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A COMBINATORIAL TRACE METHOD: COUNTING CLOSED WALKS TO ASSAY GRAPH EIGENVALUES

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ABSTRACT. The eigenvalues of the adjacency operator of a finite regular graph yield a great deal of information about the graph, including bounds on its chromatic number, diameter, girth, isoperimetric constant, etc. The second largest eigenvalue is frequently of particular interest. The sum of the kth powers of these eigenvalues equals the number of closed walks of length k in the graph, so counting the latter affords us a combinatorial technique for obtaining information about the former. In this expository paper, we first illustrate this technique with the toy example of cycle graphs. We then briefly discuss how this method has been used to prove two theorems which provide lower bounds on the second largest eigenvalue of a graph, namely the Alon-Boppana theorem (for arbitrary regular graphs) as well as a theorem of Cioabă (for Cayley graphs of abelian groups).

1. Introduction. While the eigenvalues of a finite graph do not completely determine the graph, as shown by the existence of nonisomorphic isospectral graphs, they do convey a tremendous amount of information about the graph. Virtually every graph invariant is in some way intimately related to the graph's spectrum, i.e., its multiset of eigenvalues, counted with multiplicity. Examples include the chromatic number, girth, diameter, isoperimetric constant, etc., see [**3, 10, 11**] for comprehensive surveys. For a regular graph, the second largest eigenvalue λ_1 is frequently of primary interest. The following theorem of Chung [**4**] is typical: If X is a d-regular graph with n vertices, then $\operatorname{diam}(X) \leq \lceil \log(n-1)/\log(d/\lambda) \rceil$, where λ is the eigenvalue of X with second-largest absolute value. See [**5**] for a more comprehensive survey of results along these lines.

Computing, or even finding, good bounds for the eigenvalues of a family of graphs can be quite difficult. The purpose of this expository

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paper is to discuss one technique for estimating a graph's eigenvalues. We call it the "combinatorial trace method." It works as follows.

Let X be a finite graph with vertex set $\{v_1, \ldots, v_n\}$. Let A be the adjacency matrix of X, i.e., the matrix whose (i, j)-entry is 1 if v_i is adjacent to v_j and 0 otherwise. We call the *eigenvalues of* X the eigenvalues of A; we denote them $\lambda_0, \ldots, \lambda_{n-1}$. Observe that the (i, j)-entry of A^k equals the number of walks in X of length k from v_i to v_j. We may therefore compute the trace of A^k in two different ways. On the one hand, it equals the sum of the kth powers of the eigenvalues of A. On the other hand, it equals the total number of closed walks in X of length k. Letting $e(i, k)$ denote the number of closed walks of length k in X that begin and end at v_i, we can summarize this result in the following equality:

$$\sum_{i=0}^{n-1} \lambda_i^k = \mathrm{Trace}(A^k) = \sum_{i=1}^{n} e(i, k). \tag{1}$$

The point is that the right-hand side of (1.1) is an entirely *combinatorial* expression. So we can obtain information about the eigenvalues of a graph merely by counting. This relatively trivial observation has far-reaching consequences. In Section 2, as an illustrative example, we extract information about the eigenvalues of cycle graphs via the combinatorial trace method. In Section 3, we use this technique to prove the Alon-Boppana theorem, which provides an asymptotic lower bound for the second-largest eigenvalue of a graph. In Section 4, we sketch the proof of a theorem, due to Cioabă, on the eigenvalues of a Cayley graph of an abelian group.

2. Cycle graphs. Recall that a *Cayley graph* is constructed as follows. Let G be a group. Let Γ be a *symmetric* subset of G, i.e., a subset of G such that, if $\gamma \in \Gamma$, then $\gamma^{-1} \in \Gamma$. Moreover, assume that Γ does not contain the identity element. Define $\mathrm{Cay}\,(G, \Gamma)$ to be the graph whose vertices are the elements of G, where two vertices x, y are adjacent if and only if $x = y\gamma$ for some $\gamma \in \Gamma$. Note that the symmetric condition on Γ guarantees that the adjacency relation is symmetric. We call $\mathrm{Cay}\,(G, \Gamma)$ the Cayley graph on G with respect to Γ.

We will use the combinatorial trace method and the following lemma to produce an expression for λ, the eigenvalue with second largest absolute value, when X is the 2-regular cycle graph $C_n = \mathrm{Cay}\,(\mathbf{Z}_n; \{1, -1\})$.

