ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 5, 2012

RINGS WHOSE TOTAL GRAPHS HAVE GENUS AT MOST ONE

HAMID REZA MAIMANI, CAMERON WICKHAM
AND SIAMAK YASSEMI

ABSTRACT. Let R be a commutative ring with $\mathrm{Z}(R)$ its set of zero-divisors. In this paper, we study the total graph of R, denoted by $\mathrm{T}(\Gamma(R))$. It is the (undirected) graph with all elements of R as vertices and, for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in \mathrm{Z}(R)$. We investigate properties of the total graph of R and determine all isomorphism classes of finite commutative rings whose total graph has genus at most one (i.e., a planar or toroidal graph). In addition, it is shown that, given a positive integer g, there are only finitely many finite rings whose total graph has genus g.

1. Introduction. Let R be a commutative ring with non-zero unity. Let $\mathrm{Z}(R)$ be the set of zero-divisors of R. The concept of the graph of zero divisors of R was first introduced by Beck [**6**], where he was mainly interested in colorings. In his work all elements of the ring were vertices of the graph. This investigation of colorings of a commutative ring was then continued by D.D. Anderson and Naseer in [**2**]. In [**5**], D.F. Anderson and Livingston associate a graph, $\Gamma(R)$, to R with vertices $\mathrm{Z}(R) \setminus \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in \mathrm{Z}(R) \setminus \{0\}$, vertices x and y are adjacent if and only if $xy = 0$.

An interesting question was proposed by D.F. Anderson, et al. [**4**]: For which finite commutative rings R is $\Gamma(R)$ planar? A partial answer was given in [**1**], but the question remained open for local rings of order 32. In [**12**] and then independently in [**7, 13**] it is shown that there is no ring of order 32 whose zero-divisor graph is planar.

2010 AMS *Mathematics subject classification.* Primary 05C75, 13A15.
Keywords and phrases. Total graph, genus, planar graph, toroidal graph.
The research of the first author was in part supported by a grant from IPM (No. 88050214). The research of the third author was in part supported by a grant from IPM (No. 88130213).
Received by the editors on August 17, 2009, and in revised form on January 29, 2010.
DOI:10.1216/RMJ-2012-42-5-1551

The genus of a graph is the minimal integer n such that the graph can be drawn without crossing itself on a sphere with n handles (i.e., an oriented surface of genus n). Thus, a planar graph has genus zero, because it can be drawn on a sphere without self-crossing. In [**13, 16**] the rings whose zero-divisor graphs have genus one are studied. A genus one graph is called a toroidal graph. In other words, graph G is toroidal if it can be embedded on the torus; that means the graph's vertices can be placed on a torus such that no edges cross. Usually, it is assumed that G is also nonplanar. In [**17**] it is shown that, for a positive integer g, there are only finitely many finite rings whose zero-divisor graph has genus g.

In [**3**], D.F. Anderson and Badawi introduced the total graph of R, denoted by $\mathrm{T}(\Gamma(R))$, as the graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in \mathrm{Z}(R)$.

In this paper, we investigate properties of the total graph of R and determine all isomorphism classes of finite rings whose total graph has genus at most one (i.e., a planar or toroidal graph). In addition, we show that, for a positive integer g, there are only finitely many finite rings whose total graph has genus g.

1. Main result. A *complete graph* is a graph in which each pair of distinct vertices is joined by an edge. We denote the complete graph with n vertices by K_n. A *bipartite graph* is a graph such that its vertex set can be partitioned into two subsets V_1 and V_2, and each edge joins a vertex of V_1 to a vertex of V_2. A *complete bipartite graph* is a bipartite graph such that each vertex in V_1 is joined by an edge to each vertex in V_2 and is denoted by $K_{m,n}$ when $|V_1| = m$ and $|V_2| = n$. A *clique* of a graph is a maximal complete subgraph. For a graph G, the *degree* of a vertex v in G, denoted $\deg(v)$, is the number of edges of G incident with v. The number $\delta(G) = \min\{\deg(v) \mid v \text{ is a vertex of } G\}$ is the *minimum degree* of G. For a nonnegative integer k, a graph is called *k-regular* if every vertex has degree k. Recall that a graph is said to be *connected* if, for each pair of distinct vertices v and w, there is a finite sequence of distinct vertices $v = v_1, \dots, v_n = w$ such that each pair $\{v_i, v_{i+1}\}$ is an edge. Such a sequence is said to be a *path*, and the *distance* $d(v, w)$ between connected vertices v and w is the length of the shortest path connecting them. For any graph G, the disjoint union of k copies of G is denoted kG. Let S be a nonempty subset of vertex set of graph G. The *subgraph induced by* S is the subgraph with the vertex set S and with any edges whose endpoints are both in the S and is denoted by $\langle S \rangle$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertices set V_i and edges set E_i. The cartesian product of G_1 and G_2 is denoted by $G = G_1 \times G_2$ with vertices set $V_1 \times V_2$, and (x, y) is adjacent to (x', y') if $x = x'$ and y is adjacent y' in G_2 or $y = y'$ and x is adjacent to x' in G_1.

