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IMPROVEMENT OF A CRITERION FOR STARLIKENESS

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ABSTRACT. In this paper a result concerning the starlikeness of the image of the Alexander operator is improved. The techniques of differential subordinations and the method of extreme points are used.

1. Introduction. Let $U(z_0, r)$ be the disc centered at point z_0 and of radius r defined by $U(z_0, r) = \{z \in \mathbf{C} : |z - z_0| < r\}$.

U denotes the open unit disc in $\mathbf{C}$, $U = \{z \in \mathbf{C} : |z| < 1\}$.

Let $\mathcal{A}$ be the class of analytic functions f, which are defined on the unit disc U and have the form: $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$.

The subclass of $\mathcal{A}$ consisting of functions for which the domain $f(U)$ is starlike with respect to 0, is denoted by S^*. An analytic characterization of S^* is given by

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}.$$

Another subclass of $\mathcal{A}$ with which we deal is the class of close-to-convex functions denoted by C. A function $f \in \mathcal{A}$ belongs to class C if and only if there is a starlike function $g \in S^*$, so that $\operatorname{Re}(zf'(z)/g(z)) > 0$, $z \in U$. We note that C and S^* contain univalent functions. The Alexander integral operator is defined by the equality:

$$A(f)(z) = \int_0^z \frac{f(t)}{t}\, dt.$$

However, it has been proved in [**1**] that the Alexander operator does not map the class of close-to-convex functions in the class of starlike functions; namely, $A(C) \not\subset S^*$, it is possible to determine subclasses of

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C, which are mapped in S^* by operator A. Regarding this question, the authors have proved in [**2**, pages 310–311] the following result:

Theorem 1. *Let A be the operator of Alexander, and let $g \in \mathcal{A}$ satisfy*

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geq \left| \operatorname{Im} \frac{z(zg'(z))'}{g(z)} \right|, \quad z \in U. \tag{1}$$

If $f \in \mathcal{A}$ satisfies

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in U,$$

then $F = A(f) \in S^*$.

In [**4**] an improvement of this result has been proved. The aim of this paper is to present another improvement of Theorem 1.

2. Preliminaries. In order to prove the main result, we need the following definitions and lemmas. Let f and g be analytic functions in U. The function f is said to be subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, $z \in U$ and $f(z) = g(w(z))$, $z \in U$. Recall that, if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Lemma 1 [**2**, page 22]. *Let $p(z) = a + \sum_{k=n}^{\infty} a_k z^k$ be analytic in U with $p(z) \not\equiv a$, $n \geq 1$, and let $q : U(0,1) \to \mathbf{C}$ be a univalent function with $q(0) = a$. If there are two points $z_0 \in U(0,1)$ and $\zeta_0 \in \partial U(0,1)$ so that q is defined in ζ_0, $p(z_0) = q(\zeta_0)$ and $p(U(0,r_0)) \subset q(U)$, where $r_0 = |z_0|$, then there is an $m \in [n, +\infty)$ so that*

(i) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$

and

(ii)

$$\operatorname{Re}\left(1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right) \geq m \operatorname{Re}\left(1 + \frac{\zeta_0 q''(\zeta_0)_0}{q'(\zeta_0)}\right).$$

We note that $z_0 p'(z_0)$ is the outward normal to the curve $p(\partial U(0, r_0))$ at point $p(z_0)$. ($\partial U(0, r_0)$ denotes the border of disc $U(0, r_0)$).

Lemma 2 [**2**, page 26]. *Let* $p(z) = a + \sum_{k=n}^{\infty} a_k z^k$, $p(z) \not\equiv a$ *and* $n \geq 1$. *If* $z_0 \in U$ *and*

$$\operatorname{Re} p(z_0) = \min\{\operatorname{Re} p(z) : |z| \leq |z_0|\},$$

then

(i)

$$z_0 p'(z_0) \leq -\frac{n}{2} \frac{|p(z_0) - a|^2}{\operatorname{Re}(a - p(z_0))}$$

and

(ii) $\operatorname{Re}[z_0^2 p''(z_0)] + z_0 p'(z_0) \leq 0.$

Lemma 3 [**5**]. *If* $\theta \in (0, 2\pi)$ *and* $\beta > 0$, *then the following identity holds*:

$$\int_0^\infty \frac{x(e^{\theta x} + e^{(2\pi-\theta)x})}{(\beta^2 + x^2)(e^{2\pi x} - 1)}\, dx + i\beta \int_0^\infty \frac{e^{(2\pi-\theta)x} - e^{\theta x}}{(\beta^2 + x^2)(e^{2\pi x} - 1)}\, dx = \frac{1}{2\beta} + \sum_{k=1}^{\infty} \frac{e^{i\theta k}}{k + \beta}. \tag{2}$$

