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ON ESSENTIAL SPECTRA OF LINEAR RELATIONS AND QUOTIENT INDECOMPOSABLE NORMED SPACES

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ABSTRACT. We introduce several essential spectra of a linear relation on a normed space. We investigate the closedness and the emptiness of such essential spectra. As an application we prove two results, the first of which characterizes the class of quotient indecomposable normed spaces in terms of F_- and strictly cosingular linear relations, and the second gives conditions under which a linear relation on a complex quotient indecomposable normed space is a strictly cosingular perturbation of a multiple of the identity.

1. Introduction. A Banach space E is said to be *indecomposable* if it does not contain any pair of closed infinite dimensional subspaces M, N such that $E = M \oplus N$. It is hereditarily indecomposable if every closed subspace of E is indecomposable, and it is quotient indecomposable if every quotient is indecomposable. In [**9**], Gowers and Maurey gave the first known example of a hereditarily indecomposable Banach space X_{GM}. Moreover, they showed that if E is a complex hereditarily indecomposable Banach space, then every bounded operator on E can be written as $\lambda I + S$, where $\lambda \in \mathbf{C}$ and S is strictly singular. Recently, Álvarez [**2**] extended this property to the case of multi-valued linear operators in normed spaces.

Ferenczi [**7**] proved that the space X_{GM} is a quotient indecomposable Banach space. If the dual E' of a Banach space E is hereditarily indecomposable or quotient indecomposable, then E is quotient indecomposable or hereditarily indecomposable, respectively. Since the space X_{GM} is hereditarily indecomposable and reflexive [**9**], X'_{GM} is quotient indecomposable.

Aiena and González proved in [**1**] that every bounded operator on a complex quotient indecomposable Banach space is of the form $\lambda I + S$,

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where $\lambda \in \mathbf{C}$ and S is strictly cosingular. This property together with the analogous property for hereditarily indecomposable Banach spaces [**9**] receives special attention for its connection with the invariant subspace problem. This connection is a motivation for finding general properties of hereditarily and quotient indecomposable normed spaces.

In this paper we analyse the validity of the above result of Aiena and González [**1**] in the context of linear relations.

In Section 2 we introduce three essential spectra of a linear relation. The closedness of such essential spectra is established, and we also give conditions under which these essential spectra are non-empty subsets.

A result, essentially due to Weis [**12**], characterizes the quotient indecomposable Banach spaces as those spaces F such that, for every Banach space E, any bounded operator from E into F is either F_- or strictly cosingular. In Section 3 we give a generalization of this result to multi-valued linear operators in normed spaces. This generalization will be used in conjunction with spectral properties for linear relations proved in Section 2 to obtain Theorem 20 following which generalises a similar result of Aiena and González [**1**] for bounded operators in Banach spaces.

Notations. We adhered to the notation and terminology of the book [**5**]: Let $X, Y, \ldots$ denote infinite-dimensional normed spaces over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and X' is the dual space of X. Let $\mathcal{E}(X)$ denote the class of all closed infinite codimensional subspaces of X. If $M \subset X$ and $N \subset X'$, then $M^{\perp} := \{x' \in X' : x'(x) = 0, x \in M\}$ and $N^{\top} := \{x \in X : x'(x) = 0, x' \in N\}$.

A linear relation or multi-valued linear operator $T : X \to Y$ is a mapping from a subspace $D(T) \subset X$, called the domain of T, into the collection of non-empty subsets of Y such that $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$ for all non-zero scalars α, β and $x_1, x_2 \in D(T)$. The class of such linear relations T is denoted by $LR(X, Y)$, and we write $LR(X) := LR(X, X)$. If T maps the points of its domain to singletons, then T is said to be *single valued* or simply an *operator*.

Let $T \in LR(X, Y)$. The graph $G(T)$ of T is defined by $G(T) := \{(x, y) \in X \times Y : x \in D(T), y \in Tx\}$ which is a subspace of $X \times Y$. The inverse of T is the linear relation T^{-1} defined by $G(T^{-1}) := \{(y, x) : (x, y) \in G(T)\}$. If T^{-1} is single valued, then T is called injective, that is, T is injective if and only if its null space $N(T) := T^{-1}(0) = \{0\}$, and

T is called surjective if its range $R(T) := T(D(T)) = Y$. The adjoint or conjugate T' of T is defined by $G(T') := G(-T^{-1})^{\perp}$.