Lemma 1.1. *Let x be a vertex of $\mathrm{T}(\Gamma(R))$. Then the degree of x is either $|\mathrm{Z}(R)|$ or $|\mathrm{Z}(R)| - 1$. In particular, if $2 \in \mathrm{Z}(R)$, then $\mathrm{T}(\Gamma(R))$ is a $(|\mathrm{Z}(R)| - 1)$-regular graph.*

Proof. If x is adjacent to y, then $x + y = a \in \mathrm{Z}(R)$, and hence $y = a - x$ for some $a \in \mathrm{Z}(R)$. We have two cases:

Case 1. Suppose $2x \in \mathrm{Z}(R)$. Then x is adjacent to $a - x$ for any $a \in \mathrm{Z}(R) \setminus \{2x\}$. Thus, the degree of x is $|\mathrm{Z}(R)| - 1$. In particular, if $2 \in \mathrm{Z}(R)$, then $\mathrm{T}(\Gamma(R))$ is a $(|\mathrm{Z}(R)| - 1)$-regular graph.

Case 2. Suppose $2x \notin \mathrm{Z}(R)$. Then x is adjacent to $a - x$ for any $a \in \mathrm{Z}(R)$. Thus, the degree of x is $|Z(R)|$. □

Let S_k denote the sphere with k handles, where k is a nonnegative integer, that is, S_k is an oriented surface of genus k. The genus of a graph G, denoted by $\gamma(G)$, is the minimum integer n such that the graph can embedded in S_n. A graph G is called planar if $\gamma(\Gamma(G)) = 0$, and toroidal if $\gamma(\Gamma(G)) = 1$. We note here that, if H is a subgraph of a graph G, then $\gamma(H) \le \gamma(G)$.

In the following theorem we bring some well-known formulas, see, e.g., [**14, 15**]:

Theorem 1.2. *The following statements hold*:

(a) *For $n \ge 3$ we have $\gamma(K_n) = \lceil [(n-3)(n-4)]/12 \rceil$.*

(b) *For $m, n \ge 2$ we have $\gamma(K_{m,n}) = \lceil [(m-2)(n-2)]/4 \rceil$.*

(c) *Let G_1 and G_2 be two graphs and, for each i, p_i the number of vertices of G_i. Then* $\max\{p_1\gamma(G_2) + \gamma(G_1), p_2\gamma(G_1) + \gamma(G_2)\} \le \gamma(G_1 \times G_2)$.

According to Theorem 1.2, we have $\gamma(K_n) = 0$ for $1 \le n \le 4$ and $\gamma(K_n) = 1$ for $5 \le n \le 7$ and, for other values of n, $\gamma(K_n) \ge 2$.

Lemma 1.3. *Let $\mathbf{F}_q$ denote the field with q elements. Then the total graph of $\mathbf{F}_2 \times \mathbf{F}_q$ is isomorphic to $K_2 \times K_q$. Furthermore, for any positive integer m and $q > 2$,*

$$\gamma(T(\Gamma(\mathbf{F}_{2^m} \times \mathbf{F}_q))) \ge 2^m \left\lceil \frac{(q-3)(q-4)}{12} \right\rceil.$$

