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TRANSFORMATION FORMULAS FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION WITH INTEGRAL PARAMETER DIFFERENCES

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ABSTRACT. Transformation formulas of Euler and Kummer-type are derived respectively for the generalized hypergeometric functions ${}_{r+2}F_{r+1}(x)$ and ${}_{r+1}F_{r+1}(x)$, where r pairs of numeratorial and denominatorial parameters differ by positive integers. Certain quadratic transformations for the former function, as well as a summation theorem when $x = 1$, are also considered.

1. Introduction. The generalized hypergeometric function ${}_pF_q(x)$ may be defined for complex parameters and argument by the series

$$(1.1)\qquad {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ \\ b_1, b_2, \ldots, b_q \end{matrix} \,\middle|\, x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}.$$

When $q \geq p$, this series converges for $|x| < \infty$, but when $q = p-1$, convergence occurs when $|x| < 1$. However, when only one of the numeratorial parameters a_j is a negative integer or zero, then the series always converges since it is simply a polynomial in x of degree $-a_j$. In (1.1) the Pochhammer symbol or ascending factorial $(a)_k$ is defined by $(a)_0 = 1$, and for $k \geq 1$ by $(a)_k = a(a+1)\cdots(a+k-1)$. However, for all integers k we simply write

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)},$$

where Γ is the gamma function. We shall adopt the convention of writing the finite sequence (except where otherwise noted) of parameters

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$(a_1, \dots, a_p)$ simply by (a_p) and the product of p Pochhammer symbols by

$$((a_p))_k \equiv (a_1)_k \dots (a_p)_k,$$

where an empty product $p = 0$ reduces to unity.

Let (m_r) be a nonempty sequence of positive integers. In this paper we shall derive transformation formulas for the generalized hypergeometric functions ${}_{r+2}F_{r+1}(x)$ and ${}_{r+1}F_{r+1}(x)$ whose r numeratorial and denominatorial parameters differ by positive integers (m_r). Thus, we shall show in Sections 3, 4 and 5, respectively, that

$$(1.2)\quad {}_{r+1}F_{r+1}\left(\begin{matrix} b, & (f_r + m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, x\right) = e^x\, {}_{m+1}F_{m+1}\left(\begin{matrix} \lambda, & (\xi_m + 1) \\ c, & (\xi_m) \end{matrix} \,\middle|\, -x\right),$$

where $|x| < \infty$,

$$(1.3)\quad {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r + m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, x\right) = (1-x)^{-a}\, {}_{m+2}F_{m+1}\left(\begin{matrix} a, \lambda, & (\xi_m + 1) \\ c, & (\xi_m) \end{matrix} \,\middle|\, \frac{x}{x-1}\right),$$

where $|x| < 1$, $\operatorname{Re} x < 1/2$, and

$$(1.4)\quad {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r + m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, x\right) = (1-x)^{c-a-b-m}\, {}_{m+2}F_{m+1}\left(\begin{matrix} \lambda, \lambda', & (\eta_m + 1) \\ c, & (\eta_m) \end{matrix} \,\middle|\, x\right),$$

where $|x| < 1$. In these transformation formulas the quantities m, λ and λ' are defined by

$$(1.5)\quad m \equiv m_1 + m_2 + \dots + m_r, \qquad \lambda \equiv c - b - m, \qquad \lambda' = c - a - m,$$

where, when (m_r) is empty, we define $m = 0$. Following [**8**], the (ξ_m) and (η_m) are nonvanishing zeros of certain associated parametric polynomials of degree m, which we denote generically by $Q_m(t)$, provided that certain restrictions on some of the parameters of the generalized hypergeometric functions on both sides of (1.2)–(1.4) are satisfied. The polynomial $Q_m(t)$ for transformations (1.2) and (1.3) is given by (2.4). The associated parametric polynomial for transformation (1.4) is given by (5.10). Certain generalized quadratic transformations for ${}_{r+2}F_{r+1}(x)$ are also provided in Section 6 and a summation theorem when $x = 1$ is rederived in Section 7.

When (m_r) is empty, (1.2) reduces to Kummer's transformation formula for the confluent hypergeometric function, namely

$$ {}_1F_1\left(\begin{matrix} b \\ c \end{matrix}\,\middle|\, x\right) = e^x\, {}_1F_1\left(\begin{matrix} c-b \\ c \end{matrix}\,\middle|\, -x\right), \tag{1.6}$$

where $|x| < \infty$. Similarly, (1.3) and (1.4) reduce respectively to Euler's classical first and second transformations for the Gauss hypergeometric function, namely,

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, x\right) = (1-x)^{-a}\, {}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix}\,\middle|\, \tfrac{x}{x-1}\right) \tag{1.7}$$

$$ = (1-x)^{c-a-b}\, {}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}\,\middle|\, x\right), \tag{1.8}$$

where $|x| < 1$, $\operatorname{Re} x < 1/2$ in (1.7) and $|x| < 1$ in (1.8).

In [**8**] Miller obtained the specialization $m_1 = \cdots = m_r = 1$ of transformation (1.2) by employing a summation formula for a ${}_{r+2}F_{r+1}(1)$ hypergeometric series combined with a reduction identity for a certain Kampé de Fériet function. In [**11**], an alternative, more direct derivation of this specialization was given by employing Kummer's transformation (1.6) and a generating relation for Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ defined implicitly by (2.2). The specialization alluded to in [**8, 11**] is given by

$$(1.9)\quad {}_{r+1}F_{r+1}\left(\begin{matrix} b, & (f_r+1) \\ c, & (f_r) \end{matrix}\,\middle|\, x\right) = e^x\, {}_{r+1}F_{r+1}\left(\begin{matrix} c-b-r, & (\xi_r+1) \\ c, & (\xi_r) \end{matrix}\,\middle|\, -x\right).$$

The (ξ_r) are nonvanishing zeros (provided $b \neq f_j$ $(1 \leq j \leq r)$ and $(c-b-r)_r \neq 0$) of the associated parametric polynomial $Q_r(t)$ of degree r given by

$$(1.10)\qquad Q_r(t) = \sum_{j=0}^{r} s_{r-j} \sum_{k=0}^{j} \left\{\begin{matrix} j \\ k \end{matrix}\right\} (b)_k (t)_k (c-b-r-t)_{r-k},$$

where the s_{r-j} $(0 \leq j \leq r)$ are determined by the generating relation

$$(1.11)\qquad (f_1+x)\cdots(f_r+x) = \sum_{j=0}^{r} s_{r-j} x^j.$$

When $r=1$, we have from (1.9), (1.10) and (1.11),

$$(1.12)\quad {}_2F_2\left(\begin{matrix} b, & f+1 \\ c, & f \end{matrix}\,\middle|\, x\right) = e^x\, {}_2F_2\left(\begin{matrix} c-b-1, & \xi+1 \\ c, & \xi \end{matrix}\,\middle|\, -x\right),$$

where the nonvanishing zero ξ (provided $b \neq f$, $c-b-1 \neq 0$) of

$$Q_1(t) = (b-f)t + f(c-b-1)$$

is given by

$$(1.13)\qquad \xi = \frac{f(c-b-1)}{f-b}.$$

The Kummer-type transformation (1.12) for ${}_2F_2(x)$ was obtained by Paris [**15**] who employed other methods. Paris's result generalized a transformation for ${}_2F_2(x)$ derived by Exton [**4**] and rederived in simpler ways by Miller [**10**] for the specialization $f = b/2$. Other derivations of (1.12) have been recorded in [**3, 9, 19**].

In [**12**], the Euler-type transformations (1.3) and (1.4) specialized with $m_1 = \cdots = m_r = 1$ were obtained. These specializations are given by

$$
{}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+1) \\ c, & (f_r) \end{matrix} \,\middle|\, x\right)
$$

(1.14)

$$
= (1-x)^{-a}\,{}_{r+2}F_{r+1}\left(\begin{matrix} a, & c-b-r, & (\xi_r+1) \\ c, & (\xi_r) & \end{matrix} \,\middle|\, \frac{x}{x-1}\right)
$$

(1.15)

$$
= (1-x)^{c-a-b-r}\,{}_{r+2}F_{r+1}\left(\begin{matrix} c-a-r, & c-b-r, & (\eta_r+1) \\ c, & (\eta_r) & \end{matrix} \,\middle|\, x\right).
$$

The (ξ_r) are again the nonvanishing zeros of the polynomial $Q_r(t)$ of degree r given by (1.10), where $b \neq f_j$ $(1 \leq j \leq r)$ and $(c-b-r)_r \neq 0$. The (η_r) are nonvanishing zeros of a different polynomial, also of degree r, that may be obtained from Theorem 4 specialized with $m_1 = \cdots = m_r = 1$ so that $m = r$. When $r = 1$, the transformation (1.14) reduces to the result due to Rathie and Paris [**19**]

$$
(1.16)\quad {}_3F_2\left(\begin{matrix} a, b, & f+1 \\ c, & f \end{matrix} \,\middle|\, x\right) = (1-x)^{-a}\,{}_3F_2\left(\begin{matrix} a, c-b-1, & \xi+1 \\ c, & \xi \end{matrix} \,\middle|\, \frac{x}{x-1}\right),
$$

where ξ is given by (1.13). The transformation (1.16) was subsequently obtained by Maier [**7**] who employed other methods. Maier [**7**] also obtained the specialization $r = 1$ of (1.15), namely,

$$ {}_3F_2\left(\begin{matrix} a, b, & f+1 \\ c, & f \end{matrix} \,\middle|\, x\right) = (1-x)^{c-a-b-1}\,{}_3F_2\left(\begin{matrix} c-a-1, c-b-1, & \eta+1 \\ c, & \eta \end{matrix} \,\middle|\, x\right), $$

where

$$ \eta = \frac{f(c-a-1)(c-b-1)}{ab + f(c-a-b-1)}, $$

which was also derived in [**12**].