Let X be a locally convex linear topological space. For a subset $D \subset X$ the closed convex hull of D is defined as the intersection of all closed convex sets containing D and will be denoted by $\operatorname{co}(D)$. If $D \subset V \subset X$, then D is called an extremal subset of V provided that, whenever $u = tx + (1-t)y$ where $u \in D$, $x, y \in V$ and $t \in (0,1)$, then $x, y \in D$.

An extremal subset of D consisting of only one point is called an extreme point of D. The set of the extreme points of D will be denoted by ED. Let $\mathcal{H}(U)$ be the set of analytic functions defined on U.

Lemma 4 [**1**, page 45]. *If* $J : \mathcal{H}(U) \to \mathbf{R}$ *is a real-valued, continuous convex functional and* $\mathcal{F}$ *is a compact subset of* $\mathcal{H}(U)$, *then*

$$\max\{J(f) : f \in \operatorname{co}(\mathcal{F})\} = \max\{J(f) : f \in \mathcal{F}\} = \max\{J(f) : f \in E(\operatorname{co}(\mathcal{F}))\}.$$

Let $\mathcal{P}$ be the class of analytic functions with positive real part defined by:

$$\mathcal{P} = \{f \in \mathcal{H}(U) : f(0) = 1,\ \operatorname{Re} f(z) > 0,\ z \in U\}.$$

Lemma 5 (The Herglotz formula) [**1**]. *For each $f \in \mathcal{P}$ there is a probability measure μ on interval $[0, 2\pi]$, so that*

$$f(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu(t),$$

or, in developed form,

$$f(z) = 1 + 2\int_0^{2\pi} \left(\sum_{n=1}^{\infty} z^n e^{-int}\right) d\mu(t).$$

The converse of the theorem is also valid.

Lemma 6 [**1**]. *The set of the extreme points of class $\mathcal{P}$ is*

$$E\mathcal{P} = \left\{f_t : f_t(z) = \frac{1 + ze^{-it}}{1 - ze^{-it}},\ t \in [0, 2\pi]\right\}.$$

We note that a linear operator maps an extreme point of a set in an extreme point of the image.

3. The main result.

Theorem 2. *If p is an analytic function in U, $p(0) = 1$ and*

$$\operatorname{Re} p(z) > |\operatorname{Im}(zp'(z) + p^2(z))|,\ z \in U, \tag{3}$$

then $\operatorname{Re} p(z) > 2.273|\operatorname{Im} p(z)|$, $z \in U$.

Proof. To prove the assertion we introduce the notation $\mathcal{D} = \{z \in \mathbf{C} : |\arg(z)| < (\pi/2)\tau\}$, where $\tau = (2/\pi)\arctan(1000/2273)$. Inequality $\operatorname{Re} p(z) > 2.273|\operatorname{Im} p(z)|$, $z \in U$, is equivalent to

$$p \prec q, \tag{4}$$

where

$$q : U \longrightarrow \mathcal{D}, \qquad q(z) = \left(\frac{1+z}{1-z}\right)^{\tau}$$

is univalent and $q(U) = \mathcal{D}$. (The principal branch of $(1+z/1-z)^{\tau}$ is used.)

If (4) does not hold, then Lemma 1 implies that there are two points $z_0 \in U$ and $\zeta_0 \in \mathbf{C}$, such that $|\zeta_0| = 1$, $p(U(0, |z_0|)) \subset q(U)$,

$$p(z_0) = q(\zeta_0)$$

and

$$z_0 p'(z_0) = m \zeta_0 q'(\zeta_0)$$

where $m \in \mathbf{R}$, $m \geq 1$.