If M is a subspace of $D(T)$, then $T\mid_M$ is defined by $G(T\mid_M) := \{(m, y) : m \in M, y \in Tm\}$, and, if M is a subspace of X such that $M \cap D(T) \neq \varnothing$, we write $T\mid_M := T\mid_{M\cap D(T)}$. The completion $\widetilde{T}$ of T is defined by $G(\widetilde{T}) := \widetilde{G(T)} \subset \widetilde{X} \times \widetilde{Y}$, where $\widetilde{X}$ denotes the completion of X. We define $\alpha(T) := \dim N(T)$; $\beta(T) := \dim Y/R(T)$; $\overline{k}(T) := \alpha(\widetilde{T}) - \alpha(T')$ if $\alpha(\widetilde{T})$ or $\alpha(T')$ are not both infinite and $k(T) := \alpha(T) - \beta(T)$ if either $\alpha(T)$ or $\beta(T)$ are finite.

For a given closed subspace M of X, let Q_M denote the natural quotient map from X onto X/M, J_M denotes the injection from M into X and J_X is the natural injection from X into its completion. We shall denote $Q_{\overline{T(0)}}$ by Q_T. Clearly $Q_T T$ is single valued. For $x \in D(T)$, $\|Tx\| := \|Q_T Tx\|$ and the norm of T is defined by $\|T\| := \|Q_T T\|$.

For a given linear relation $T \in LR(X, Y)$, we define the quantities $\Gamma'(T)$ and $\Delta'(T)$ as follows:

If Y is finite dimensional, then $\Gamma'(T) = \Delta'(T) = 0$, and, if Y is infinite dimensional, then

$$\Gamma'(T) := \inf\{\|Q_M J_Y T\| : M \in \mathcal{E}(\widetilde{Y})\}$$
$$\Delta'(T) := \sup\{\Gamma'(Q_M T) : M \in \mathcal{E}(Y)\}.$$

A linear relation T is said to be *closed* if its graph is closed, *continuous* if $\|T\| < \infty$, *bounded* if it is everywhere defined and *continuous, open* if its inverse is continuous equivalently if $\gamma(T) := \sup\{\lambda \geq 0 : \lambda\, d(x, N(T)) \leq \|Tx\|, x \in D(T)\} > 0$, *partially continuous* if a finite codimensional subspace M of X exists for which $T\mid_M$ is continuous, F_+ if there is a finite codimensional subspace M of X such that $T\mid_M$ is injective and open, ϕ_+ if $R(T)$ is closed and $N(T)$ is finite dimensional, F_- if T' is F_+, ϕ_- if $R(T)$ is a closed finite codimensional subspace of Y and *strictly cosingular* if $\Delta'(T) = 0$.

The classes of partially continuous, F_+, ϕ_+, F_-, ϕ_- and strictly cosingular linear relations from X into Y will be denoted by $PB(X, Y)$, $F_+(X, Y)$, $\phi_+(X, Y)$, $F_-(X, Y)$, $\phi_-(X, Y)$ and $SC(X, Y)$, respectively.

Let $S, T \in LR(X, Y)$, and let $\alpha \in \mathbf{K}$. Relations $S + T$ and αT are defined by $G(S + T) := \{(x, y) \in X \times Y : y = s + t, (x, s) \in$

$G(S), (x,t) \in G(T)\}$ and $G(\alpha T) := \{(x, \alpha y) \in X \times Y : (x,y) \in G(T)\}$. Let $U \in LR(X,Y)$ and $V \in LR(Y,Z)$ where $R(U) \cap D(V) \neq \varnothing$. The composition or product VU is the linear relation defined by $G(VU) := \{(x,z) \in X \times Z :$ there exists $y \in Y, (x,y) \in G(U), (y,z) \in G(V)\}$.

Linear relations made their first appearance in functional analysis in von Neumann [**11**], motivated by the need to consider adjoints of non-densely defined linear differential operators. The adjoints of such operators are linear relations. One main reason why multi-valued linear operators are more a convenient means than operators is that one can define the inverse, the closure and the completion for a linear relation.

The articles of Baskakov and Chernyshov, [**3, 4**], survey some applications of the spectral theory of linear relations to important problems of operator theory. We cite some of them:

1. The pseudoresolvent theory of operators. We note that any pseudoresolvent of a single valued relation is the resolvent of a certain linear relation.

2. The spectral theory of ordered pairs of operators. Many properties of the spectrum of the pair (G, F) of closed operators are obtained as an application of spectral properties of the linear relations $F^{-1}G$ and GF^{-1}.