2. Preliminary results. In this section we record several preliminary results that we shall utilize in the sequel. Lemmas 1 and 3 and Theorem 1 below are proved in [**8**].

Lemma 1. *Consider the polynomial in n of degree $\mu \geq 1$ given by*

$$ P_\mu(n) \equiv a_0 n^\mu + a_1 n^{\mu-1} + \cdots + a_{\mu-1} n + a_\mu, $$

where $a_0 \neq 0$ and $a_\mu \neq 0$. Then we may write

$$ P_\mu(n) = a_\mu \frac{((\xi_\mu + 1))_n}{((\xi_\mu))_n}, $$

where (ξ_μ) are the nonvanishing zeros of the polynomial $Q_\mu(t)$ defined by

$$ Q_\mu(t) \equiv a_0(-t)^\mu + a_1(-t)^{\mu-1} + \cdots + a_{\mu-1}(-t) + a_\mu. $$

Lemma 2. *Consider the generalized hypergeometric function ${}_{r+1}F_{s+1}$ $((c_{r+1}); (d_{s+1}) \mid z)$ whose series representation determined by* (1.1) *converges for z in an appropriate domain. Then* [**20**, page 166]

$$ (2.1) \quad {}_{r+1}F_{s+1}\left(\begin{matrix} (c_{r+1}) \\ (d_{s+1}) \end{matrix} \,\middle|\, z\right) = e^z \sum_{n=0}^{\infty} {}_{r+2}F_{s+1}\left(\begin{matrix} -n, (c_{r+1}) \\ (d_{s+1}) \end{matrix} \,\middle|\, 1\right) \frac{(-z)^n}{n!}, $$

provided the summation converges.

The notation $\left\{ {n \atop k} \right\}$ will be employed to denote Stirling numbers of the second kind. These nonnegative integers represent the number of ways to partition n objects into k nonempty sets and arise for nonnegative integers n in the generating relation [**5**, page 262]

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (-1)^k (-x)_k, \qquad \left\{ {n \atop 0} \right\} = \delta_{0n}, \tag{2.2}$$

where δ_{0n} is the Kronecker symbol.

Lemma 3. *For nonnegative integers j, define*

$$S_j \equiv \sum_{n=0}^{\infty} n^j \, \frac{\Lambda_n}{n!}, \qquad S_0 \equiv \sum_{n=0}^{\infty} \frac{\Lambda_n}{n!},$$

where the infinite sequence (Λ_n) is such that S_j converges for all j. Then

$$S_j = \sum_{k=0}^{j} \left\{ {j \atop k} \right\} \sum_{n=0}^{\infty} \frac{\Lambda_{n+k}}{n!}.$$

We shall also utilize the following summation theorem for the generalized hypergeometric series ${}_{r+2}F_{r+1}(1)$, whose r numeratorial and denominatorial parameters differ by positive integers.

Theorem 1. *For nonnegative integer n and positive integers (m_r),*

$${}_{r+2}F_{r+1}\left(\begin{matrix} -n, b, & (f_r + m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, 1 \right) = \frac{(\lambda)_n}{(c)_n} \, \frac{((\xi_m + 1))_n}{((\xi_m))_n}, \tag{2.3}$$

where $m = m_1 + \cdots + m_r$, $\lambda = c - b - m$, $(\lambda)_m \neq 0$ and $b \neq f_j$ $(1 \leq j \leq r)$. The (ξ_m) are the nonvanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree m given by

$$Q_m(t) = \sum_{j=0}^{m} \sigma_{m-j} \sum_{k=0}^{j} \left\{ {j \atop k} \right\} (b)_k (t)_k (\lambda - t)_{m-k}, \tag{2.4}$$

where the σ_j $(0 \le j \le m)$ *are determined by the generating relation*

$$(2.5)\qquad (f_1+x)_{m_1}\cdots(f_r+x)_{m_r}=\sum_{j=0}^{m}\sigma_{m-j}x^j.$$

Note that, when $m_1 = \cdots = m_r = 1$, then $m = r$ so that by (1.11) $\sigma_j = s_j$ $(0 \le j \le r)$ and $Q_m(t)$ reduces to $Q_r(t)$, which is the polynomial of degree r given by (1.10).

The following Theorem 2 concerns a specialization of a hypergeometric function in two variables, called the *Kampé de Fériet function*; for an introduction to the latter, see [**20**, pages 63–64]. Since the proof of Theorem 2 is very similar to that given in [**8**, Theorem 2], we shall omit its proof.

Theorem 2. *Suppose* $b \ne f_j$ $(1 \le j \le r)$ *and* $(c-b-r)_r \ne 0$. *Then we have the reduction formula for the Kampé de Fériet function*

$$(2.6)\qquad F^{p:r+1;0}_{q:r+1;0}\left(\begin{matrix}(a_p): & b, & (f_r+m_r); & -\\ (b_q): & c, & (f_r); & -\end{matrix}\,\middle|\, -x, x\right) = {}_{p+m+1}F_{q+m+1}\left(\begin{matrix}c-b-m, & (a_p), & (\xi_m+1)\\ c, & (b_q), & (\xi_m)\end{matrix}\,\middle|\, x\right),$$

where $m \equiv m_1 + \cdots + m_r$ *and the solid horizontal line indicates an empty parameter sequence. The* (ξ_m) *are the nonvanishing zeros of the associated parametric polynomial* $Q_m(t)$ *of degree* m *given by (2.4).*

Finally, the following lemma expresses a ${}_{r+s}F_{r+1}(x)$ hypergeometric function, where, in the sequel, $s = 1, 2$ and r pairs of numeratorial and denominatorial parameters differ by positive integers, in terms of a finite sum of ${}_sF_1(x)$ functions. This lemma will prove fundamental to our discussion.

Lemma 4. *For a nonnegative integer* s, *let* (a_s) *denote a parameter sequence containing* s *elements, where, when* $s = 0$, *the sequence is* *empty. Let* (a_s+k) *denote the sequence when* k *is added to each element of* (a_s)*. Let* $\mathcal{F}(x)$ *denote the generalized hypergeometric function with* r *numeratorial and denominatorial parameters differing by the positive integers* (m_r)*, namely,*

$$\mathcal{F}(x) \equiv {}_{r+s}F_{r+1}\left(\begin{matrix}(a_s), & (f_r+m_r) \\ c, & (f_r)\end{matrix}\,\middle|\, x\right), \tag{2.7}$$

where, by (1.1)*, convergence of the series representation for the latter occurs in an appropriate domain depending upon the values of* s *and the elements of the parameter sequence* (a_s)*. Then*

$$\mathcal{F}(x) = \frac{1}{A_0}\sum_{k=0}^{m} x^k A_k \frac{((a_s))_k}{(c)_k}\,{}_sF_1\left(\begin{matrix}(a_s+k) \\ c+k\end{matrix}\,\middle|\, x\right), \tag{2.8}$$

where $m = m_1 + \cdots + m_r$*, the coefficients* A_k *are defined by*

$$\begin{aligned} A_k &\equiv \sum_{j=k}^{m} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \sigma_{m-j}, \\ A_0 &= (f_1)_{m_1} \cdots (f_r)_{m_r}, \\ A_m &= 1, \end{aligned} \tag{2.9}$$

and the σ_j $(0 \le j \le m)$ *are generated by relation* (2.5)*.*

Proof. Now

$$\frac{((f_r+m_r))_n}{((f_r))_n} = \frac{(f_1+n)_{m_1}}{(f_1)_{m_1}} \cdots \frac{(f_r+n)_{m_r}}{(f_r)_{m_r}},$$

where the numeratorial expression on the right-hand side is a polynomial in n of degree m which can be written in the form

$$(f_1+n)_{m_1} \cdots (f_r+n)_{m_r} = \sum_{j=0}^{m} \sigma_{m-j} n^j$$

by (2.5). By (1.1), upon expanding $\mathcal{F}(x)$ as a power series in x, we obtain

$$\mathcal{F}(x) = \sum_{n=0}^{\infty} \frac{((a_s))_n}{(c)_n} \frac{((f_r+m_r))_n}{((f_r))_n} \frac{x^n}{n!}$$
$$= \frac{1}{A_0} \sum_{j=0}^{m} \sigma_{m-j} \sum_{n=0}^{\infty} n^j \frac{((a_s))_n}{(c)_n} \frac{x^n}{n!}$$

upon interchanging the order of summation. Application of Lemma 3 to the n-summation followed by use of the identity

$$(\alpha)_{k+n} = (\alpha)_k(\alpha+k)_n = (\alpha)_n(\alpha+n)_k \tag{2.10}$$

then yields

$$\mathcal{F}(x) = \frac{1}{A_0} \sum_{j=0}^{m} \sigma_{m-j} \sum_{k=0}^{j} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \sum_{n=0}^{\infty} \frac{((a_s))_{n+k}}{(c)_{n+k}} \frac{x^{n+k}}{n!}$$
$$= \frac{1}{A_0} \sum_{k=0}^{m} x^k A_k \sum_{n=0}^{\infty} \frac{((a_s))_{n+k}}{(c)_{n+k}} \frac{x^n}{n!}$$
$$= \frac{1}{A_0} \sum_{k=0}^{m} x^k A_k \frac{((a_s))_k}{(c)_k} \sum_{n=0}^{\infty} \frac{((a_s+k))_n}{(c+k)_n} \frac{x^n}{n!},$$

where we have interchanged the order of the j and k-summations and introduced the coefficients A_k defined by (2.9). Identification of the summation over n as ${}_sF_1((a_s+k);\, c+k \mid x)$ then completes the proof. □

3. The Kummer-type transformation (1.2). If, in (2.6), we set $p = q = 0$, we immediately obtain (1.2). Also, by setting $s = r$ and $c_{r+1} = b$, $(c_r) = (f_r + m_r)$, $d_{r+1} = c$, $(d_r) = (f_r)$ in identity (2.1) and using the summation formula (2.3) of Theorem 1, we can derive (1.2). However, we provide below a more insightful derivation of the Kummer-type transformation (1.2) that utilizes Kummer's transformation (1.6) for the confluent hypergeometric function ${}_1F_1(x)$, together with Lemmas 1 and 4.