If $\arg \zeta_0 = \beta$, then $q(\zeta_0) = |\cot(\beta/2)|^{\tau}(\cos(\tau\pi/2) \pm i \sin(\tau\pi/2))$, and

$$\zeta_0 q'(\zeta_0) = \frac{-\tau}{2\sin^2(\beta/2)} \left|\cot\frac{\beta}{2}\right|^{\tau-1} \left(\cos\frac{(\tau-1)\pi}{2} \pm i \sin\frac{(\tau-1)\pi}{2}\right).$$

We are considering the case

$$q(\zeta_0) = \left|\cot\frac{\beta}{2}\right|^{\tau} \left(\cos\frac{\tau\pi}{2} + i \sin\frac{\tau\pi}{2}\right).$$

The other case is similar.

In this case, condition (3) becomes

$$\left|\cot\frac{\beta}{2}\right|^{\tau} \cos\frac{\tau\pi}{2} \geq \left|\frac{m\tau|\cot(\beta/2)|^{\tau-1}\cos(\tau\pi/2)}{2\sin^2(\beta/2)} + \left|\cot\frac{\beta}{2}\right|^{2\tau} \sin\tau\pi\right|$$

and, using the notation $t = |\cot(\beta/2)|$, it will be equivalent to

$$m\tau t^2 + 4t^{\tau+1}\sin\frac{\tau\pi}{2} - 2t + m\tau \leq 0. \tag{5}$$

Condition $m \geq 1$ implies that

$$\tau t^2 + 4t^{\tau+1}\sin\frac{\tau\pi}{2} - 2t + \tau \leq m\tau t^2 + 4t^{\tau+1}\sin\frac{\tau\pi}{2} - 2t + m\tau.$$

An elementary analysis of the behavior of the function

$$\varphi : [0, +\infty) \to \mathbf{R}, \quad \varphi(t) = \tau t^2 + 4t^{\tau+1} \sin \frac{\tau\pi}{2} - 2t + \tau,$$
$$\left(\tau = \frac{2}{\pi} \arctan \frac{1}{2.273}\right)$$

shows that the mapping φ has a global minimum at point $x_0 = 0.5289\ldots$ and

$$\min_{x\in[0,\infty)} \varphi(x) = \varphi(x_0) = 0.0000021\ldots.$$

Thus, $\varphi(t) > 0$, $t \in [0, \infty)$, and this contradicts (5). The contradiction implies the subordination: $p \prec q$. □

Corollary 1. *If $g \in \mathcal{A}$, then condition* (1) *implies the inequality*:

$$\mathrm{Re}\, \frac{zg'(z)}{g(z)} \geq 2.273 \left|\mathrm{Im}\, \frac{zg'(z)}{g(z)}\right|, \quad z \in U.$$

Proof. Indeed, if we denote $p(z) = (zg'(z))/(g(z))$, then

$$\frac{z(zg'(z))'}{g(z)} = zp'(z) + p^2(z)$$

and condition (1) becomes:

$$\mathrm{Re}\, p(z) > |\mathrm{Im}\,(zp'(z) + p^2(z))|, \quad z \in U.$$

Now, according to Theorem 2, the conclusion follows. □

Theorem 3. *If $g \in \mathcal{A}$ is a function which satisfies the condition*

$$\mathrm{Re}\, \frac{zg'(z)}{g(z)} > 2.273 \left|\mathrm{Im}\, \frac{zg'(z)}{g(z)}\right|, \quad z \in U, \tag{6}$$

then

$$\mathrm{Re}\, \frac{g(z)}{z} > \frac{100}{83} \left|\mathrm{Im}\, \frac{g(z)}{z}\right|, \quad z \in U. \tag{7}$$

Proof. Let $p(z) = (g(z)/z)$, and $q(z) = (1+z)/(1-z)^\tau$, where $\tau = (2/\pi)\arctan(83/100)$. As in the proof of Theorem 2, we observe that inequality (7) is equivalent to the subordination $p \prec q$. If this subordination does not hold, then according to Lemma 1 there are two points $z_0 \in U$ and $\zeta_0 \in \mathbf{C}$, such that $|\zeta_0| = 1$, $p(U(0,|z_0|)) \subset q(U)$,

$$p(z_0) = q(\zeta_0)$$

and

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$$

where $m \in \mathbf{R}$, $m \geq 1$.