3. The solvability of the Cauchy problem. Let us consider the Cauchy problem

$$x(0) = x_0 \in X$$

for homogeneous linear differential equation

$$Fx'(t) = Gx(t), \quad t \in [0, \infty)$$

with the pair of closed operators G, F between Banach spaces under the condition $N(F) \neq \{0\}$. The spectral theory of multi-valued linear operators plays an important role in the solvability and in the construction of solutions to the above equation.

4. The study of linear bundles. Let $T, S : X \to Y$ be bounded operators. The map $P(\lambda) := T + \lambda S$, $\lambda \in \mathbf{C}$ is called a linear bundle. It is known that many problems of mathematical physics are reduced to the study of the reversibility conditions of operators $P(\lambda)$, $\lambda \in \mathbf{C}$. The investigation of linear bundles is reduced to the study of spectral properties of linear relations $S^{-1}T$ and TS^{-1}.

It is interesting to note that the investigation of essential spectra of a linear relation may provide a useful guide to the study of operators in normed spaces since every continuous operator between normed spaces is the inverse of an injective F_+-relation, and the class of all bounded Fredholm operators in Banach spaces coincides with the class of inverses of closed Fredholm linear relations which are both surjective and injective.

2. Essential spectra of linear relations. Throughout this section T will denote an element of $LR(X)$ where X is a complex normed space. We shall write $\lambda - T := \lambda I - T$ and $T_\lambda := (\lambda - \widetilde{T})^{-1}$.

Definition 1 ([**5**, VI.1.1]). The resolvent set of T is the set

$$\rho(T) := \{\lambda \in \mathbf{C} : T_\lambda \text{ is everywhere defined and single valued}\}.$$

The spectrum of T is the set $\sigma(T) := \mathbf{C} \setminus \rho(T)$.

It is clear from the closed graph theorem for operators that T_λ is a bounded single valued defined on $\widetilde{X}$ if and only if $\lambda \in \rho(T)$. Therefore, our definition of resolvent set coincides with the standard definition for bounded or closed operators in Banach spaces.

There are many definitions of essential spectra in operator theory. Five of these are studied in Edmunds and Evans [**6**]. We generalize three of these to linear relations.

Definition 2. The essential resolvents $\rho_{e+}(T)$, $\rho_{e-}(T)$ and $\rho_e(T)$ of T are defined as follows:

$$\begin{aligned}
\rho_{e+}(T) &:= \{\lambda \in \mathbf{C} : \lambda - \widetilde{T} \in \phi_+(X)\}\\
\rho_{e-}(T) &:= \{\lambda \in \mathbf{C} : \lambda - \widetilde{T} \in \phi_-(X)\}\\
\rho_e(T) &:= \{\lambda \in \mathbf{C} : \lambda - \widetilde{T} \in \phi_+(X) \cap \phi_-(X) \text{ and } k(\lambda = \widetilde{T}) = 0\}.
\end{aligned}$$

The essential spectra of T are the sets $\sigma_{e+}(T) := \mathbf{C} \setminus \rho_{e+}(T)$, $\sigma_{e-}(T) := \mathbf{C} \setminus \rho_{e-}(T)$ and $\sigma_e(T) := \mathbf{C} \setminus \rho_e(T)$.

Proposition 3 ([**5**, V.1.7, V.1.9, V.2.4, V.15.1], [**10**, 3.7]). *We have*:

(i) *If T is closed and X is complete, then $T \in F_+(X)$ if and only if $T \in \phi_+(X)$.*

(ii) *$T \in F_-(X)$ if and only if $\widetilde{T} \in \phi_-(\widetilde{X})$. In such a case $\overline{k}(T) = k(\widetilde{T})$.*

(iii) *$T \in F_+(X)$ if and only if $\widetilde{T} \in \phi_+(\widetilde{X})$. In such a case $\overline{k}(T) = k(\widetilde{T})$.*

(iv) *If T is single valued, then T is strictly cosingular if and only if there is no $M \in \mathcal{E}(Y)$ such that $(Q_M T)'$ has a continuous inverse.*

Corollary 4. *Let $\lambda \in \mathbf{C}$. Then*

(i) *$\lambda \in \rho_{e+}(T)$ if and only if $\lambda - T \in F_+(X)$.*

(ii) *$\lambda \in \rho_{e-}(T)$ if and only if $\lambda - T \in F_-(X)$.*

(iii) *$\lambda \in \rho_e(T)$ if and only if $\lambda - T \in F_+(X) \cap F_-(X)$ and $\overline{k}(\lambda - T) = 0$.*

We conclude from Proposition 3 and Corollary 4 that the definition of essential spectra of Edmunds and Evans [**6**] coincides with our definition (Definition 2) when T is a bounded or closed operator and X is complete.