For positive integers (m_r), define

$$F(x) \equiv {}_{r+1}F_{r+1}\left(\begin{matrix} b, & (f_r+m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, x \right).$$

Then, from (2.8) with $s = 1$ and $a_1 = b$, we have

$$F(x) = \frac{1}{A_0} \sum_{k=0}^{m} x^k A_k \, \frac{(b)_k}{(c)_k} \; {}_1F_1 \left(\begin{matrix} b+k \\ \\ c+k \end{matrix} \middle| \, x \right),$$

where $|x| < \infty$ and $m = m_1 + \cdots + m_r$. Application of Kummer's transformation (1.6) to each of the ${}_1F_1(x)$ functions then yields

$$\begin{aligned} F(x) &= \frac{e^x}{A_0} \sum_{k=0}^{m} x^k A_k \, \frac{(b)_k}{(c)_k} \, {}_1F_1 \left(\begin{matrix} c-b \\ \\ c+k \end{matrix} \middle| \, -x \right) \\ &= \frac{e^x}{A_0} \sum_{k=0}^{m} (-1)^k A_k \, \frac{(b)_k}{(c)_k} \sum_{n=k}^{\infty} \frac{(c-b)_{n-k}}{(c+k)_{n-k}} \, \frac{(-x)^n}{(1)_{n-k}}, \end{aligned}$$

where an obvious adjustment of the summation index has been made. Upon noting the identities

$$\frac{1}{(1)_{n-k}} = \frac{(-1)^k(-n)_k}{n!}, \qquad (\alpha + k)_{n-k} = \frac{(\alpha)_n}{(\alpha)_k}, \tag{3.1}$$

we have

$$F(x) = \frac{e^x}{A_0} \sum_{k=0}^{m} A_k \, (b)_k \sum_{n=0}^{\infty} (-n)_k \, \frac{(c-b)_{n-k}}{(c)_n} \, \frac{(-x)^n}{n!},$$

where we have replaced the summation index in the inner sum by $n = 0$ since $(-n)_k = 0$ when $n < k$. Noting the easily established identity

$$(c-b)_{n-k} = \frac{(\lambda)_n (\lambda + n)_{m-k}}{(\lambda)_m}, \tag{3.2}$$

where $\lambda = c - b - m$, we then obtain

$$\begin{aligned} F(x) &= \frac{e^x}{A_0 \, (\lambda)_m} \sum_{k=0}^{m} A_k (b)_k \sum_{n=0}^{\infty} \frac{(\lambda)_n \, (-x)^n}{(c)_n \, n!} \, (-n)_k (\lambda + n)_{m-k} \\ &= \frac{e^x}{A_0 \, (\lambda)_m} \sum_{n=0}^{\infty} \frac{(\lambda)_n \, (-x)^n}{(c)_n \, n!} \sum_{k=0}^{m} A_k (b)_k (-n)_k (\lambda + n)_{m-k}, \end{aligned} \tag{3.3}$$

where we have interchanged summations.

With the definition

$$P_m(n) \equiv \sum_{k=0}^{m} A_k (b)_k (-n)_k (\lambda+n)_{m-k}$$
$$= \sum_{j=0}^{m} \sigma_{m-j} \sum_{k=0}^{j} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} (b)_k (-n)_k (\lambda+n)_{m-k}, \tag{3.4}$$

it is shown in [**8**] that $P_m(n)$ is a polynomial in n of degree m having the form

$$P_m(n) = (f_1 - b)_{m_1} \cdots (f_r - b)_{m_r} n^m + \cdots + A_0\,(\lambda)_m,$$

where the remaining intermediate coefficients of powers of n in $P_m(n)$ (when $m > 1$) are determined by the expression on the right-hand side of (3.4). Now, assuming $b \neq f_j$ $(1 \leq j \leq r)$ and $(\lambda)_m \neq 0$, we may invoke Lemma 1, thus obtaining

$$P_m(n) = A_0\,(\lambda)_m \, \frac{((\xi_m + 1))_n}{((\xi_m))_n}, \tag{3.5}$$

where the (ξ_m) are the nonvanishing zeros of the associated parametric polynomial of degree m given by (2.4).

Finally, combining (3.3), (3.4) and (3.5), we find

$$F(x) = e^x \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(c)_n} \frac{((\xi_m+1))_n}{((\xi_m))_n} \frac{(-x)^n}{n!},$$

which is the Kummer-type transformation (1.2). □

4. The first Euler-type transformation (1.3). In this section we shall provide two derivations of the Euler-type transformation formula given by (1.3). The first proof relies on the reduction formula for the Kampé de Fériet function given in Theorem 2. The second proof utilizes Lemma 4 and (1.7) and is similar to the derivation of the Kummer-type transformation (1.2) given in Section 3.

Proof. I. Let (m_r) be a sequence of nonnegative integers, and consider

$$F(y) \equiv (1-y)^{-a}\, {}_{r+2}F_{r+1} \left(\begin{matrix} a, b, & (f_r + m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, \frac{y}{y-1} \right),$$

where $b \neq f_j$ $(1 \leq j \leq r)$ and $(c-b-r)_r \neq 0$, so that

$$F(y) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k\, k!} \frac{((f_r+m_r))_k}{((f_r))_k} (-y)^k (1-y)^{-a-k}.$$

Since, for $|y| < 1$,

$$(1-y)^{-a-k} = \sum_{n=0}^{\infty} \frac{(a+k)_n}{n!}\, y^n$$

upon noting the identity (2.10), we have

$$\begin{aligned} F(y) &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (a)_{k+n} \frac{(b)_k}{(c)_k} \frac{((f_r+m_r))_k}{((f_r))_k} \frac{(-y)^k}{k!} \frac{y^n}{n!} \\ &= F^{1:r+1;0}_{0:r+1;0}\left(\begin{matrix} a & : & b, & (f_r+m_r) & ; & - \\ - & : & c, & (f_r) & ; & - \end{matrix} \,\middle|\, -y, y \right). \end{aligned}$$

Now applying Theorem 2 with $p=1$, $q=0$ and $a_1 = a$, we find

$$F(y) = {}_{m+2}F_{m+1}\left(\begin{matrix} a, c-b-m, & (\xi_m+1) \\ c, & (\xi_m) \end{matrix} \,\middle|\, y \right),$$

so that

$$\begin{aligned} (1-y)^{-a}\, {}_{r+2}F_{r+1} &\left(\begin{matrix} a, b, & (f_r+m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, \frac{y}{y-1} \right) \\ &= {}_{m+2}F_{m+1}\left(\begin{matrix} a, c-b-m, & (\xi_m+1) \\ c, & (\xi_m) \end{matrix} \,\middle|\, y \right), \end{aligned}$$

where $m = m_1 + \cdots + m_r$. The (ξ_m) are the nonvanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree m given by (2.4). Finally, letting $y = x/(x-1)$, we deduce (1.3). This evidently completes the first proof. □

Proof. II. Let (m_r) be a sequence of nonnegative integers, and consider

$$F(x) \equiv {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, x\right), \tag{4.1}$$

where $b \neq f_j$ $(1 \leq j \leq r)$ and $(c-b-r)_r \neq 0$. Then, from (2.8) with $s = 2$ and $a_1 = a$, $a_2 = b$, we have

$$F(x) = \frac{1}{A_0}\sum_{k=0}^{m} x^k A_k \,\frac{(a)_k(b)_k}{(c)_k}\, {}_2F_1\left(\begin{matrix} a+k, b+k \\ c+k \end{matrix} \,\middle|\, x\right), \tag{4.2}$$

where $|x| < 1$. The coefficients A_k and the integer m are defined, respectively, by (2.9) and (1.5).

Application of Euler's transformation (1.7) to the above ${}_2F_1(x)$ functions then yields

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a+k, b+k \\ c+k \end{matrix} \,\middle|\, x\right) &= (1-x)^{-a-k}\, {}_2F_1\left(\begin{matrix} a+k, c-b \\ c+k \end{matrix} \,\middle|\, \tfrac{x}{x-1}\right) \\ &= (1-x)^{-a-k}\sum_{n=k}^{\infty} \frac{(a+k)_{n-k}(c-b)_{n-k}}{(c+k)_{n-k}\,(n-k)!}\left(\frac{x}{x-1}\right)^{n-k}, \end{aligned} \tag{4.3}$$

where an obvious adjustment of the summation index has been made. Noting the identities (3.1) and (3.2), we may write (4.3) as

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a+k, b+k \\ c+k \end{matrix} \,\middle|\, x\right) &= x^{-k}(1-x)^{-a}\,\frac{(c)_k}{(a)_k(\lambda)_m} \\ &\times \sum_{n=0}^{\infty} \frac{(a)_n(\lambda)_n}{(c)_n\, n!}\left(\frac{x}{x-1}\right)^{n} (-n)_k(\lambda+n)_{m-k}, \end{aligned} \tag{4.4}$$

where the summation index $n = k$ has been replaced by $n = 0$ since $(-n)_k = 0$ when $n < k$. Now substitution of (4.4) in (4.2) yields

$$F(x) = \frac{(1-x)^{-a}}{A_0(\lambda)_m} \sum_{n=0}^{\infty} \frac{(a)_n(\lambda)_n}{(c)_n\, n!} \left(\frac{x}{x-1}\right)^n \sum_{k=0}^{m} A_k(b)_k(-n)_k(\lambda+n)_{m-k},$$

where the order of summation has been interchanged. Finally, recalling (3.4) and (3.5), we see that

$$\begin{aligned} &{}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r + m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, x\right) \\ &\qquad = (1-x)^{-a} \sum_{n=0}^{\infty} \frac{(a)_n(\lambda)_n}{(c)_n\, n!} \frac{((\xi_m+1))_n}{((\xi_m))_n} \left(\frac{x}{x-1}\right)^n, \end{aligned}$$

which evidently completes the proof of the transformation (1.3). □

We summarize the results of Sections 3 and 4 in the following:

Theorem 3. *Let (m_r) be a nonempty sequence of positive integers and $m \equiv m_1 + \cdots + m_r$. Then, if $b \neq f_j$ $(1 \leq j \leq r)$, $(\lambda)_m \neq 0$, where $\lambda \equiv c - b - m$, we have the transformation formulas*

$$\begin{aligned} (4.5)\quad &{}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r + m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, x\right) \\ &\qquad = (1-x)^{-a}\; {}_{m+2}F_{m+1}\left(\begin{matrix} a, \lambda, & (\xi_m + 1) \\ c, & (\xi_m) \end{matrix} \,\middle|\, \frac{x}{x-1}\right), \end{aligned}$$

where $|x| < 1$, $\operatorname{Re} x < 1/2$, and

(4.6)

$${}_{r+1}F_{r+1}\left(\begin{matrix} b, & (f_r + m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, x\right) = e^x\; {}_{m+1}F_{m+1}\left(\begin{matrix} \lambda, & (\xi_m + 1) \\ c, & (\xi_m) \end{matrix} \,\middle|\, -x\right),$$

where $|x| < \infty$. The (ξ_m) are the nonvanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree m given by

$$Q_m(t) = \sum_{j=0}^{m} \sigma_{m-j} \sum_{k=0}^{j} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} (b)_k(t)_k(\lambda - t)_{m-k},$$

where the σ_j $(0 \le j \le m)$ *are determined by the generating relation* (2.5)*.*

We remark that the Kummer-type transformation formula (4.6) may be employed to quickly provide an upper bound for the number of zeros of the generalized hypergeometric function considered by Ki and Kim [**6**], namely,

$$w(x) \equiv {}_{r+1}F_{r+1}\left(\begin{array}{c}(f_{r+1}+m_{r+1})\\ \\ (f_{r+1})\end{array}\middle|\, x\right),$$

where $|x| < \infty$ and (m_{r+1}) is a sequence of positive integers such that $M \equiv m_1 + \cdots + m_{r+1}$. Thus, we have the following:

Corollary 1. *The entire function* $w(x)$ *has at most* M *zeros in the complex plane.*

Proof. In (4.6) with $m = m_1 + \cdots + m_r$, let $b = f_{r+1} + m_{r+1}$ and $c = f_{r+1}$. Then $\lambda = -M$ and $(-M)_m \neq 0$, so that

$$w(x) = e^x\, {}_{m+1}F_{m+1}\left(\begin{array}{c}-M,\ (\xi_m+1)\\ \\ f_{r+1},\ (\xi_m)\end{array}\middle|\, -x\right). \tag{4.7}$$

Since $w(x)$ is proportional to a polynomial in $-x$ of degree at most M, the proof of the corollary is evident. □

In fact, we can show that [**13**]

$${}_{m+1}F_{m+1}\left(\begin{array}{c}-M,\ (\xi_m+1)\\ \\ f_{r+1},\ (\xi_m)\end{array}\middle|\, -x\right) = \frac{1}{A_0}\sum_{k=0}^{M} A_k x^k, \tag{4.8}$$

where the A_k $(0 \le k \le M)$ are defined in an analogous manner to that in (2.9). Thus, the zeros of the entire function $w(x)$ are characterized completely by (4.7) and (4.8), whereas Ki and Kim [**6**] only show the existence of at most M zeros for $w(x)$. See also the fourth example in Section 8, where we consider the specialization of $w(x)$, namely, (8.2).

5. The second Euler-type transformation (1.4). Before establishing the second Euler-type transformation (1.4), we shall prove a preliminary lemma. This lemma addresses the form of the associated parametric polynomial $Q_m(t)$ for this transformation and is intended to streamline the derivation of the main theorem.

Lemma 5. *Let m be a positive integer. Consider the polynomial in n defined by*

$$P_m(n) \equiv \sum_{k=0}^{m} B_k \sum_{s=0}^{p} \frac{(-p)_s}{s!} \Lambda_{k,s}(n), \tag{5.1}$$

where

$$\Lambda_{k,s}(n) \equiv (\lambda+n)_{p-s}(\lambda'+n)_{p-s}(-n)_{k+s}(1-c-n)_s, \tag{5.2}$$

$p \equiv m-k$, $\lambda \equiv c-b-m$, $\lambda' \equiv c-a-m$ and the coefficients B_k $(0 \le k \le m)$ are arbitrary complex numbers. Then $P_m(n)$ is a polynomial in n of degree m that takes the form

$$P_m(n) = \alpha_0 n^m + \cdots + \alpha_{m-1} n + \alpha_m,$$

provided that $(1+a+b-c)_m \neq 0$ and $\alpha_0 \neq 0$, where

$$\alpha_0 = (-1)^m \sum_{k=0}^{m} B_k \frac{(1+a+b-c)_m}{(1+a+b-c)_k} \tag{5.3}$$

and

$$\alpha_m = B_0 (\lambda)_m (\lambda')_m. \tag{5.4}$$

Proof. It is evident that $P_m(n)$ is a polynomial in n of degree at most $2m$. By employing the identities (2.10) and

$$(\alpha)_{-k} = \frac{(-1)^k}{(1-\alpha)_k}, \tag{5.5}$$

we may write

$$
\begin{aligned}
P_m(n) &= \sum_{k=0}^{m} B_k(-n)_k(\lambda+n)_p(\lambda'+n)_p \\
&\qquad \times \sum_{s=0}^{p} \frac{(-p)_s(k-n)_s(1-c-n)_s}{(1-\lambda-p-n)_s(1-\lambda'-p-n)_s s!} \\
&= \sum_{k=0}^{m} B_k(-n)_k(\lambda+n)_p(\lambda'+n)_p\, G_{p,k}(n),
\end{aligned} \tag{5.6}
$$

where the s-summation has been expressed as a ${}_3F_2(1)$ hypergeometric series that we define as

$$
G_{p,k}(n) \equiv {}_3F_2\left(\begin{matrix} -p,\ k-n,\ 1-c-n \\ 1-\lambda-p-n,\ 1-\lambda'-p-n \end{matrix}\,\middle|\, 1\right). \tag{5.7}
$$

The degree of the polynomial $P_m(n)$ can then be obtained by employing Sheppard's transformation [**2**, page 141] given by

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} -p, a, b \\ d, e \end{matrix}\,\middle|\, 1\right) \\
&\qquad = \frac{(d-a)_p(e-a)_p}{(d)_p(e)_p}\, {}_3F_2\left(\begin{matrix} -p,\ a,\ 1-\sigma \\ 1+a-d-p,\ 1+a-e-p \end{matrix}\,\middle|\, 1\right),
\end{aligned}
$$

where p is a nonnegative integer and $\sigma = d+e-a-b+p$ is the parametric excess.[1] Application of this transformation to $G_{p,k}(n)$ given by (5.7) then yields

$$
\begin{aligned}
G_{p,k}(n) = \frac{(1-\lambda-p-k)_p(1-\lambda'-p-k)_p}{(1-\lambda-p-n)_p(1-\lambda'-p-n)_p}& \\
{}_3F_2\left(\begin{matrix} -p,\ -n+k,\ 1-\sigma \\ \lambda+k,\ \lambda'+k \end{matrix}\,\middle|\, 1\right)&,
\end{aligned}
$$

where now $1-\sigma = c-a-b-m$. Employing the identity (5.5) we obtain from this and (5.6), the alternative representation

$$(5.8)\quad P_m(n) = \sum_{k=0}^{m} B_k(-n)_k(\lambda+k)_p(\lambda'+k)_p \; {}_3F_2\left(\begin{matrix} -p,\ -n+k,\ 1-\sigma \\ \lambda+k,\ \lambda'+k \end{matrix}\,\middle|\, 1\right).$$

Since n appears only in a single numeratorial parameter of the ${}_3F_2(1)$ series on the right-hand side of (5.8), we see that ${}_3F_2(1)$ is a polynomial in n of degree $p = m-k$ only if $\sigma \neq 1, 2, \ldots, p$; that is, provided $(1+a+b-c)_m \neq 0$. As $(-n)_k$ is a polynomial in n of degree k, it follows that $P_m(n)$ is a polynomial in n of degree $k+p=m$ and hence must have the form given in the statement of the lemma.

The coefficient α_0 can be determined as follows. The highest power of n in the ${}_3F_2(1)$ series in (5.8) arises from the last term when it is expressed as an s-summation, that is, when $s = p$

$$\frac{(-1)^p(-n+k)_p(1-\sigma)_p}{(\lambda+k)_p(\lambda'+k)_p} = \frac{(1-\sigma)_p}{(\lambda+k)_p(\lambda'+k)_p} n^p + \cdots .$$

Thus, from (5.8) we find the coefficient of n^m in the polynomial $P_m(n)$, namely,

$$\alpha_0 = \sum_{k=0}^{m} (-1)^k B_k (1-\sigma)_{m-k}$$

which yields (5.3). Finally, when $n = 0$, the only contribution to the double sum in (5.1) arises from $k = s = 0$. Thus, since $P_m(0) = \alpha_m$, we deduce (5.4). The proof of the lemma is evidently complete. □

As we shall see below, when

$$B_k = (-1)^k A_k (a)_k (b)_k \qquad (0 \leq k \leq m),$$

where the A_k $(0 \leq k \leq m)$ are given by (2.9), the associated parametric polynomial $Q_m(t)$ for the transformation (1.4) may be obtained from either (5.1), (5.6) or (5.8) by replacing n in the latter by $-t$, so that in each case $Q_m(t) = P_m(-t)$.