Proof. We induct on m. If $m = 1$, let $V_0 = \{(0, y) \mid y \in \mathbf{F}_q\}$, and let $V_1 = \{(1, y) \mid y \in \mathbf{F}_q\}$. Then the subgraphs G_i of the total graph of $\mathbf{F}_2 \times \mathbf{F}_q$ induced by each of the V_i is K_q. Now, for each $y \in \mathbf{F}_q$, there is an edge between $(0, y) \in G_0$ and $(1, -y) \in G_1$. Furthermore, these are the only other edges in the total graph. Identifying $(1, -y)$ with $(1, y)$, we can replace G_1 with an isomorphic copy G_1'; under this isomorphism, the edge between $(0, y) \in G_0$ and $(1, -y) \in G_1$ is the edge between $(0, y) \in G_0$ and $(1, y) \in G_1'$. Thus, the total graph of $\mathbf{F}_2 \times \mathbf{F}_q$ has vertex set $\{(x, y) \mid x \in \mathbf{F}_2 \text{ and } y \in \mathbf{F}_q\}$, with an edge between (x, y) and (x', y') if $x = x'$ and $y \ne y'$, or $y = y'$ and $x \ne x'$. That is, it is the graph $K_2 \times K_q$. Parts (a) and (c) of Theorem 1.2 now yield

$$\gamma(T(\Gamma(\mathbf{F}_2 \times \mathbf{F}_q))) = \gamma(K_2 \times K_q) \ge 2 \left\lceil \frac{(q-3)(q-4)}{12} \right\rceil.$$

If $m > 1$, we can partition $\mathbf{F}_{2^m}$ into two sets, S_1 and S_2, each of cardinality 2^{m-1}; let f be a bijection from S_1 to S_2. Since each element of a field of characteristic 2 is its own inverse, then the subgraph of $T(\Gamma(\mathbf{F}_{2^m} \times \mathbf{F}_q))$ induced by $S_i \times \mathbf{F}_q$ is isomorphic to $T(\Gamma(\mathbf{F}_{2^{m-1}} \times \mathbf{F}_q))$. For any $y \in \mathbf{F}_q$ and $s \in S_1$, the element (s, y) is adjacent to $(f(s), -y)$. We thus have a copy of $K_2 \times T(\Gamma(\mathbf{F}_{2^{m-1}} \times \mathbf{F}_q))$ as a subgraph of $T(\Gamma(\mathbf{F}_{2^m} \times \mathbf{F}_q))$. Part (c) of Theorem 1.2 and the induction hypothesis now yield

$$\begin{aligned} \gamma(T(\Gamma(\mathbf{F}_{2^m} \times \mathbf{F}_q))) &\ge \gamma(K_2 \times T(\Gamma(\mathbf{F}_{2^{m-1}} \times \mathbf{F}_q))) \\ &\ge 2\gamma(T(\Gamma(\mathbf{F}_{2^{m-1}} \times \mathbf{F}_q))) \\ &\ge 2^m \left\lceil \frac{(q-3)(q-4)}{12} \right\rceil. \qquad \square \end{aligned}$$

A *subdivision* of a graph is a graph obtained from it by replacing edges with pairwise internally disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. The Kuratowski theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (see [**3**, page 153]). In addition, every planar graph has a vertex v such that $\deg(v) \leq 5$.

Theorem 1.4. *For any positive integer g, there are finitely many finite rings R whose total graph has genus g.*

Proof. Let R be a finite ring. If R is local, then $Z(R)$ is the maximal ideal of R and $|R| \leq |Z(R)|^2$. If R is not local, then $R = R_1 \times R_2 \times \cdots \times R_n$ where each of the R_i's is a local ring and $n \geq 2$ [**10**]. Suppose that $|R_1| \leq |R_2| \leq \cdots \leq |R_n|$, and set $R_1^* = 0 \times R_2 \times \cdots R_n$. Since $|R| = |R_1||R_1^*|$, we conclude that $|R| \leq |R_1^*|^2$. Let S denote either $Z(R)$ if R is local or R_1^* if R is not local. Then every pair of elements of S is adjacent in $\mathrm{T}(\Gamma(R))$, and hence we have a complete graph $K_{|S|}$ in the structure of $\mathrm{T}(\Gamma(R))$. This implies that $\gamma(K_{|S|}) \leq g$. Therefore, $\lceil ((|S|-3)(|S|-4))/12 \rceil \leq g$, so $|S| \leq (7+\sqrt{49+48(g-1)})/2$, and hence $|R| \leq ((7+\sqrt{49+48(g-1)})/2)^2$. $\square$