It is known (see, for example [**1**, 4.9]), that for every bounded operator T on a complex Banach space, the sets $\sigma(T)$, $\sigma_{e+}(T)$, $\sigma_{e-}(T)$ and $\sigma_e(T)$ are closed and non-empty. The corresponding properties for multi-valued linear operators will now be investigated.

Theorem 5. *For every $T \in LR(X)$, the sets $\sigma_{e+}(T)$, $\sigma_{e-}(T)$, $\sigma_e(T)$ and $\sigma(T)$ are closed.*

Proof. Assume that $\lambda \in \rho_{e+}(T) \cup \rho_{e-}(T) \cup \rho_e(T)$. Since $R(\lambda - \widetilde{T})$ is a closed subspace of $\widetilde{X}$, it follows from the open mapping theorem for linear relations [**5**, III.4.2] and from [**5**, III.4.6] that $0 < \gamma(\lambda - \widetilde{T}) = \gamma(\lambda - \widetilde{T'})$. If $|\eta - \lambda| < \gamma(\lambda - \widetilde{T})$, then by Proposition 3 and [**5**, V.3.2 and V.5.1], $\eta - \widetilde{T} \in \phi_+$. Similarly, if $\lambda - \widetilde{T} \in \phi_-$ and $|\eta - \lambda| < \gamma(\lambda - \widetilde{T'})$, then by Proposition 3 and [**5**, V.5.12], $\eta - \widetilde{T} \in \phi_-$. Therefore, $\rho_{e+}(T)$ and $\rho_{e-}(T)$ are open. Furthermore, $\overline{k}(\eta - T) = k(\eta - \widetilde{T}) = k(\lambda - \widetilde{T})$ whenever $|\eta - \lambda| < \gamma(\lambda - \widetilde{T})$ by virtue of [**5**, V.15.6]. Hence, $\rho_e(T)$ is open, as desired. Finally, that $\sigma(T)$ is closed was proved by Cross [**5**, VI.1.3]. $\square$

Proposition 6. $\sigma_{e+}(T) \cup \sigma_{e-}(T) \subset \sigma_e(T) \subset \sigma(T)$.

Proof. The result follows from Corollary 4 upon noting that the resolvent set of T coincides with the set $\{\lambda \in \mathbf{C} : \lambda - T$ is injective, open and has dense range$\}$ and that every open and injective (respectively, open with dense range) linear relation is F_+ by virtue of [**5**, V.5.1] (respectively, is F_- by [**5**, III.4.6 and V.5.2]). ◻

Example 7. A closed densely defined single valued exists whose essential spectra are empty sets.

Let $X = l_2$, and let K be the injective operator defined by

$$K : (\alpha_1, \alpha_2, \dots , \alpha_n, \dots) \in l_2 \longrightarrow (0, \alpha_1, \alpha_2/2, \dots , \alpha_n/n, \dots) \in l_2.$$

In [**5**, VI.2.7], the author proves that the inverse of K is single valued and has empty spectrum. This fact, combined with Proposition 6, yields the fact that $\sigma_{e+}(K^{-1}) = \sigma_{e-}(K^{-1}) = \sigma_e(K^{-1}) = \varnothing$. ◻

For $\mu \in \rho(T)$ and $\lambda \neq \mu$, we write $S := (\mu - \lambda)((\mu - \lambda)^{-1} - T_\mu)$.

Lemma 8. *Let $\mu \in \rho(T)$, and let $\lambda \neq \mu$. Then*

(i) $(\mu - \widetilde{T})x = (\mu - \lambda)x + \widetilde{T}(0)$ *if and only if* $x = (\mu - \lambda)T_\mu x$, $x \in D(\widetilde{T}) \setminus \{0\}$.

(ii) $\lambda - \widetilde{T} = S(\mu - \widetilde{T})$.

(iii) $N(\lambda - \widetilde{T}) = N(S)$ and $R(\lambda - \widetilde{T}) = R(S)$.