We now establish an extension of the second Euler transformation (1.8) given in the following.

Theorem 4. *Suppose*[2] $(1+a+b-c)_m \neq 0$ *and* $(\lambda)_m \neq 0$, $(\lambda')_m \neq 0$. *Then*

$$(5.9)\quad {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+m_r) \\ c, & (f_r) \end{matrix}\,\middle|\, x\right) = (1-x)^{c-a-b-m}\,{}_{m+2}F_{m+1}\left(\begin{matrix} \lambda, \lambda', & (\eta_m+1) \\ c, & (\eta_m) \end{matrix}\,\middle|\, x\right)$$

valid in $|x|<1$, *where* $\lambda = c-b-m$ *and* $\lambda' = c-a-m$. *The* (η_m) *are the nonvanishing zeros of the associated parametric polynomial* $Q_m(t)$ *of degree* $m = m_1 + \cdots + m_r$, *given by*

$$(5.10)\quad Q_m(t) = \sum_{k=0}^{m} (-1)^k A_k (a)_k (b)_k (t)_k (\lambda - t)_p (\lambda' - t)_p\, G_{p,k}(-t),$$

where $p \equiv m-k$, *the coefficients* A_k *are defined by* (2.9) *and* $G_{p,k}(-t)$ *is defined by* (5.7).

Proof. Our starting point is the expansion (4.2) which expresses the hypergeometric function $F(x)$ defined by (4.1) as a finite series of ${}_2F_1(x)$ functions. To each of the latter functions we apply the second Euler transformation (1.8) to find

$$x^k\,{}_2F_1\left(\begin{matrix} a+k, b+k \\ c+k \end{matrix}\,\middle|\, x\right) = x^k (1-x)^{c-a-b-k}\,{}_2F_1\left(\begin{matrix} c-a, c-b \\ c+k \end{matrix}\,\middle|\, x\right)$$
$$= (1-x)^{c-a-b-m} \sum_{s=0}^{p} \frac{(-p)_s}{s!} \times \sum_{n=0}^{\infty} \frac{(c-a)_n (c-b)_n}{(c+k)_n} \frac{x^{n+k+s}}{n!},$$

where we have defined $p \equiv m-k$ and used the binomial theorem to expand the factor $(1-x)^p$. If we now change the summation index

$n \mapsto n+k+s$ and make use of (2.10), (3.1) and the identity (5.5), the right-hand side of the above equation can be written as

$$(1-x)^{c-a-b-m} \sum_{s=0}^{p} \frac{(-p)_s}{s!} \sum_{n=k+s}^{\infty} \frac{(c-a)_{n-k-s}(c-b)_{n-k-s}}{(c+k)_{n-k-s}(1)_{n-k-s}} x^n$$
$$= (1-x)^{c-a-b-m} \frac{(-1)^k (c)_k}{(\lambda)_m (\lambda')_m}$$
$$\times \sum_{s=0}^{p} \frac{(-p)_s}{s!} \sum_{n=0}^{\infty} \frac{(\lambda)_n (\lambda')_n}{(c)_n} \Lambda_{k,s}(n) \frac{x^n}{n!},$$

where we have introduced the coefficients $\Lambda_{k,s}(n)$ defined by (5.2) and have replaced the inner summation index $n = k+s$ by $n = 0$ since $(-n)_{k+s} = 0$ for $n < k+s$. Hence, from (4.2), we obtain

$$F(x) = \frac{(1-x)^{c-a-b-m}}{A_0 (\lambda)_m (\lambda')_m} \sum_{n=0}^{\infty} \frac{(\lambda)_m (\lambda')_m}{(c)_n} \frac{x^n}{n!} P_m(n) \tag{5.11}$$

upon interchanging the order of summation, where we have defined

$$P_m(n) \equiv \sum_{k=0}^{m} (-1)^k A_k (a)_k (b)_k \sum_{s=0}^{p} \frac{(-p)_s}{s!} \Lambda_{k,s}(n). \tag{5.12}$$

Now setting $B_k = (-1)^k A_k (a)_k (b)_k$ in Lemma 5, we see that $P_m(n)$ is a polynomial in n of degree m having the form

$$P_m(n) = \alpha_0 n^m + \cdots + \alpha_{m-1} n + \alpha_m,$$

where, from (5.3) and (5.4),

$$\begin{aligned} \alpha_0 &= (-1)^m (1+a+b-c)_m \sum_{k=0}^{m} \frac{(-1)^k A_k (a)_k (b)_k}{(1+a+b-c)_k}, \\ \alpha_m &= A_0 (\lambda)_m (\lambda')_m. \end{aligned} \tag{5.13}$$

Assuming that the coefficient $\alpha_0 \neq 0$ and $(\lambda)_m \neq 0$, $(\lambda')_m \neq 0$, we may then invoke Lemma 1 to obtain

$$P_m(n) = A_0 \, (\lambda)_m (\lambda')_m \, \frac{((\eta_m + 1))_n}{((\eta_m))_n}, \tag{5.14}$$

where, from (5.6) with B_k defined as above, the (η_m) are the nonvanishing zeros of the associated parametric polynomial given by (5.10).

Then, provided $\alpha_0 \neq 0$, $\alpha_m \neq 0$ by Lemma 1, the zeros (η_m) of the associated parametric polynomial $Q_m(t)$ are nonvanishing. This requires that $(\lambda)_m \neq 0$ and $(\lambda')_m \neq 0$ for the coefficient $\alpha_m \neq 0$; a necessary condition for $\alpha_0 \neq 0$ is $(1+a+b-c)_m \neq 0$ since, if this is satisfied, then $(1+a+b-c)_k \neq 0$ for $k < m$, so that the k-summation in (5.13) exists as a finite value. A sufficient condition for $\alpha_0 \neq 0$ is that the finite sum in (5.13) does not vanish. With these restrictions, it then follows from (5.11) and (5.14) that

$$F(x) = (1-x)^{c-a-b-m} \sum_{n=0}^{\infty} \frac{(\lambda)_n(\lambda')_n}{(c)_n} \frac{((\eta_m+1))_n}{((\eta_m))_n} \frac{x^n}{n!},$$

thereby establishing Theorem 4. □

6. Quadratic transformations. In this section we derive generalizations of two well-known quadratic transformation formulas for the Gauss hypergeometric function, which we state in the following theorem.

Theorem 5. *Let (m_r) denote a sequence of positive integers such that $m \equiv m_1 + \cdots + m_r$. Then we have the generalized quadratic transformation*

$$(6.1) \quad {}_{r+2}F_{r+1}\left(\begin{matrix} a, a+(1/2), & (f_r+m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, \frac{x^2}{(1\mp x)^2}\right)$$
$$= (1\mp x)^{2a} {}_{2m+2}F_{2m+1}\left(\begin{matrix} 2a, \; c-m-(1/2), & (\xi_{2m}+1) \\ 2c-1, & (\xi_{2m}) \end{matrix} \,\middle|\, \pm 2x\right),$$

where, provided $(c-m-(1/2))_m \neq 0$, the (ξ_{2m}) are the nonvanishing zeros of the associated parametric polynomial $Q_{2m}(t)$ of degree $2m$ given by

$$(6.2) \qquad Q_{2m}(t) = \sum_{k=0}^{m} \frac{A_k}{2^{2k}} (t)_{2k} \left(c-m-\frac{1}{2}-t\right)_{m-k}.$$

In addition, we have the second generalized quadratic transformation

$$(6.3)\quad \begin{aligned} &{}_{r+2}F_{r+1}\left(\begin{matrix} a, a+(1/2), & (f_r+m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, \tfrac{4x}{(1+x)^2}\right) \\ &= (1+x)^{2a}{}_{2m+2}F_{2m+1}\left(\begin{matrix} 2a,\ 2a-c+1, & (\eta_{2m}+1) \\ c, & (\eta_{2m}) \end{matrix} \,\middle|\, x\right), \end{aligned}$$

where, provided $(2a-c+1)_m \neq 0$*, the* (η_{2m}) *are the nonvanishing zeros of the associated parametric polynomial of degree* $2m$ *given by*

$$(6.4)\qquad Q_{2m}(t) = \sum_{k=0}^{m} \frac{(-1)^k A_k}{(2a-c+1)_k}\,(t)_k(2a-t)_k.$$

The coefficients A_k *are defined by* (2.9) *and the transformations* (6.1) *and* (6.3) *hold in neighborhoods of* $x = 0$.

When $r = 0$, then $m = 0$ so that (6.1) and (6.3) reduce to the well-known quadratic transformation formulas due to Kummer given by

$$(6.5)\quad {}_2F_1\left(\begin{matrix} a, a+(1/2) \\ c \end{matrix} \,\middle|\, \tfrac{x^2}{(1\mp x)^2}\right) = (1\mp x)^{2a}{}_2F_1\left(\begin{matrix} 2a,\ c-(1/2) \\ 2c-1 \end{matrix} \,\middle|\, \pm 2x\right)$$

and

$$(6.6)\quad {}_2F_1\left(\begin{matrix} a, a+(1/2) \\ c \end{matrix} \,\middle|\, \tfrac{4x}{(1+x)^2}\right) = (1+x)^{2a}{}_2F_1\left(\begin{matrix} 2a,\ 2a-c+1 \\ c \end{matrix} \,\middle|\, x\right),$$

which are, respectively, slight variations of those given in [**1**, Section 15.3, (19) and (20)].