Theorem 1.5. *Let R be a finite ring such that $\mathrm{T}(\Gamma(R))$ is planar. Then the following hold*:

(a) *If R is a local ring, then R is a field or R is isomorphic to the one of the* 9 *following rings*:

$$\mathrm{Z}_4,\ \frac{\mathrm{Z}_2[X]}{(X^2)},\ \frac{\mathrm{Z}_2[X]}{(X^3)},\ \frac{\mathrm{Z}_2[X,Y]}{(X,Y)^2},\ \frac{\mathrm{Z}_4[X]}{(2X,X^2)},$$
$$\frac{\mathrm{Z}_4[X]}{(2X,X^2-2)},\ \mathrm{Z}_8,\ \mathbf{F}_4[X](X^2),\ \frac{\mathrm{Z}_4[X]}{(X^2+X+1)}.$$

(b) *If R is not local ring, then R is an infinite integral domain or R is isomorphic to $\mathrm{Z}_2 \times \mathrm{Z}_2$ or Z_6.*

Proof. Any planar graph has a vertex v with $\deg(v) \leq 5$. So, if the total graph of R is planar, then $\delta(\mathrm{T}(\Gamma(R))) \leq 5$. By Lemma 1.1, $\delta(\mathrm{T}(\Gamma(R))) = |Z(R)|$ or $|Z(R)|-1$, and hence $|Z(R)| \leq 6$.

(a) Assume that R is a local ring, and let $n = |Z(R)|$ and $m = |R/Z(R)|$. If $2 \in Z(R)$, then $\mathrm{T}(\Gamma(R)) \cong mK_n$ ([**3**, Theorems 2.1 and 2.2]). Hence, $|Z(R)| \leq 4$. Also, $|R| = 2^k$ since $2 \in Z(R)$. So $|R| = 16, 8, 4$, or 2. According to Corbas and Williams [**8**], there are two nonisomorphic rings of order 16 with maximal ideals of order 4, namely, $\mathbf{F}_4[x]/(x^2)$ and $\mathrm{Z}_4[x]/(x^2+x+1)$ (see also Redmond [**11**]), so for these rings we have $\mathrm{T}(\Gamma(R)) \cong 4K_4$. Since K_4 is planar we conclude that the total graphs of these rings are planar. In [**8**] it is also shown that there are 5 local rings of order 8 (except F_8). In all of these rings, we have $|Z(R)| = 4$ and hence $\mathrm{T}(\Gamma(R)) \cong 2K_4$. Also, there are two non-isomorphic local rings of order 4; these are Z_4 and $\mathrm{Z}_2[X]/(X^2)$. For both we have $\mathrm{T}(\Gamma(R)) \cong 2K_2$, and thus they are planar. Note that, if $|Z(R)| = 1$, then R is a field and hence the total graph is planar. If $2 \notin Z(R)$, then $\mathrm{T}(\Gamma(R)) \cong K_n \cup ((m-1)/2)K_{n,n}$ ([**3**, Theorem 2.2]). This implies $n \leq 2$, and thus R either has order 4 or is a field.

(b) Suppose that R is not local ring. Since R is finite, then there are finite local rings R_i such that $R = R_1 \times \cdots \times R_t$ where $t \geq 2$. Since $|Z(R)| \leq 6$, then we have the following candidates:

$$\mathrm{Z}_2 \times \mathrm{Z}_2,\ \mathrm{Z}_6,\ \mathrm{Z}_2 \times \mathbf{F}_4,\ \mathrm{Z}_2 \times \mathrm{Z}_4,\ \mathrm{Z}_2 \times \frac{\mathrm{Z}_2[X]}{(X^2)},$$
$$\mathrm{Z}_2 \times \mathrm{Z}_5,\ \mathrm{Z}_3 \times \mathrm{Z}_3,\ \mathrm{Z}_3 \times \mathbf{F}_4.$$

The total graph of $\mathrm{Z}_2 \times \mathrm{Z}_2$ is isomorphic to the cycle C_4, and this graph is planar. By Lemma 1.3, the total graph of $\mathrm{Z}_6 \cong \mathrm{Z}_2 \times \mathrm{Z}_3$ is isomorphic to $K_2 \times K_3$, which is also planar.