If $\arg\zeta_0 = \beta$, then $q(\zeta_0) = |\cot(\beta/2)|^\tau(\cos(\tau\pi/2) \pm i\sin(\tau\pi/2))$, and

$$\zeta_0 q'(\zeta_0) = \frac{-\tau}{2\sin^2(\beta/2)}\left|\cot\frac{\beta}{2}\right|^{\tau-1}\left(\cos\frac{(\tau-1)\pi}{2} \pm i\sin\frac{(\tau-1)\pi}{2}\right).$$

Using these equalities we get $(g(z_0)/z_0) = |\cot(\beta/2)|^\tau(\cos(\tau\pi/2) \pm i\sin(\tau\pi/2))$ and $g'(z_0) - (g(z_0)/z_0) = (-\tau m)/(2\sin^2(\beta/2))|\cot(\beta/2)|^{\tau-1}$ $(\cos((\tau-1)\pi/2) \pm i\sin((\tau-1)\pi/2))$. Thus,

$$\frac{z_0 g'(z_0)}{g(z_0)} = 1 \mp \frac{\tau m i}{|\sin\beta|},$$

and condition (6) becomes: $1 > (2.273\tau m)/|\sin\beta|$. but this is a contradiction because $2.273\tau > 1$. Consequently, we have $p \prec q$. □

Theorem 4. *Let $g \in \mathcal{A}$ be a function with* $\mathrm{Re}\,(g(z)/z) > (100/83)|\mathrm{Im}\,(g(z)/z)|$, *$z \in U$. If $f \in \mathcal{A}$ and*

$$\mathrm{Re}\,\frac{zf'(z)}{g(z)} > 0, \quad z \in U, \tag{8}$$

then

$$\mathrm{Re}\,\frac{f(z)}{z} > 0.134, \quad z \in U.$$

Proof. Let p be the function defined by equality $p(z) = (f(z)/z - \alpha)/(1-\alpha)$, $\alpha = 0.134$. If the inequality $\mathrm{Re}\,p(z) > 0$ does not hold for each

$z \in U$, then Lemma 2 implies that there are two real numbers $s, t \in \mathbf{R}$ and a complex number $z_0 \in U$ such that

$$p(z_0) = is$$
$$z_0 p'(z_0) = t \leq -\frac{1}{2}(s^2 + 1).$$

The above equalities are equivalent to $f(z_0)/z_0 = \alpha + is(1 - \alpha)$, $f'(z_0) = \alpha + t(1 - \alpha) + is(1 - \alpha)$. If $g(z_0)/z_0 = a + ib$, then according to the conditions of the theorem we have $a > (100/83)|b|$, and

$$\operatorname{Re} \frac{z_0 f'(z_0)}{g(z_0)} = \operatorname{Re} \frac{\alpha + t(1 - \alpha) + is(1 - \alpha)}{a + ib} = \frac{a[\alpha + t(1 - \alpha)] + s(1 - \alpha)b}{a^2 + b^2}.$$

Inequality $\operatorname{Re}(z_0 f'(z_0))/(g(z_0)) > 0$ is equivalent to

$$a[\alpha + t(1 - \alpha)] + s(1 - \alpha)b > 0.$$

On the other hand, in the case of $\alpha \leq 0.134$, we have

$$a[\alpha + t(1 - \alpha)] + s(1 - \alpha)b \leq |b|\left[-\frac{50}{83}(1 - \alpha)s^2 \pm s(1 - \alpha) + \frac{50}{83}(-1 + 3\alpha)\right] \leq 0,$$

for all $s \in \mathbf{R}$. This contradiction shows that $\operatorname{Re}(f(z)/z) > 0.134$, for all $z \in U$. □

We also need the following result before we can prove the improvement of Theorem 1.