Proof. We shall first establish (6.1). Let us define

$$(6.7)\quad F(x) \equiv {}_{r+2}F_{r+1}\left(\begin{matrix} a, a+(1/2), & (f_r+m_r) \\ c, & (f_r) \end{matrix} \,\middle|\, X\right), \qquad X \equiv \frac{x^2}{(1\mp x)^2}.$$

Then use of the expansion (2.8) with $s = 2$ and $a_1 = a$, $a_2 = a + (1/2)$ yields

(6.8)

$$F(x) = \frac{1}{A_0} \sum_{k=0}^{m} A_k \frac{(a)_{2k}}{2^{2k}(c)_k} X^k \, {}_2F_1\left(\begin{matrix} a+k, \; a+k+(1/2) \\ c+k \end{matrix} \middle| X \right),$$

where we have employed the duplication formula

$$(\alpha)_{2k} = 2^{2k}(\alpha)_k(\alpha + (1/2))_k. \tag{6.9}$$

Application of the quadratic transformation (6.5) to each of the ${}_2F_1(X)$ functions then yields

$$\begin{aligned} F(x) = \frac{(1 \mp x)^{2a}}{A_0} \sum_{k=0}^{m} A_k \frac{(a)_{2k}}{2^{2k}(c)_k} x^{2k} \\ \times {}_2F_1\left(\begin{matrix} 2a+2k, \; c+k-(1/2) \\ 2c+2k-1 \end{matrix} \middle| \pm 2x \right) \\ = \frac{(1 \mp x)^{2a}}{A_0} \sum_{k=0}^{m} A_k \frac{(a)_{2k}}{2^{4k}(c)_k} \\ \times \sum_{n=2k}^{\infty} \frac{(2a+2k)_{n-2k}(c+k-(1/2))_{n-2k}}{(2c+2k-1)_{n-2k}} \frac{(\pm 2x)^n}{(1)_{n-2k}}, \end{aligned}$$

where an obvious adjustment of the summation index has been made.

We now make use of (3.1) with k replaced by $2k$ and (6.9) together with the identity

$$(c' + k)_{n-2k} = \frac{(c'-m)_n}{(c'-m)_m} \frac{(c'-m+n)_{m-k}}{(c')_k}, \qquad c' \equiv c - \frac{1}{2}.$$

Thus, we obtain, after some reduction,

$$\begin{aligned} F(x) = \frac{(1 \mp x)^{2a}}{A_0(c'-m)_m} \sum_{k=0}^{m} 2^{-2k} A_k \\ \times \sum_{n=2k}^{\infty} \frac{(2a)_n (c'-m)_n}{(2c')_n} \frac{(\pm 2x)^n}{n!} (-n)_{2k} (c'-m+n)_{m-k} \\ = \frac{(1 \mp x)^{2a}}{A_0(c'-m)_m} \sum_{n=0}^{\infty} \frac{(2a)_n (c'-m)_n}{(2c')_n} \frac{(\pm 2x)^n}{n!} P_{2m}(n), \end{aligned}$$

where we have interchanged the order of summation, replaced the summation index $n = 2k$ by $n = 0$ since $(-n)_{2k} = 0$ for $n < 2k$, and defined

$$P_{2m}(n) \equiv \sum_{k=0}^{m} \frac{A_k}{2^{2k}} (-n)_{2k}(c' - m + n)_{m-k}.$$

Since, by (2.9), $A_m = 1$, it is clear that $P_{2m}(n)$ is a polynomial in n of degree $2m$ and has the form

$$P_{2m}(n) = 2^{-2m} n^{2m} + \cdots + A_0(c' - m)_m.$$

We can then invoke Lemma 1 to obtain

$$P_{2m}(n) = A_0(c' - m)_m \frac{((\xi_{2m} + 1))_n}{((\xi_{2m}))_n},$$

where, provided $(c' - m)_m \neq 0$, the (ξ_{2m}) are the nonvanishing zeros of the associated parametric polynomial given by (6.2). It then follows that

$$F(x) = (1 \mp x)^{2a} \sum_{n=0}^{\infty} \frac{(2a)_n(c - m - (1/2))_n}{(2c - 1)_n} \frac{((\xi_{2m} + 1))_n}{((\xi_{2m}))_n} \frac{(\pm 2x)^n}{n!},$$

thereby establishing the first part of Theorem 6.

The second quadratic transformation formula (6.3) can be established in a similar manner. We again let $F(x)$ be given by (6.7), where X is now defined by $X \equiv 4x/(1 + x)^2$. Then, from (6.8) and the quadratic transformation (6.6), we find *mutatis mutandis* that

$$\begin{aligned} F(x) = &\frac{(1 + x)^{2a}}{A_0} \\ &\times \sum_{k=0}^{m} A_k \frac{(2a)_{2k}}{(c)_k} x^k {}_2F_1 \left(\begin{matrix} 2a + 2k, \; 2a - c + k + 1 \\ c + k \end{matrix} \middle| \, x \right) \\ = &\frac{(1 + x)^{2a}}{A_0} \sum_{k=0}^{m} A_k \frac{(2a)_{2k}}{(c)_k} \\ &\qquad \times \sum_{n=k}^{\infty} \frac{(2a + 2k)_{n-k}(2a - c + k + 1)_{n-k}}{(c + k)_{n-k}} \frac{x^n}{(1)_{n-k}} \\ = &\frac{(1 + x)^{2a}}{A_0} \sum_{n=0}^{\infty} \frac{(2a)_n(2a - c + 1)_n}{(c)_n} \frac{x^n}{n!} P_{2m}(n), \end{aligned}$$

where now

$$(6.10)\qquad P_{2m}(n) \equiv \sum_{k=0}^{m} \frac{(-1)^k A_k}{(2a-c+1)_k}\,(-n)_k(2a+n)_k.$$

The polynomial $P_{2m}(n)$ is clearly of degree $2m$ and possesses the form

$$P_{2m}(n) = \frac{n^{2m}}{(2a-c+1)_m} + \cdots + A_0.$$

Provided $(2a-c+1)_m \neq 0$, we may invoke Lemma 1, thus giving

$$P_{2m}(n) = A_0\,\frac{((\eta_{2m}+1))_n}{((\eta_{2m}))_n},$$

where the (η_{2m}) are the nonvanishing zeros of the associated parametric polynomial $Q_{2m}(t)$ given by (6.4). It then follows that

$$F(x) = (1+x)^{2a} \sum_{n=0}^{\infty} \frac{(2a)_n(2a-c+1)_n}{(c)_n}\,\frac{((\eta_{2m}+1))_n}{((\eta_{2m}))_n}\,\frac{x^n}{n!},$$

which establishes (6.3) and so completes the proof of Theorem 6. □

In the case $r = 1$, $m_1 = 1$, we see with $f_1 = f$ that the associated parametric polynomials $Q_2(t)$ given by (6.2) and (6.4) are, respectively,

$$\frac{1}{4}t^2 + \left(\frac{1}{4} - f\right)t + f\left(c - \frac{3}{2}\right) \quad \text{and} \quad \frac{t^2 - 2at + f(2a-c+1)}{2a-c+1}.$$

The zeros of these polynomials are, respectively,

$$\xi_{1,2} = 2f - \frac{1}{2} \pm \left[\left(2f - \frac{1}{2}\right)^2 - 4f\left(c - \frac{3}{2}\right)\right]^{1/2}$$

and

$$\eta_{1,2} = a \pm [a^2 - f(2a-c+1)]^{1/2}.$$

Thus, from (6.1) and (6.3), we obtain the quadratic transformations

$$\begin{aligned}(6.11)\quad &{}_3F_2\left(\begin{matrix} a,\ a+(1/2), & f+1 \\ c, & f\end{matrix}\,\middle|\, \tfrac{x^2}{(1\mp x)^2}\right) \\ &= (1\mp x)^{2a}\,{}_4F_3\left(\begin{matrix} 2a,\ c-(3/2), & \xi_1+1, & \xi_2+1 \\ 2c-1, & \xi_1, & \xi_2\end{matrix}\,\middle|\, \pm 2x\right)\end{aligned}$$

provided $c \neq 3/2$, and

$$\begin{aligned}(6.12)\quad &{}_3F_2\left(\begin{matrix} a,\ a+(1/2), & f+1 \\ c, & f\end{matrix}\,\middle|\, \tfrac{4x}{(1+x)^2}\right) \\ &= (1+x)^{2a}\,{}_4F_3\left(\begin{matrix} 2a,\ 2a-c+1, & \eta_1+1, & \eta_2+1 \\ c, & \eta_1, & \eta_2\end{matrix}\,\middle|\, x\right),\end{aligned}$$

provided $c \neq 2a+1$. The transformations (6.11) and (6.12) were found in an equivalent form by Rakha et al. in [**17, 18**].

We note that when $c = 2a+1$ in (6.11) and $c = 2a$ in (6.12) the ${}_4F_3$ functions reduce to lower order ${}_3F_2$ functions. Furthermore, when $c = 2a+p+1$ in (6.12) with p a positive integer, we obtain

$$\begin{aligned}&{}_3F_2\left(\begin{matrix} a,\ a+(1/2), & f+1 \\ 2a+p+1, & f\end{matrix}\,\middle|\, \tfrac{4x}{(1+x)^2}\right) \\ &= (1+x)^{2a}\,{}_4F_3\left(\begin{matrix} -p,\ 2a, & \eta_1+1, & \eta_2+1 \\ 2a+p+1, & \eta_1, & \eta_2\end{matrix}\,\middle|\, x\right),\end{aligned}$$

where $\eta_{1,2} = a \pm (a^2+pf)^{1/2}$, and the right-hand side of this transformation is a polynomial in x of degree p. We compare this with Whipple's quadratic transformation [**2**, p. 130] expressed in the form

$$\begin{aligned}&{}_3F_2\left(\begin{matrix} a,\ a+(1/2), & f+b \\ 2a+b+1, & f\end{matrix}\,\middle|\, \tfrac{4x}{(1+x)^2}\right) \\ &= (1+x)^{2a}\,{}_3F_2\left(\begin{matrix} -b,\ 2a,\ 2a-f+1 \\ 2a+b+1,\ f\end{matrix}\,\middle|\, x\right),\end{aligned}$$

where $b \neq -1 - 2a$ is otherwise arbitrary. In the particular cases $b = 1$ and $p = 1$, it is easily seen that the right-hand sides of both transformations reduce to

$$(1+x)^{2a}\left(1 + \frac{a(2a-f+1)}{(a+1)f}x\right).$$

It is worth mentioning that, in general, when the result of a transformation is proportional to a polynomial $S_p(x)$ of degree p, then it not essential to determine the zeros of the associated parametric polynomial $Q_\mu(t)$ of degree μ for the transformation in order to compute the coefficients of powers of x in $S_p(x)$, since these coefficients may be obtained directly by use of $P_\mu(n) = Q_\mu(-n)$ itself. Thus, in the specialization $c = 2a + p + 1$ discussed above, $P_{2m}(n)$ given by (6.10) may be used with the result for $F(x)$ directly preceding it in order to compute the coefficients of x^n $(0 \leq n \leq p)$ in the expression for $F(x)$.