Let R be a ring with $|R| = n$. The subgraph of the total graph of $\mathrm{Z}_2 \times R$ induced by the set $\{0\} \times R$ is a copy of K_n. The edge $(1,0) - (0,0)$, together with the paths $(1,0) - (1,-r) - (0,r)$ for each $r \in R$ yield a subdivision of K_{n+1} in the total graph of $\mathrm{Z}_2 \times R$. Thus, the total graphs of $\mathrm{Z}_2 \times \mathbf{F}_4$, $\mathrm{Z}_2 \times \mathrm{Z}_4$, $\mathrm{Z}_2 \times \mathrm{Z}_2[X]/(X^2)$ and $\mathrm{Z}_2 \times \mathrm{Z}_5$ are not planar. Also, the total graph of $\mathrm{Z}_3 \times R$ contains a subgraph which is isomorphic to $K_{3,n}$ (consider the induced subgraph $\langle S \rangle$ where $S = \{(1,r) \mid r \in R\} \cup \{(2,r) \mid r \in R\}$). Thus, the total graphs of $\mathrm{Z}_3 \times \mathrm{Z}_3$ and $\mathrm{Z}_3 \times \mathbf{F}_4$ are not planar. $\square$

Theorem 1.6. *Let R be a finite ring such that* $\mathrm{T}(\Gamma(R))$ *is toroidal. Then the following statements hold*:

(a) *If R is a local ring, then R is isomorphic to* Z_9, *or* $\mathrm{Z}_3[X]/(X^2)$.

(b) *If R is not a local ring, then R is isomorphic to one of the following rings*:

$$\mathrm{Z}_2 \times \mathbf{F}_4,\ \mathrm{Z}_3 \times \mathrm{Z}_3,\ \mathrm{Z}_2 \times \mathrm{Z}_4,\ \mathrm{Z}_2 \times \frac{\mathrm{Z}_2[x]}{(x^2)},\ \mathrm{Z}_2 \times \mathrm{Z}_2 \times \mathrm{Z}_2.$$

Proof. For any graph G with ν vertices and genus g, we have $\delta(G) \leq 6 + (12g - 12)/\nu$. If $\gamma(G) = 1$, then $\delta(G) \leq 6$ and equality holds if and only if G is a triangulation of the torus and 6-regular (see [**16**, Proposition 2.1]). If R is a finite ring with toroidal total graph, then $\delta(\mathrm{T}(\Gamma(R))) \leq 6$, and by Lemma 1.1 $\delta(\mathrm{T}(\Gamma(R))) = |Z(R)|$ or $|Z(R)| - 1$. Thus, we conclude that $|Z(G)| \leq 7$.

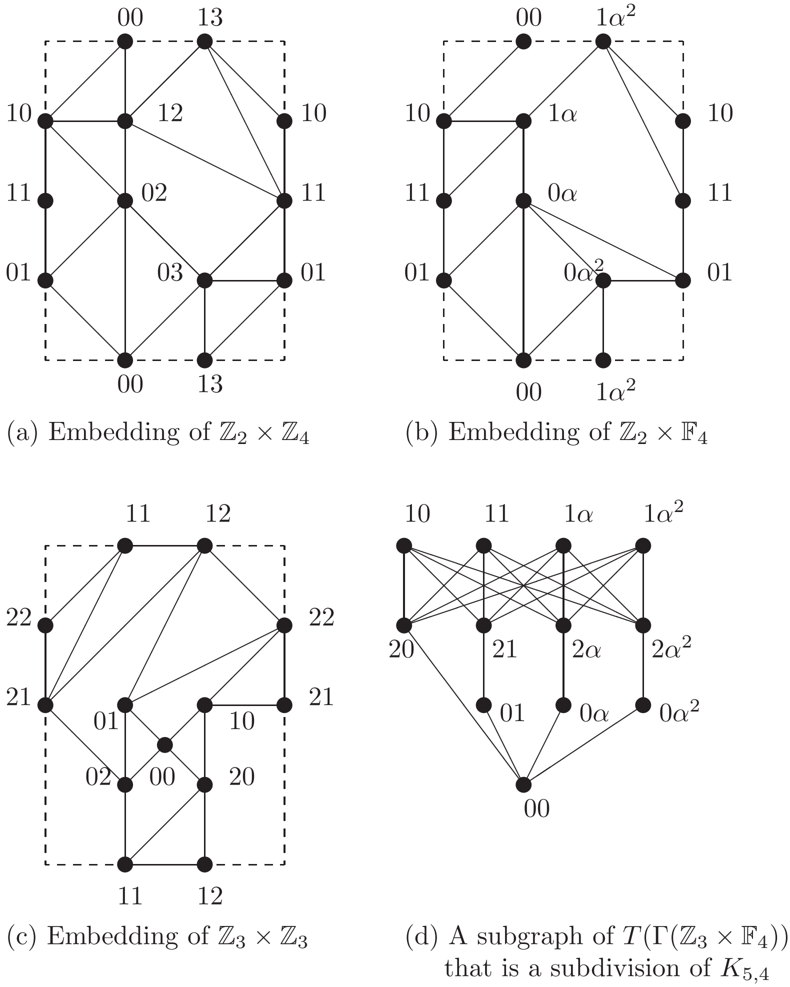
(a) Let R be a local ring. If $2 \in Z(R)$, then $\mathrm{T}(\Gamma(R))$ is a disjoint union of copies of the complete graph K_n, where $|Z(R)| = n$. Hence, $5 \leq n \leq 7$. But, in this case, $|Z(R)|$ is a power of 2, and thus there are no such local rings. Now suppose that $2 \notin Z(R)$. Then $T(\Gamma(R)) \cong K_n \cup ((m-1)/2)K_{n,n}$, where $n = |Z(R)|$ and $m = |R/Z(R)|$. Thus, $3 \leq n \leq 4$, and since $2 \notin Z(R)$, we must have $n = 3$. There are two local rings, Z_9 and $\mathrm{Z}_3[X]/(X^2)$, such that the cardinality of the set of zero-divisors is 3; both of these rings have total graph $K_3 \cup K_{3,3}$ which is toroidal.