Proof. (i) Assume that $(\mu - \widetilde{T})x = (\mu - \lambda)x + \widetilde{T}(0)$. Then, since $\mu - \widetilde{T}$ is injective (as $\rho(T) = \rho(\widetilde{T})$ by [**5**, VI.1.1]) we have that

$$x = T_\mu(\mu - \widetilde{T})x = (\mu - \lambda)T_\mu x + T_\mu(\mu - \widetilde{T})(0) = (\mu - \lambda)T_\mu x.$$

Conversely, suppose that $x = (\mu - \lambda)T_\mu x$. Then

$$\begin{aligned}(\mu - \widetilde{T})x &= (\mu - \lambda)(\mu - \widetilde{T})(\mu - \widetilde{T})^{-1}x \\ &= (\mu - \lambda)(\{x\} \cap R(\mu - \widetilde{T}) + (\mu - \widetilde{T})(0))\end{aligned}$$

(by [**5**, I.3.1 (d)])

$$= (\mu - \lambda)x + \widetilde{T}(0) \quad (\text{as } R(\mu - \widetilde{T}) = \widetilde{X}).$$

(ii) We have that

$$\begin{aligned}(\mu - \widetilde{T}) &= (\mu - \lambda)((\mu - \lambda)^{-1} - T_\mu)(\mu - \widetilde{T}) \\ &= (I - (\mu - \lambda)T_\mu)(\mu - \widetilde{T}) \\ &= (T_\mu(\mu - \widetilde{T}) - (\mu - \lambda)T_\mu)(\mu - \widetilde{T})\end{aligned}$$

(as $\mu - \widetilde{T}$ is injective)

$$= (T_\mu(\mu - \widetilde{T} - (\mu - \lambda)))(\mu - \widetilde{T})$$

(by [**5**, I.4.2 (e)] as $D(T_\mu) = \widetilde{X}$)

$$\begin{aligned}&= T_\mu(\mu - \widetilde{T})(\mu - \widetilde{T} - (\mu - \lambda)) \\ &= \mu - \widetilde{T} - (\mu - \lambda) \\ &= \lambda - \widetilde{T}.\end{aligned}$$

(iii) Since $R(\mu - \widetilde{T}) = \widetilde{X}$ and $\lambda - \widetilde{T} = S(\mu - \widetilde{T})$ ((ii)) we obtain trivially that $R(\lambda - \widetilde{T}) = R(S)$.

Now, let $x \in D(\widetilde{T}) \setminus \{0\}$. Then

$$\begin{aligned}x \in N(\lambda - \widetilde{T}) &\Longleftrightarrow (\lambda - \widetilde{T})x \\ &= (\lambda - \widetilde{T})(0) \Longleftrightarrow (\lambda - \widetilde{T})x \\ &= (\lambda - \mu)x + (\mu - \widetilde{T})x \\ &= (\lambda - \mu)(0) + (\mu - \widetilde{T})(0) \Longleftrightarrow (\mu - \widetilde{T})x \\ &= (\mu - \lambda)x + (\mu - \widetilde{T})(0) \\ &= (\mu - \lambda)x + \widetilde{T}(0) \Longleftrightarrow x \\ &= (\mu - \lambda)T_\mu x\end{aligned}$$

(by (i)) $\Longleftrightarrow (I-(\mu-\lambda)T_\mu)x$

$$= 0 \Longleftrightarrow x \in N(S).$$

Therefore, $N(\lambda - \widetilde{T}) = N(S)$, as desired. □

A condition will now be found for T to have non-empty essential spectra.

Theorem 9. *Let $T \in LR(X)$ be partially continuous with $\rho(T) \neq \varnothing$ and $\dim T(0) < \infty$. Then the spectrum and the essential spectra of T are non-empty subsets.*

Proof. We first note that, by virtue of [**5**, V.11.3], we have that T is partially continuous with $\dim T(0) < \infty$ if and only if $\widetilde{T}$ is continuous with $\dim \widetilde{T}(0) < \infty$.

We shall verify that $\sigma_{e+}(T) \neq \varnothing$. Let $\mu \in \rho(T)$. Then $\dim N(T_\mu) = \dim \widetilde{T}(0) < \infty$, and, since $\mu - \widetilde{T}$ is continuous, it follows from [**5**, III.4.2] that $R(T_\mu) = D(\mu - \widetilde{T}) = D(\widetilde{T})$ is a closed subspace of $\widetilde{X}$. Hence, the bounded operator T_μ is ϕ_+, that is, $0 \in \rho_{e+}(T_\mu)$ and since $\sigma_{e+}(T_\mu) \neq \varnothing$, there exists $0 \neq \eta \in \sigma_{e+}(T_\mu)$. Let $\lambda := \mu - (1/\eta)$. We shall see that $\lambda \in \sigma_{e+}(T)$. Assume that $\lambda - \widetilde{T} \in \phi_+$, and thus it follows from Lemma 8 that $S \in \phi_+$; therefore, $(\mu - \lambda)^{-1} \in \rho_{e+}(T_\mu)$ which contradicts $\eta \in \sigma_{e+}(T_\mu)$. Hence, $\sigma_{e+}(T)$ is non-empty and consequently $\sigma_e(T)$ is also non-empty.