Finally, we make an observation concerning the derivation of the generalized quadratic transformations (6.1) and (6.3). A quadratic transformation for ${}_2F_1(\alpha, \beta; \gamma \mid x)$ exists if and only if any of the quantities

$$\pm(1-\gamma), \qquad \pm(\alpha-\beta), \qquad \pm(\alpha+\beta-\gamma)$$

are such that either one of them equals 1/2 or two of them are equal [**1**, page 560]. It has been possible to obtain the transformations (6.1) and (6.3) since the corresponding Gauss functions that appear in expansion (6.8) satisfy a condition of the type $\alpha - \beta = -1/2$ for $0 \leq k \leq m$. An example where it is does not seem possible to apply a quadratic transformation to each of the Gauss functions in (6.8) is given by

$$(6.13)\quad {}_{r+2}F_{r+1}\left(\begin{matrix} a,\ b, & (f_r+m_r) \\ a+b+(1/2), & (f_r) \end{matrix}\ \middle|\ X\right), \qquad X \equiv 4x(1-x).$$

In this case, the third condition above for the functions ${}_2F_1(a + k, b + k; a + b + k + (1/2) \mid X)$, with $0 \leq k \leq m$, has the form $\alpha + \beta - \gamma = k - (1/2)$; that is, a quadratic transformation only exists when $k = 0$ and $k = 1$. Consequently, we are compelled to take $r = 1$, $m = 1$ in (6.13). Thus, omitting details for brevity, we find by a similar analysis described in [**13**]

$$
{}_3F_2\left(\begin{matrix} a,\ b, & f+1 \\ a+b+(1/2), & f \end{matrix}\,\middle|\, X\right) = (1-2x)^{-1}\,{}_4F_3\left(\begin{matrix} 2a-1,\ 2b-1, & \xi_1+1, & \xi_2+1 \\ a+b+(1/2), & \xi_1, & \xi_2 \end{matrix}\,\middle|\, x\right),
$$

where X is defined in (6.13),

$$
\xi_{1,2} = A+\frac{1}{2} \pm \left[\left(A+\frac{1}{2}\right)^2 - 2Af\right]^{1/2}, \qquad A = \frac{(2a-1)(2b-1)}{2(a+b-f)-1},
$$

and it is supposed that $a, b \neq 1/2$, $f \neq a+b-(1/2)$.

7. Summation theorems. In this section we shall show that Lemma 4 may be employed to quickly and efficiently obtain the following summation theorem.

Theorem 6. *Suppose* (m_r) *is a sequence of positive integers such that* $m \equiv m_1 + \cdots + m_r$. *Then, provided that* $\mathrm{Re}\,(c-a-b) > m$, *we have*

$$
(7.1)\quad {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+m_r) \\ c, & (f_r) \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{m} \frac{A_k}{A_0}\, \frac{(-1)^k (a)_k (b)_k}{(1+a+b-c)_k},
$$

where the A_k $(0 \leq k \leq m)$ *are defined by* (2.9). *Moreover when* $c = b+1$, *then* (7.1) *reduces to the Karlsson-Minton summation formula given by*

$$
(7.2)\quad {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+m_r) \\ b+1, & (f_r) \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(1+b)\Gamma(1-a)}{\Gamma(1+b-a)}\, \frac{(f_1-b)_{m_1}\cdots(f_r-b)_{m_r}}{(f_1)_{m_1}\cdots(f_r)_{m_r}},
$$

where $\mathrm{Re}\,(-a) > m_1 + \cdots + m_r - 1$.

Proof. In (2.8), let $x = 1$, $s = 2$, $a_1 = a$, $a_2 = b$ where for convergence of $\mathcal{F}(1)$ we must have $\operatorname{Re}(c - a - b) > m$. Thus, we obtain

$$(7.3)\quad {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r + m_r) \\ c, & (f_r) \end{matrix}\,\middle|\, 1\right) = \frac{1}{A_0}\sum_{k=0}^{m} A_k \frac{(a)_k (b)_k}{(c)_k}\, {}_2F_1\left(\begin{matrix} a+k,\ b+k \\ c+k \end{matrix}\,\middle|\, 1\right).$$

Note that each ${}_2F_1(1)$ converges since $\operatorname{Re}(c - a - b) > m \geq k \geq 0$. Thus, employing the Gauss summation theorem given by

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \operatorname{Re}(c-a-b) > 0$$

and the identity (5.5), we find for nonnegative integers k that

$${}_2F_1\left(\begin{matrix} a+k,\ b+k \\ c+k \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \frac{(-1)^k (c)_k}{(1+a+b-c)_k}.$$

Combining this with (7.3), we then obtain (7.1).

Now set $c = b + 1$ in (7.1), thus giving

$$(7.4)\quad {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r + m_r) \\ b+1, & (f_r) \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(1+b)\Gamma(1-a)}{\Gamma(1+b-a)} \frac{1}{A_0}\sum_{k=0}^{m}(-1)^k A_k (b)_k,$$

where the A_k $(0 \leq k \leq m)$ are given by

$$(7.5)\qquad A_k = \sum_{j=k}^{m} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \sigma_{m-j}, \quad A_0 = (f_1)_{m_1} \cdots (f_r)_{m_r}$$

and the σ_j $(0 \le j \le m)$ are defined by (2.5). However,

$$\begin{aligned}\sum_{k=0}^{m}(-1)^k A_k(b)_k &= \sum_{k=0}^{m}\sum_{j=k}^{m}\left\{ {j \atop k} \right\} \sigma_{m-j}(-1)^k(b)_k \\ &= \sum_{j=0}^{m}\sigma_{m-j}\sum_{k=0}^{j}\left\{ {j \atop k} \right\}(-1)^k(b)_k,\end{aligned}$$

where, by (2.2),

$$\sum_{k=0}^{j}\left\{ {j \atop k} \right\}(-1)^k(b)_k = (-b)^j.$$

Thus, using (2.5), we have

$$\sum_{k=0}^{m}(-1)^k A_k(b)_k = \sum_{j=0}^{m}\sigma_{m-j}\,(-b)^j = (f_1-b)_{m_1}\cdots(f_r-b)_{m_r}$$

which, when combined with (7.4) and (7.5), yields (7.2). This evidently completes the proof of Theorem 6. □

We remark that the summation formula (7.1) has previously been deduced in [**14**], where a slightly more complex result is recorded. For previous work pertaining to the Karlsson-Minton summation formula (7.2), see the references cited in [**14**].

8. Examples and concluding remarks. We now present some examples of the theorems developed in this paper; the cases $r = 1$ and $m = 1$ have already been mentioned. Consider first the case $r = 2$ with $m_1 = m_2 = 1$, so that the associated parametric polynomial for the transformations (1.2) and (1.3) is given by [**8**]

$$Q_2(t) = \alpha t^2 - ((\alpha+\beta)\lambda+\beta)t + f_1 f_2\lambda(\lambda+1), \tag{8.1}$$

where $\lambda = c - b - 2$ and

$$\alpha = (f_1-b)(f_2-b), \qquad \beta = f_1 f_2 - b(b+1).$$

If we choose $b=1$, $c=1/3$, $f_1=2/3$ and $f_2=1/2$, then

$$Q_2(t)=\frac{1}{6}\left(t^2-14t+\frac{80}{9}\right),$$

so that the zeros are $\xi_1=2/3$ and $\xi_2=40/3$. We then have the first Euler and Kummer-type transformation formulas

$$\begin{aligned}{}_4F_3\left(\begin{matrix}a, & 1, & \frac{5}{3}, & \frac{3}{2}\\ & \frac{1}{3}, & \frac{2}{3}, & \frac{1}{2}\end{matrix}\,\middle|\, x\right) &= (1-x)^{-a}{}_4F_3\left(\begin{matrix}a, & -\frac{8}{3}, & \frac{5}{3}, & \frac{43}{3}\\ & \frac{1}{3}, & \frac{2}{3}, & \frac{40}{3}\end{matrix}\,\middle|\, \frac{x}{x-1}\right),\\ {}_3F_3\left(\begin{matrix}1, & \frac{5}{3}, & \frac{3}{2}\\ \frac{1}{3}, & \frac{2}{3}, & \frac{1}{2}\end{matrix}\,\middle|\, x\right) &= e^x{}_3F_3\left(\begin{matrix}-\frac{8}{3}, & \frac{5}{3}, & \frac{43}{3}\\ \frac{1}{3}, & \frac{2}{3}, & \frac{40}{3}\end{matrix}\,\middle|\, x\right),\end{aligned}$$

where a is a free parameter.