(b) Assume that R is not a local ring. Since $|\mathrm{Z}(R)| \leq 7$, we have the following candidates for R by Theorem 1.5 (b):

$$\mathrm{Z}_2 \times \mathbf{F}_4,\ \mathrm{Z}_2 \times \mathrm{Z}_4,\ \mathrm{Z}_2 \times \frac{\mathrm{Z}_2[x]}{(x^2)},$$
$$\mathrm{Z}_2 \times \mathrm{Z}_5,\ \mathrm{Z}_3 \times \mathrm{Z}_3,\ \mathrm{Z}_3 \times \mathbf{F}_4,\ \mathbf{F}_4 \times \mathbf{F}_4,\ \mathrm{Z}_2 \times \mathrm{Z}_2 \times \mathrm{Z}_2.$$

By Theorem 1.5, $\gamma(\mathrm{T}(\Gamma(\mathrm{Z}_2 \times \mathbf{F}_4)))$, $\gamma(\mathrm{T}(\Gamma(\mathrm{Z}_2 \times \mathrm{Z}_4)))$, $\gamma(\mathrm{T}(\Gamma(\mathrm{Z}_3 \times \mathrm{Z}_3)))$ are all at least 1. The embeddings in Figure 1, parts (a), (b) and (c), show explicitly that $\gamma(\mathrm{T}(\Gamma(\mathrm{Z}_2 \times \mathbf{F}_4))) = \gamma(\mathrm{T}(\Gamma(\mathrm{Z}_2 \times \mathrm{Z}_4))) = \gamma(\mathrm{T}(\Gamma(\mathrm{Z}_3 \times \mathrm{Z}_3))) = 1$. Since $\mathrm{T}(\Gamma(\mathrm{Z}_2 \times \mathrm{Z}_2[x]/(x^2))) \cong \mathrm{T}(\Gamma(\mathrm{Z}_2 \times \mathrm{Z}_4))$, then $\gamma(\mathrm{T}(\Gamma(\mathrm{Z}_2 \times \mathrm{Z}_2[x]/(x^2)))) = 1$.

If we partition the elements of $\mathrm{Z}_2 \times \mathrm{Z}_2 \times \mathrm{Z}_2$ by the four sets $V_1 = \{(0,0,0),(1,1,1)\}$, $V_2 = \{(1,0,0),(0,1,1)\}$, $V_3 = \{(0,1,0),(1,0,1)\}$ and $V_4 = \{(0,0,1),(1,1,0)\}$, it is clear that $\mathrm{T}(\Gamma(\mathrm{Z}_2 \times \mathrm{Z}_2 \times \mathrm{Z}_2)) = K_{2,2,2,2}$. Hence, $\gamma(\mathrm{Z}_2 \times \mathrm{Z}_2 \times \mathrm{Z}_2) = 1$ by [**9**, Corollary 4].