Lemma 2.1. *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be positive reals with a_1 appearing m times. Then*

$$a_1 = \lim_{k\to\infty} (a_1^k + \cdots + a_n^k)^{1/k}.$$

Proof. If $a_1 = a_2 = \cdots = a_n$, the result follows easily. Otherwise, define $L = \lim_{k\to\infty}(a_1^k + \cdots + a_n^k)^{1/k}$. Then

$$\begin{aligned}\log L &= \lim_{k\to\infty} \frac{\log(a_1^k + \cdots + a_n^k)}{k} \\ &= \lim_{k\to\infty} \frac{a_1^k(\log a_1) + \cdots + a_n^k(\log a_n)}{a_1^k + \cdots + a_n^k}.\end{aligned}$$

Divide by a_1^k and note that, for $a_i < a_1$, we have $a_i/a_1 < 1$ which implies that $(a_i/a_1)^k \to 0$.

$$\begin{aligned}\log L &= \lim_{k\to\infty} \frac{m\log(a_1) + \sum_{i>m}(a_i/a_1)^k \log(a_i)}{m + \sum_{i>m}(a_i/a_1)^k} \\ &= \frac{m \log a_1}{m}.\end{aligned}$$

It follows that $L = a_1$. □

This lemma can be used to select λ, the second largest eigenvalue in absolute value, from trace (A^k) by subtracting out the trivial eigenvalue d (the degree of the graph) and, if the graph is bipartite, the other trivial eigenvalue $-d$:

$$\begin{aligned}\lambda &= \lim_{k\to\infty} \left(\sum_{\text{nontrivial}} \lambda^{2k} \right)^{1/2k} \\ &= \lim_{k\to\infty} \left(\text{Trace}\,(A^{2k}) - \sum_{\text{trivial}} \lambda^{2k} \right)^{1/2k}.\end{aligned}$$

The even exponent $2k$ ensures that the eigenvalue powers are positive so that we may apply the lemma. Let $p(l)$ denote the total number of closed walks of length l in C_n. Then, letting $l = 2k$,

$$p(2k) = \begin{cases} d^{2k} + \lambda_1^{2k} + \cdots + \lambda_{n-1}^{2k} & \text{if } n \text{ is odd,} \\ d^{2k} + \lambda_1^{2k} + \cdots + \lambda_{n-2}^{2k} + (-d)^{2k} & \text{if } n \text{ is even.} \end{cases} \tag{2}$$

We've already remarked that $p(2k)$ is the trace of A^{2k}. The lemma immediately gives us the following:

Corollary 2.2. *Let C_n and λ be as above. Then*

$$\lambda = \begin{cases} \lim_{k\to\infty}[p(2k) - 2^{2k}]^{1/2k} & \text{if } n \text{ is odd,} \\ \lim_{k\to\infty}[p(2k) - 2^{2k+1}]^{1/2k} & \text{if } n \text{ is even.} \end{cases} \tag{3}$$

We now find an expression for $p(k)$ for $C_n = \mathrm{Cay}\,(\mathbf{Z}_n;\pm 1)$. Let $e(k)$ be the number of closed walks of length k based at vertex 0 in $\mathrm{Cay}\,(\mathbf{Z}_n;\pm 1)$ corresponding to the identity element in $\mathbf{Z}_n$. The symmetry of C_n gives $p(k) = ne(k)$, so it is enough to find $e(k)$. There is a bijective correspondence between walks of length k and k-tuples of 1's and -1's, corresponding to clockwise and counterclockwise steps in C_n, with the following condition: The k-tuple consists of i 1's and $(k-i)$ (-1)'s with the property that $i-(k-i) \equiv 0 \pmod{n}$ where $i-(k-i)$ is the winding number of the closed walk. We can choose the location of the i 1's in $\binom{k}{i}$ ways, and, summing over all i, we obtain $e(k) = \sum_{0\le i\le k, n|2i-k} \binom{k}{i}$. The combinatorial trace formula (1.1) says that this sum is equal to the sum of the k-th powers of the eigenvalues of C_n.

The following proposition establishes this equality algebraically without mentioning graphs.

Proposition 2.3. *Let $n \ge 3$, $k \ge 1$. Then*

$$p(k) = n \sum_{\substack{0\le i\le k \\ n|2i-k}} \binom{k}{i} = \sum_{j=0}^{n-1} \left[2\cos\frac{2\pi j}{n}\right]^k. \tag{4}$$

Proof. Write $2\cos(2\pi j/n) = \xi^j + \xi^{-j}$ where ξ is the primitive n-th root of unity $e^{2\pi i/n}$. Then

$$\sum_{j=0}^{n-1}(\xi^j+\xi^{-j})^k = \sum_{j=0}^{n-1}\left(\sum_{i=0}^{k}\binom{k}{i}\xi^{ij}\xi^{-(k-i)j}\right)$$
$$= \sum_{j=0}^{n-1}\left(\sum_{i=0}^{k}\binom{k}{i}\xi^{(2i-k)j}\right)$$
$$= \sum_{i=0}^{k}\binom{k}{i}\sum_{j=0}^{n-1}\left[\xi^{2i-k}\right]^j.$$