Our second example has $r=1$ where we consider in turn the cases with $m_1=2$ and $m_1=3$. When $m_1=2$, then $\lambda=c-b-2$, and the associated parametric polynomial $Q_2(t)$ for the first Euler and Kummer-type transformations takes the form

$$Q_2(t)=At^2+Bt+C,$$

where

$$\begin{aligned}A&=(f-b)_2,\\ B&=(b)_2+2b\lambda(f+1)-(2\lambda+1)(f)_2,\\ C&=(f)_2(\lambda)_2.\end{aligned}$$

We remark that the latter $Q_2(t)$ is easily seen to reduce to (8.1) in which $f_1=f$ and $f_2=f+1$. In the particular case $b=5/3$, $c=4/3$ and $f=1/3$, we find

$$Q_2(t)=\frac{1}{81}(36t^2-348t+112),$$

so that $\xi_1=1/3$ and $\xi_2=28/3$. When $m_1=3$, the cubic polynomial $Q_3(t)$ with $b=1$, $c=7/4$ and $f=2$ reduces to

$$Q_3(t)=-\frac{1}{8}(48t^3+192t^2+234t+135),$$

so that $\xi_1 = -5/2$ and $\xi_{2,3} = -3/4 \pm 3/4i$. Hence, with $m_1 = 2$ and $m_1 = 3$, respectively, we obtain from (1.3) the first Euler-type transformation formulas

$$ {}_3F_2\left(\begin{matrix} a, & \frac{5}{3}, & \frac{7}{3} \\ & \frac{4}{3}, & \frac{1}{3} \end{matrix} \middle| \, x \right) = (1-x)^{-a} \, {}_3F_2\left(\begin{matrix} a, & -\frac{7}{3}, & \frac{31}{3} \\ & \frac{1}{3}, & \frac{28}{3} \end{matrix} \middle| \, \frac{x}{x-1} \right), $$

$$ {}_3F_2\left(\begin{matrix} a, & 1, & 5 \\ & \frac{7}{4}, & 2 \end{matrix} \middle| \, x \right) = (1-x)^{-a} $$

$$ \times \, {}_5F_4\left(\begin{matrix} a, & -\frac{9}{4}, & -\frac{3}{2}, & \frac{1}{4}+\frac{3}{4}i, & \frac{1}{4}-\frac{3}{4}i \\ & \frac{7}{4}, & -\frac{5}{2}, & -\frac{3}{4}+\frac{3}{4}i, & -\frac{3}{4}-\frac{3}{4}i \end{matrix} \middle| \, \frac{x}{x-1} \right) $$

and from (1.2) the Kummer-type transformation formulas

$$ {}_2F_2\left(\begin{matrix} \frac{5}{3}, & \frac{7}{3} \\ \frac{4}{3}, & \frac{1}{3} \end{matrix} \middle| \, x \right) = e^x \, {}_2F_2\left(\begin{matrix} -\frac{7}{3}, & \frac{31}{3} \\ \frac{1}{3}, & \frac{28}{3} \end{matrix} \middle| \, -x \right), $$

$$ {}_2F_2\left(\begin{matrix} 1, & 5 \\ \frac{7}{4}, & 2 \end{matrix} \middle| \, x \right) = e^x \, {}_4F_4\left(\begin{matrix} -\frac{9}{4}, & -\frac{3}{2}, & \frac{1}{4}+\frac{3}{4}i, & \frac{1}{4}-\frac{3}{4}i \\ \frac{7}{4}, & -\frac{5}{2}, & -\frac{3}{4}+\frac{3}{4}i, & -\frac{3}{4}-\frac{3}{4}i \end{matrix} \middle| \, -x \right). $$

We remark that, in the case $m_1 = 2$, a contraction of the order of hypergeometric functions on the right-hand side has been possible since $c = \xi_1 + 1 = 4/3$.

As a third example, we consider the second Euler-type transformation (1.4) with $r = 2$ and $m_1 = m_2 = 1$. With the parameters $a = 1/3$, $b = 1/2$, $c = 1$ and $f_1 = 1/4$, $f_2 = 2$, so that $\lambda = -3/2$ and $\lambda' = -4/3$, we find from (5.10) the associated parametric polynomial given by

$$ Q_2(t) = \frac{1}{72}\left(\frac{15}{2}t^2 + 23t + 12\right), $$

which has the zeros $\eta_1 = -2/3$, $\eta_2 = -12/5$. This yields the second Euler-type transformation formula

$$ {}_4F_3\left(\begin{matrix} \frac{1}{3}, & \frac{1}{2}, & \frac{5}{4}, & 3 \\ & 1, & \frac{1}{4}, & 2 \end{matrix} \middle| \, x \right) = (1-x)^{-11/6} \, {}_4F_3\left(\begin{matrix} -\frac{3}{2}, & -\frac{4}{3}, & \frac{1}{3}, & -\frac{7}{5} \\ & 1, & -\frac{2}{3}, & -\frac{12}{5} \end{matrix} \middle| \, x \right). $$

Finally, we give a fourth example by setting in (1.2) $m_j = 1$, $f_j = c$ $(1 \le j \le r)$ and $b = c+1$. Thus, $m = r$, $\lambda = -r-1$, and we have

$$(8.2)\quad {}_{r+1}F_{r+1}\left(\begin{matrix} c+1, & \cdots, & c+1 \\ c, & \cdots, & c \end{matrix}\,\middle|\, x\right) = e^x\, {}_{r+1}F_{r+1}\left(\begin{matrix} -r-1, & (\xi_r+1) \\ c, & (\xi_r) \end{matrix}\,\middle|\, -x\right),$$

where the (ξ_r) are the nonvanishing zeros of the transformation's respective associated parametric polynomial of degree r. However, we shall show that the polynomial of degree $r+1$ on the right-hand side of (8.2) may be written explicitly. For, since

$$\left(\frac{(c+1)_n}{(c)_n}\right)^p = \left(1+\frac{n}{c}\right)^p = c^{-p}\sum_{k=0}^{p}\binom{p}{k} n^k c^{p-k},$$

for positive integer p, we have

$${}_pF_p\left(\begin{matrix} c+1, & \cdots, & c+1 \\ c, & \cdots, & c \end{matrix}\,\middle|\, x\right) = c^{-p}\sum_{k=0}^{p}\binom{p}{k} c^{p-k}\sum_{n=0}^{\infty} n^k\,\frac{x^n}{n!},$$

where we have interchanged the order of summation. Now, employing Lemma 3, we see that

$$\sum_{n=0}^{\infty} n^k\,\frac{x^n}{n!} = \sum_{j=0}^{k}\left\{\begin{matrix} k \\ j \end{matrix}\right\}\sum_{n=0}^{\infty}\frac{x^{n+j}}{n!} = e^x\sum_{j=0}^{k}\left\{\begin{matrix} k \\ j \end{matrix}\right\} x^j,$$

so that

$$(8.3)\quad {}_pF_p\left(\begin{matrix} c+1, & \cdots, & c+1 \\ c, & \cdots, & c \end{matrix}\,\middle|\, x\right) = c^{-p}e^x\, R_p(c;x),$$

where we have defined the polynomial of degree p

$$R_p(c;x) \equiv \sum_{k=0}^{p}\binom{p}{k} c^{p-k}\sum_{j=0}^{k}\left\{\begin{matrix} k \\ j \end{matrix}\right\} x^j.$$

Interchanging the order of summation in the latter, we may write

$$R_p(c;x) = \sum_{j=0}^{p} \sum_{k=j}^{p} c^{p-k} \binom{p}{k} \left\{ {k \atop j} \right\} x^j. \tag{8.4}$$

Although (8.3) is indicated in [**16**, Section 7.12.4, page 593], Prudnikov et al. do not provide the explicit formula (8.4) for $R_p(c;x)$ but only give a recurrence relation by which these polynomials may be computed. Thus, from (8.2) and (8.3), we have

$$ {}_{r+1}F_{r+1}\left(\begin{matrix} -r-1, & (\xi_r+1) \\ c, & (\xi_r) \end{matrix} \,\middle|\, -x \right) = c^{-r-1} R_{r+1}(c;x),$$

where the (ξ_r) are the nonvanishing zeros of the associated parametric polynomial alluded to above.

We remark that when $c = 1$, since [**5**, (6.15), page 265],

$$\sum_{k=0}^{p} \binom{p}{k} \left\{ {k \atop j} \right\} = \left\{ {p+1 \atop j+1} \right\},$$

we find

$$R_p(1;x) = \sum_{j=0}^{p} \left\{ {p+1 \atop j+1} \right\} x^j,$$

so that, from (8.3),

$$ {}_pF_p\left(\begin{matrix} 2, \dots, 2 \\ 1, \dots, 1 \end{matrix} \,\middle|\, x \right) = e^x \sum_{j=0}^{p} \left\{ {p+1 \atop j+1} \right\} x^j. \tag{8.5}$$

Equation (8.5) is recorded in [**16**] in an equivalent form along with the particular cases $1 \le p \le 7$.

The analogous special case when $m_j = 1$, $f_j = c$ $(1 \le j \le r)$ and $a = b = c$ in the transformations (1.3) and (1.4), so that $\lambda = -r$ in both cases, is discussed in [**12**], where it is shown that explicit representations for the polynomials of degree r on the right-hand sides of these transformations can be derived.

In this investigation we have developed an essentially elementary algebraic method for obtaining transformation and summation formulas, respectively, for generalized hypergeometric functions and series of unit arguments with integral parameter differences. The salient feature employed herein is Lemma 4, whereby, under mild restrictions, such hypergeometric functions and series can be written in a useful way as a finite sum of Gauss or confluent functions. We have provided several examples to indicate the efficiency and power of this method.

ENDNOTES

1. We must assume $\sigma \neq 1$ for otherwise this transformation degenerates to a summation formula.

2. The following are necessary conditions for the nonvanishing of the (η_m); sufficient conditions are given below.

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