(a) Embedding of $\mathbb{Z}_2 \times \mathbb{Z}_4$

(b) Embedding of $\mathbb{Z}_2 \times \mathbb{F}_4$

(c) Embedding of $\mathbb{Z}_3 \times \mathbb{Z}_3$

(d) A subgraph of $T(\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4))$ that is a subdivision of $K_{5,4}$

FIGURE 1. Embeddings in the torus and a subgraph of $T(\Gamma(\mathrm{Z}_3 \times \mathbf{F}_4))$.

By Lemma 1.3, $\gamma(T(\Gamma(\mathrm{Z}_2 \times \mathrm{Z}_5))) \geq 2$ and thus is not toroidal. By (the proof of) Lemma 1.3, $\gamma(T(\Gamma(\mathbf{F}_4 \times \mathbf{F}_4))) \geq 2\gamma(T(\Gamma(\mathrm{Z}_2 \times \mathbf{F}_4)))$, and so by Theorem 1.5, $T(\Gamma(\mathbf{F}_4 \times \mathbf{F}_4))$ is not toroidal.

Finally, Figure 1 (d) shows a subgraph of $T(\Gamma(\mathrm{Z}_3 \times \mathbf{F}_4))$ that is a subdivision of $K_{5,4}$, and thus by Theorem 1.2, $\gamma(T(\Gamma(\mathrm{Z}_3 \times \mathbf{F}_4))) \geq 2$. $\square$

Acknowledgments. The authors wish to thank the referee for very detailed and useful comments.

REFERENCES

1. S. Akbari, H.R. Maimani and S. Yassemi, *When a zero-divisor graph is planar or a complete r-partite graph*, J. Algebra **270** (2003), 169–180.

2. D.D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra **159** (1993), 500–514.

3. D.F. Anderson and A. Badawi, *The total graph of a commutative ring*, J. Algebra **320** (2008), 2706–2719.

4. D.F. Anderson, A. Frazier, A. Lauve and P.S. Livingston, *The zero-divisor graph of a commutative ring*, II, in *Ideal theoretic methods in commutative algebra* Dekker, New York, 2001.

5. D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), 434–447.

6. I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), 208–226.

7. R. Belshoff and J. Chapman, *Planar zero-divisor graphs*, J. Algebra **316** (2007), 471–480.

8. B. Corbas and G.D. Williams, *Ring of order p^5* II. *Local rings*, J. Algebra **231** (2000), 691–704.

9. D.L. Craft, *On the genus of joins and composition of graphs*, Discrete Math. **178** (1998), 25–50.

10. B.R. McDonald, *Finite rings with identity*, Pure Appl. Math. **28**, Marcel Dekker, Inc., New York, 1974.

11. S.P. Redmond, *On zero-divisor graphs of small finite commutative rings*, Discrete Math. **307** (2007), 1155–1166.

12. Neal O. Smith, *Planar zero-divisor graphs*, Internat. J. Commutative Rings **2** (2002), 177–188.

13. H. Wang, *Zero-divisor graphs of genus one*, J. Algebra **304** (2006), 666–678.

14. A.T. White, *Graphs, groups and surfaces*, North-Holland, Amsterdam, 1973.

15. ———, *The genus of the Cartesian product of two graphs*, J. Combin. Theory **11** (1971), 89–94.

16. C. Wickham, *Classification of rings with genus one zero-divisor graphs*, Comm. Algebra **36** (2008), 325–345.

17. C. Wickham, *Rings whose zero-divisor graphs have positive genus*, J. Algebra **321** 2009, 377–383.

Department of Mathematics, Shahid Rajaee Teacher Training University, Tehran, Iran and Institute for Research in Fundamental Sciences (IPM), Tehran, Iran
Email address: maimani@ipm.ir

Mathematics Department, Missouri State University, Springfield, MO 65897
Email address: cwickham@missouristate.edu

Department of Mathematics, University of Tehran, Tehran, Iran and Institute for Research in Fundamental Sciences (IPM), Tehran, Iran
Email address: yassemi@ipm.ir