It only remains to show that $\sigma_{e-}(T) \neq \varnothing$. But, since $\sigma_e(T)$ is closed (Theorem 5) and non-empty, to see that $\sigma_{e-}(T) \neq \varnothing$ it is sufficient to verify that the boundary of $\sigma_e(T)$, $\sigma_e(T)^b$, is contained in $\sigma_{e-}(T)$. Let $\lambda \in \sigma_e(T)^b = \overline{\sigma_e(T)} \cap \overline{\mathbf{C} \setminus \sigma_e(T)} = \sigma_e(T) \cap \mathbf{C} \setminus \sigma_e(T)^0$, and suppose that $\lambda \notin \sigma_{e-}(T)$. Then $\lambda - \widetilde{T} \in \phi_-$, so that $0 < \gamma(\lambda - \widetilde{T}) = \gamma(\lambda - \widetilde{T}')$ by [**5**, III.4.2 and III.4.6] and so, from [**5**, V.5.12, V.15.6], we deduce that $\eta - \widetilde{T} \in \phi_-$ and $k(\eta - \widetilde{T}) = k(\lambda - \widetilde{T})$ whenever $|\eta - \lambda| < \gamma(\lambda - \widetilde{T}')$.

Let us consider two possibilities for $k(\lambda - \widetilde{T})$:

(a) $k(\lambda - \widetilde{T}) = 0$. In that case $\alpha(\lambda - \widetilde{T}) = \beta(\lambda - \widetilde{T}) < \infty$, which implies that $\lambda \in \rho_e(T)$, contradicting $\lambda \in \sigma_e(T)^b \subset \sigma_e(T)$.

(b) $k(\lambda - \widetilde{T}) \neq 0$. In such a situation, $k(\eta - \widetilde{T}) \neq 0$ if $|\eta - \lambda| < \gamma(\lambda - \widetilde{T})$, and therefore $\lambda \in \sigma_e(T)^0$, contrary to the assumption $\lambda \in \sigma_e(T)^0 \subset \mathbf{C} \setminus \sigma_e(T)^0$. Now that the spectrum of T is non-empty, it follows immediately from Proposition 6. $\square$

3. Linear relations on quotient indecomposable normed spaces. The following elementary results help to define Definition 14 below.

Lemma 10. *Let M be a closed subspace of X. We have*:

(i) *$(X/M)'$ is isometrically isomorphic to $M^\perp$.*

(ii) *If N is a subspace of X containing M, then N is closed if and only if N/M is closed in X/M. Furthermore, if N is closed, then X/N is isometrically isomorphic to $(X/M)/(N/M)$ with $Q_N = Q_{N/M}Q_M$ and $(N/M)^\perp$ is isometrically isomorphic to $N^\perp$.*

(iii) *For any closed subspace A of X/M, the closed subspace N of X given by $N := Q_M^{-1}(A)$ satisfies $M \subset N$ and $(X/M)/A$ is isometrically isomorphic to X/N.*

Proof. See, for example, [**5**, IV.5.2]. $\square$

Lemma 11. *Let M and N be closed subspaces of X. Then*:

(i) $\overline{R(Q_M J_N)} = (M^\perp \cap N^\perp)^\top$.

(ii) *$(Q_M J_N)'$ is open if and only if $M^\perp + N^\perp$ is closed.*

Proof. (i) $\overline{R(Q_M J_N)} = (R(Q_M J_N)^\perp)^\top = N((Q_M J_N)')^\top$ ([**5**, III.1.4]) $= N(Q_{N^\perp} J_{M^\perp})^\top$ ([**5**, III.1.9]) $= (M^\perp \cap N^\perp)^\top$.

(ii) $(Q_M J_N)'$ open $\Leftrightarrow$ $R((Q_M J_N)')$ closed (by the open mapping theorem for operators) $\Leftrightarrow$ $R(Q_{N^\perp} J_{M^\perp})$ closed ([**5**, III.1.9]) $\Leftrightarrow$ $M^\perp + N^\perp$ closed (Lemma 10). $\square$

Lemma 12. *Let M and N be closed subspaces of a Banach space E. Then $M + N$ is closed if and only if $M^\perp + N^\perp$ is closed if and only if $M + N = (M^\perp \cap N^\perp)^\top$.*

Proof. This result is well known. A proof can be found in [**5**, III.3.9]. ◻

Corollary 13. *For a Banach space E the following properties are equivalent*:

(i) *E is quotient indecomposable.*

(ii) *There are no closed infinite codimensional subspaces M and N of E such that $E = M + N$.*

(iii) *There are no closed infinite codimensional subspaces M and N of E such that $Q_M J_N$ has dense range and $(Q_M J_N)'$ is open.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) was proved in [**1, 8**] and that (ii) is equivalent to (iii) follows immediately from Lemmata 11 and 12. ◻

Corollary 13 suggests the following notion.

