpha}\right)x^2 \\
&\quad - \left(1 - \frac{0.25}{\alpha^3} - \frac{0.25}{\alpha}\right)\int_{-\infty}^{t} e^{\alpha(s-t)}x^4(s)\,\mathrm{d}s \\
&\quad - \left(1 - \frac{0.5}{\alpha}\right)\int_{-\infty}^{t} e^{\alpha(s-t)}x^2(s)\,\mathrm{d}s \\
&\le 0, \quad \text{for } \alpha > 1.
\end{aligned}
\tag{58}
$$

By the Itô formula, we have

$$
\begin{aligned}
V(t, D(t, x_t)) &= V(t_0, D(t_0, x_{t_0})) + \int_{t_0}^{t} \mathcal{L}V(s, D(s, x_s))\,\mathrm{d}s \\
&\quad + \int_{t_0}^{t} V_D(s, D(s, x_s)) \cdot g(s, x_s)\,\mathrm{d}\omega(s), \quad t \in [t_0, \beta).
\end{aligned}
\tag{59}
$$

From the definition of V, we observe that, for any $t \ge t_0$,

$$
\mathbf{E}|D(t, x_t)|^2 \le \mathbf{E}V(t, D(t, x_t)) \le \mathbf{E}V(t_0, D(t_0, x_{t_0})) \triangleq H. \tag{60}
$$

Clearly, H is a positive constant independent of t. Since D satisfies the lower Lipschitz condition, from (45)–(47), we can get

$$
K_0|x(t)|_\Omega \le K_0\|x_t\|_\Omega \le H^{1/2} + |D(t_0, x_{t_0})|_\Omega + K_0\|x_{t_0}\|_\Omega. \tag{61}
$$

Therefore, we get the boundedness of the solution $x(t)$. From Theorem 5.1, the solution of equation (52) is a global unique existence for any initial data $(t_0, \xi) \in R \times L^p(\Omega, BC)$.

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