Now, when $n \mid 2i-k$, then $\xi^{2i-k}=1$ and $\sum_{j=0}^{n-1}\xi^{(2i-k)j}=n$. When $n \nmid 2i-k$, then $\xi^{2i-k}\neq 1$; therefore, the same sum equals 0 and we are left with

$$p(k) = n\sum_{\substack{0\le i\le n\\ n\mid 2i-k}}\binom{k}{i} = \sum_{j=0}^{n-1}\left[2\cos\frac{2\pi j}{n}\right]^k. \qquad \square$$

(We remark that Benjamin, Chen and Tucker [**2**] also obtained this result using a different graph-theoretic method.) The adjacency matrix of the cycle graph C_n is a circulant matrix: the rows are just the cyclic permutations of the first row. For an appropriate choice of ordering of vertices, the adjacency matrix A of C_n is

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

and the eigenvalues are $\lambda_a = 2\cos(2\pi a/n)$ for $0 \le a \le n-1$. The second largest eigenvalue $\lambda(C_n)$ (in absolute value) is $2\cos(\pi/n)$ for n odd (C_n nonbipartite) and is $2\cos(2\pi/n)$ for n even (C_n bipartite). Furthermore, we can choose $k = Mn$ to be a multiple of n and produce an equation with a sum of evenly spaced binomial coefficients on the left-hand side and a sum of powers of cosine on the right-hand side.

Summarizing:

$$\lambda(C_n) = \begin{cases} \lim_{k\to\infty} \left[ne(2k) - 2^{2k}\right]^{1/2k} = 2\cos(\pi/n) & \text{if } n \text{ is odd} \\ \lim_{k\to\infty} \left[ne(2k) - 2^{2k+1}\right]^{1/2k} = 2\cos(2\pi/n) & \text{if } n \text{ is even.} \end{cases}$$

3. The Alon-Boppana theorem. The Alon-Boppana theorem gives an essential lower bound on the size of $\lambda(X)$ by counting special types of walks in the universal cover of X that project down to closed walks in X.

Theorem 3.1 (Alon-Boppana). *If X_n is a sequence of d-regular graphs with $|X_n| \to \infty$, then $\liminf_{n\to\infty} \lambda(X_n) \geq 2\sqrt{d-1}$.*

The idea is to give a lower bound on Trace (A^k) by counting the number of walks in the universal covering graph of X that begin at a certain vertex and that never visit it until the very end. The universal covering graph T_{v_0} of X with basepoint v_0 is an infinite d-regular tree whose vertices are non-backtracking walks $(v_0 v_1 \cdots v_n)$ in X based at v_0. An edge connects two vertices of T_{v_0} if one vertex, when considered to be a walk in X, extends the walk of the other.

More specifically, let $\rho_v(k)$ be the number of walks of length k from (v) to (v) in the universal covering graph. Projecting each such walk will produce a walk of the same length in X that is not a cycle. Since some walks in X can be cycles, not all walks of length k in X have been accounted for so we only obtain a lower bound: $\sum_v \rho_v(k) \leq \sum A_{ii}^k = \sum \lambda_i^k$.

Let $\rho'_v(k)$ be the number of walks of length k in T_{v_0} beginning at (v) and ending at (v) for the first time (the walk does not revisit (v) until the end). Then

$$n\rho'_{v_0}(2k) = \sum_v \rho'_v(2k) \leq \sum_v \rho_v(2k) \leq \sum_i A_{ii}^k$$

where $n = |X|$. Finding $\rho'_v(2k)$ yields a lower bound estimate on the $\operatorname{tr} A^{2k}$, and the following lemma does this.

Lemma 3.2. *The number of walks of length $2k$ in T_{v_0} that start at v_0 and end at v_0 for the first time is*

$$\rho'_{v_0}(2k) = \frac{1}{k}\binom{2k-2}{k-1} d(d-1)^{k-1}.$$

Proof. We will provide a sketch of the proof of the lemma and refer the interested reader to [**12**]. The underlying idea is that a walk on the d-regular universal covering graph of the above type is essentially composed of k steps outward from (v_0) and k steps inward. A walk can be identified with a $2k$-tuple of 1's and -1's corresponding to outward and inward motions of the walk. Each step outward (after the first) can be picked in $d-1$ ways, but each step inward is completely determined since T_{v_0} is a tree. This accounts for the $d(d-1)^{k-1}$ term. The remaining term is obtained by counting all possible $2k$-tuples $(a_1, a_2, \ldots, a_{2k})$ of 1's and -1's such that $a_1 = 1$ and $a_{2k} = -1$. To ensure the walk returns to (v_0) only at the very end, we require the sum of first l terms always be ≥ 1 for $0 \leq l \leq 2k$. This is equivalent to requiring $a_2 + \cdots + a_{l-1} \geq 0$. The device to compute this sum is the $2(k-1)$-st Catalan number $1/k\binom{2(k-1)}{k-1}$ which counts precisely the number of strings of $(k-1)$ 1's and $k-1$ (-1)'s such that any sum of the first l terms is ≥ 0 for $1 \leq l \leq 2k-2$. $\square$