Definition 14. A normed space X is said to be quotient indecomposable (QI for short) if there are no closed infinite codimensional subspaces M and N of X such that $Q_M J_N$ has dense range and its conjugate is open.

Proposition 15. *Let X be a quotient indecomposable normed space. Then*

(i) *$\widetilde{X}$ is quotient indecomposable.*

(ii) *For every $M \in \mathcal{E}(X)$, X/M is quotient indecomposable.*

Proof. (i) Suppose that $\widetilde{X}$ is not QI. Then, by Corollary 13, there are $A, B \in \mathcal{E}(\widetilde{X})$ such that $\widetilde{X} = A + B$. Thus $A^\perp + B^\perp$ is closed and $A^\perp \cap B^\perp = \{0\}$ by virtue of [**12**, 2.2]. Then it is easy to prove that $M := A \cap X$ and $N := B \cap X$ are closed infinite codimensional subspaces of X such that $M^\perp + N^\perp$ is closed and $M^\perp \cap N^\perp = \{0\}$. It follows from Lemma 11 that X is not quotient indecomposable contradicting the hypothesis.

(ii) Assume that there are $U, V \in \mathcal{E}(X/M)$ such that $Q_U J_V$ has dense range and its conjugate is open. We deduce from Lemma 10 that $A, B \in \mathcal{E}(X)$ exist such that $U := A/M$, $V := B/M$, $A^\perp + B^\perp$ is closed and $A^\perp \cap B^\perp = \{0\}$, and thus X is not QI which contradicts the hypothesis. $\square$

Proposition 16 ([**12**]). *A Banach space F is QI if and only if, for every Banach space E and every bounded operator, $T : E \to F$ is $T \in \phi_+$ or $T \in SC$.*

We extend (Theorem 19 below) this result to multi-valued linear operators. To this end, we first establish some auxiliary results.

Proposition 17. *Let $T : X \to F$ be a bounded single valued where F is a QI Banach space. Then $\Gamma'(T) = \Delta'(T)$.*

Proof. Clearly, $\Gamma'(T) \leq \Delta'(T) < \infty$. Suppose that F is a quotient indecomposable Banach space and let $0 < \varepsilon < 1$ and $M, N \in \mathcal{E}(F)$. Then $Q_M J_N$ does not have dense range with open adjoint and hence $(Q_M J_N)' = Q_{N^\perp} J_{M^\perp}$ is not injective and open. Now, arguing as in the proof of [**5**, V.5.5] there are sequences (y'_n) in $M^\perp$, (x'_n) in $N^\perp$ and (y_n) in F such that $1 = \|y'_n\|$, $y'_n(y_m) = \delta_{n,m}$ and $\|y_n\| \|y'_n - x'_n\| < \varepsilon/2^n$, $n \in \mathbf{N}$.

Now we can define the nuclear operator

$$K : y \in F \longrightarrow K(y) := \sum_{i=1}^{\infty} (y'_n - x'_n)(y) y_n \in F.$$

Then it is clear that K is a compact single valued and its norm does not exceed ε, so that $I - K$ is an isomorphism. Moreover, $(I - K)' y'_n = x'_n$, $n \in \mathbf{N}$, and consequently $(I - K)(sp(x'_n))^\top = (sp(y'_n))^\top$. As $y'_n(y_m) = \delta_{n,m}$, $sp(y'_n)$ is infinite dimensional so that $(sp(x'_n))^\top$ and $(sp(y'_n))^\top$ are infinite codimensional closed subspaces of F/N and F/M, respectively. Furthermore, $I - K$ induces an isomorphism ψ^{-1} from $W := F/(sp(x'_n))^\top$ onto $V := F/(sp(y'n))^\top$ such that $\psi Q_V = Q_W(I - K)$. Finally, we obtain by direct computation that $\|(I - K)^{-1}\| < 1 + 2\varepsilon$ and that, if T is a bounded operator from X into F, then $\|\psi^{-1}\| = \|\psi^{-1} Q_W\| = \|Q_W(I - K)^{-1}\| < 1 + 2\varepsilon$. Consequently, $\|Q_V T\| = \|\psi^{-1} \psi Q_V T\| \leq (1 + 2\varepsilon) \|Q_W(I - K)T\| \leq$

$\|Q_W T\| + \varepsilon_1$ where $\varepsilon_1 := \varepsilon(3+2\varepsilon)\|T\|$. Since we can take $\varepsilon_1 > 0$ arbitrarily small, it follows that $\Delta'(T) \leq \Gamma'(T)$, as desired. □

We recall one method of reducing an arbitrary linear relation to the bounded case.