Theorem 5. *If $f \in \mathcal{A}$, $F = A(f)$ and*

$$\operatorname{Re} \frac{f(z)}{z} > 0.134, \quad z \in U, \tag{9}$$

then

$$\operatorname{Re} \frac{F(z)}{z} \geq \frac{83}{100}\left|\operatorname{Im} \frac{F(z)}{z}\right|, \quad z \in U. \tag{10}$$

Proof. We begin with the observation that $\operatorname{Re}(f(z)/z) > \gamma = 0.134$, $z \in U$, is equivalent to $(f(z)/z - \gamma)/(1-\gamma) \in \mathcal{P}$. Thus, according to the Herglotz formula,

$$\frac{f(z)}{z} \in \left\{ 1 + 2(1-\gamma) \int_0^{2\pi} \left(\sum_{n=1}^{\infty} z^n e^{-int} \right) d\mu(t) : \right.$$
$$\left. \mu \text{ probability measure on } [0, 2\pi] \right\},$$

and, consequently, $F(z) \in \mathcal{B}$, where

$$\mathcal{B} = \left\{ z + 2(1-\gamma) \int_0^{2\pi} \left(\sum_{n=1}^{\infty} \frac{z^{n+1}}{n+1} e^{-int} \right) d\mu(t) : \right.$$
$$\left. \mu \text{ probability measure on } [0, 2\pi] \right\}.$$

Let $z_0 \in U$ be an arbitrary fixed point, and let p_{z_0} be the functional defined by

$$p_{z_0} : \mathcal{B} \longrightarrow \mathbf{R}, \qquad p_{z_0}(g) = \frac{83}{100} \left| \operatorname{Im} \frac{g(z_0)}{z_0} \right| - \operatorname{Re} \frac{g(z_0)}{z_0}.$$

If we prove that $p_{z_0}(g) \le 0$ for each $g \in \mathcal{B}$ in the case of every arbitrary fixed point z_0, then inequality (10) follows. Since the functional p_{z_0} is convex, according to Lemma 4 we have to check $p_{z_0}(g) \le 0$ only for the extreme points of class $\mathcal{B}$. It follows from Lemma 6 that the extreme points of this class are

$$F_t(z) = z + 2(1-\gamma) \sum_{n=1}^{\infty} \frac{z^{n+1}}{n+1} e^{-int}, \quad t \in [0, 2\pi].$$

For $z_0 = r_0 e^{i\theta_0}$, the inequality $p_{z_0}(F_t) \le 0$ is equivalent to

$$\frac{83}{100} \left| \sum_{n=1}^{\infty} \frac{r_0^n \sin n(\theta_0 - t)}{n+1} \right| \le \frac{1}{2(1-\gamma)} + \sum_{n=1}^{\infty} \frac{r_0^n \cos n(\theta_0 - t)}{n+1},$$
$$r_0 \in [0, 1); \quad \theta_0, \quad t \in [0, 2\pi].$$

Denoting $\theta_0 - t = \beta$, we get

$$\frac{83}{100}\left|\sum_{n=1}^{\infty}\frac{r_0^n \sin n\beta}{n+1}\right| \leq \frac{1}{2(1-\gamma)} + \sum_{n=1}^{\infty}\frac{r_0^n \cos n\beta}{n+1}, \tag{11}$$

and we must prove this inequality in case of $\beta \in [0, 2\pi]$. Replacing β by $2\pi - \beta$, we get the same inequality. This shows that we must prove (11) only in the cases of $\beta \in [0, \pi]$ and $r_0 \in [0, 1)$. Since

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{r_0^n \sin n\beta}{n+1} &= \operatorname{Im}\int_0^1 \frac{r_0 t e^{i\beta}}{1 - r_0 t e^{i\beta}}\,dt \\ &= \int_0^1 \frac{r_0 t \sin\beta}{1 + r_0^2 t^2 - 2 r_0 t \cos\beta}\,dt \geq 0, \quad \beta \in [0, \pi],\end{aligned}$$

inequality (11) is equivalent to

$$\begin{gathered}\frac{83}{100}\sum_{n=1}^{\infty}\frac{r_0^n \sin n\beta}{n+1} - \frac{1}{2(1-\gamma)} - \sum_{n=1}^{\infty}\frac{r_0^n \cos n\beta}{n+1} < 0, \\ \beta \in [0, \pi], \quad r_0 \in [0, 1).\end{gathered} \tag{12}$$