This result furnishes us with a nice example of the combinatorial trace method to estimate $\lambda(X)$: counting walks in X directly is hard, but counting simpler walks in the universal cover is easier. The definition of $\lambda(X)$ implies in the case that X is not bipartite that

$$(n-1)\lambda(X)^{2k} \geq \sum_{i=1}^{n-1} \lambda_i(X)^{2k} \geq n\rho'_e(2k) - d^{2k}.$$

A similar result holds for X bipartite. The theorem is hereafter proved by straightforward analytic estimates of the right hand side.

4. Counting closed walks in Cayley graphs on abelian groups. In [**6**], Cioabă uses the combinatorial trace method to obtain the following theorem.

Theorem 4.1. *For each $\varepsilon > 0$ and positive integer $d \geq 2$, there exists a positive constant C such that if G is any finite abelian group and Γ is any symmetric subset of G not containing the identity element such that $|\Gamma| = d$, then the number of eigenvalues of* $\mathrm{Cay}\,(G, \Gamma)$ *greater than or equal to $d - \varepsilon$ is at least $C \cdot |G|$.*

Cioabă used the combinatorial trace method to give a proof of a theorem of Greenberg [**7**] and an elementary proof of the following theorem of Serre using the combinatorial trace method [**8**]:

Theorem 4.2. *For each $\varepsilon > 0$ and $d \geq 3$, there exists $c = c(\varepsilon, d) > 0$ such that every d-regular graph, X, the number of eigenvalues μ of X with $\mu \geq (2 - \varepsilon)\sqrt{d-1}$ is at least $c|X|$.*

It is not unduly difficult to see that Theorem 4.1 has the following consequence, a strengthening of the Alon-Boppana theorem in the special case of Cayley graphs on abelian groups.

Corollary 4.3. *Let $d \geq 2$ be a fixed integer. Let (X_n) be a sequence of finite d-regular graphs, where each X_n is a Cayley graph on an abelian group, and $|X_n| \to \infty$. Then $\lambda_1(X_n) \to d$.*

We remark that Corollary 4.3 is equivalent to the following statement: No sequence of finite abelian groups yields an expander family [**11**]. One can also prove Corollary 4.3 without using the combinatorial trace method, for example using diameters as in [**1**], or using Kazhdan constants as in [**13**]. In [**9**], Friedman, Murty and Tillich provide a more precise estimate for the slowest possible growth rate of the second largest eigenvalue of a sequence of Cayley graphs on abelian groups. We now briefly sketch the proof of Theorem 4.1. Let $X = \mathrm{Cay}\,(G, \Gamma)$, where G is a finite abelian group. Let $g \in G$. Let $e_g(2r)$ be the number of paths of length $2r$ that begin and end at g. Cioabă shows that

$$(5) \quad e_g(2r) \geq \sum_{p=0}^{r} \binom{2r}{2p}\binom{2p}{p} \sum_{i_1+\cdots+i_s=p} \binom{p}{i_1, \ldots, i_s}^2 \sum_{2j_1+\cdots+2j_t=2r-2p} \binom{2r-2p}{2j_1, \ldots 2j_t},$$

where Γ contains t elements of order 2 and $2s$ elements of order > 2. The inequality in (4.1) is obtained by first observing that $e_g(2r)$ equals the number of ways to express the identity element as a product of $2r$

elements of Γ. We then get the right-hand side of (4.1) by counting the number of such products where each element in Γ not of order 2 appears the same number of times as its inverse, and each element of order 2 appears an even number of times. The binomial coefficient $\binom{2r}{2p}$, for example, counts the number of ways in which we can assign $2p$ locations for the elements not of order 2 to go. Next, Cioabă applies some combinatorial estimates to (4.1) to show that

$$e_g(2r) > \frac{d^2 r}{2^d(2r+1)\binom{d+r-1}{d-1}}, \tag{6}$$

where $d = 2s + t = |\Gamma|$. Now the combinatorial trace method kicks in. As in our proof of the Alon-Boppana theorem in Section 3, a lower bound on the number of closed paths of fixed length yields a lower bound on the graph's largest eigenvalues. See [**6**] for details.

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