Definition 18 ([**5**, IV.3.1]). Given $T \in LR(X,Y)$, let X_T denote the vector space $D(T)$ normed by $\|x\|_T := \|x\| + \|Tx\|$, $x \in D(T)$. Let $G_T \in LR(X_T, X)$ be the identity injection of X_T onto $D(T)$. Then $TG_T \in LR(X_T, Y)$ is a bounded linear relation.

Theorem 19. *For a normed space Y the following properties are equivalent*:

(i) *For every normed space X and every partially continuous linear relation $T \in LR(X,Y)$ with* $\dim T(0) < \infty$ *is $T \in F_-(X,Y) \cup SC(X,Y)$.*

(ii) *Y is quotient indecomposable.*

Proof. Assume (i) holds, and let $M, N \in \mathcal{E}(Y)$. Since $J_N \notin F_-$ (as $\dim Y/N = \infty$), $J_N \in SC$ by the hypothesis, and it follows from the definition of quotient indecomposable normed space and Proposition 3 that Y is QI.

In order to prove that (ii) implies (i), it is enough to consider the case when Y is complete. To see this, let $T \notin F_-$. Then $J_Y T \notin F_-$ and $\widetilde{Y}$ is a quotient indecomposable Banach space (Proposition 15). Hence, if we assume that for $\widetilde{Y}$ (ii) $\Rightarrow$ (i), we have that $J_Y T \in SC$, that is, $\Delta'(J_Y T) = 0$, and since $\Delta'(T) \leq \Delta'(J_Y T)$ ([**5**, IV.5.12]), we derive that $\Delta'(T) = 0$. Accordingly, suppose that Y is a quotient indecomposable Banach space, and let $T \in PB(X,Y)$ such that $\dim T(0) < \infty$. Furthermore, the proof can be reduced to the single valued case. Indeed, it is sufficient to apply the following equivalences:

$$T \in PB \Longleftrightarrow Q_T T \in PB;$$

$T \in F_- \Leftrightarrow Q_T T \in F_-$ ([**5**, V.5.2]) $\Leftrightarrow \Gamma' Q_T T) = \Gamma'(T) > 0$ ([**5**, IV.5.6 and V.5.17] since $\dim T(0) < \infty$); $T \in SC \Leftrightarrow \Delta'(T) = 0 \Leftrightarrow \Delta'(Q_T T) = 0$ (again, from [**5**, IV.5.6]) $\Leftrightarrow Q_T T \in SC$.

Hence, we can assume without loss of generality that $T \in PB(X,Y)$ is single valued and Y is complete. First suppose that T is bounded. In such a case the result follows trivially from Proposition 17. For the general case, consider the bounded operator TG_T. Then, from what has been proved, $TG_T \in F_- \cup SC$, and we have the following chain of implications:

$$\begin{aligned} T \notin F_- \Longrightarrow TG_T \notin F_- \ ([\mathbf{5}, \text{V.5.24}]) \Longrightarrow TG_T \\ \in SC \Longleftrightarrow \Delta'(TG_T) = 0 \Longrightarrow \Delta'(T) = 0. \end{aligned}$$

The last implication is obtained upon noting that, as T is a partially continuous single valued $\Delta'(T) = \Delta'((TG_T)G_T^{-1}) \leq \Delta'(TG_T)\Delta'(J_{X_T}G_T^{-1})$ with $\Delta'(J_{X_T}G_T^{-1}) < \infty$ by virtue of [**10**, 3.7, 3.13, 4.6], and thus the theorem is proved. □

Theorem 20. *Let X be a complex QI normed space. Then every partially continuous linear relation $T \in LR(X)$ with non-empty resolvent set and* $\dim T(0) < \infty$ *can be described as $T = \lambda + S$ where $\lambda \in \mathbf{C}$ and S is a strictly cosingular linear relation.*

Proof. Combine Theorems 9 and 19. □

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