The function

$$\Phi(r, \beta) = \frac{83}{100}\sum_{n=1}^{\infty}\frac{r^n \sin n\beta}{n+1} - \frac{1}{2(1-\gamma)} - \sum_{n=1}^{\infty}\frac{r^n \cos n\beta}{n+1}$$

is harmonic on $U_h = \{z \in \mathbf{C} : |z| < 1, \operatorname{Im} z > 0\}$. Thus, according to the maximum principle for harmonic functions, we must check the inequality $\Phi(r, \beta) < 0$ only on the frontier of U_h, namely, in the case of $z = e^{i\beta}$, $\beta \in [0, \pi]$, and in case of $z = x \in (-1, 1)$, then (12) follows. According to Lemma 3, we have:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sin n\beta}{n+1} &= \int_0^{\infty}\frac{e^{(2\pi-\beta)x} - e^{\beta x}}{(1+x^2)(e^{2\pi x} - 1)}\,dx \\ \sum_{n=1}^{\infty}\frac{\cos n\beta}{n+1} &= \int_0^{\infty}\frac{x(e^{(2\pi-\beta)x} + e^{\beta x})}{(1+x^2)(e^{2\pi x} - 1)}\,dx - \frac{1}{2}.\end{aligned}$$

These integral representations show that the functions $v, u : [0, \pi] \to \mathbf{R}$,

$$v(\beta) = \frac{83}{100} \sum_{n=1}^{\infty} \frac{\sin n\beta}{n+1},$$
$$u(\beta) = \frac{1}{2(1-\gamma)} + \sum_{n=1}^{\infty} \frac{\cos n\beta}{n+1}$$

are strictly decreasing. Consequently, if we prove the inequalities

$$u(\beta_k) > v(\beta_{k-1}), \quad \text{for } \beta_k = \frac{k\pi}{100}, \quad k = \overline{1, 100}, \tag{13}$$

then the monotony of functions u and v implies that

$$u(\beta) \geq u(\beta_k) > v(\beta_{k-1}) \geq v(\beta), \quad \beta \in [\beta_{k-1}, \beta_k], \ k = \overline{1, 100},$$

and so the inequality $u(\beta) > v(\beta)$ follows for every $\beta \in [0, \pi]$. Since

$$\sum_{n=1}^{\infty} \frac{e^{in\beta}}{1+n} = -1 + e^{-i\beta} \sum_{n=1}^{\infty} \frac{e^{in\beta}}{n} = -1 + e^{-i\beta} \log \frac{1}{1-e^{i\beta}},$$

it follows that

$$u(\beta) = \frac{1}{2(1-\gamma)} - 1 + \cos\beta \ln \frac{1}{2\sin(\beta/2)} + \frac{\pi-\beta}{2} \sin\beta,$$
$$v(\beta) = \frac{83}{100} \left(-\sin\beta \ln \frac{1}{2\sin(\beta/2)} + \frac{\pi-\beta}{2} \cos\beta \right),$$

and in the case of $\gamma = 0.134$, inequality (13) can be checked easily using a computer program.

In the second case $z = x \in (-1, 1)$, $\gamma = 0.134$, and inequality (11) is equivalent to

$$\frac{1}{2(1-\gamma)} + \sum_{n=1}^{\infty} \frac{x^n}{n+1} \geq 0, \quad x \in (-1, 1).$$

This inequality holds because

$$\frac{1}{2(1-\gamma)} + \sum_{n=1}^{\infty} \frac{x^n}{n+1} = \frac{1}{2(1-\gamma)} - 1 + \frac{1}{x} \ln \frac{1}{1-x}$$
$$\geq \frac{1}{2(1-\gamma)} - 1 + \ln 2 > 0, \quad x \in (-1, 1). \qquad \square$$

Now we are able to prove the improvement of Theorem 1.

Theorem 6. *If* $f, g \in \mathcal{A}$ *and*

$$\operatorname{Re}\frac{g(z)}{z} > \frac{100}{83}\left|\operatorname{Im}\frac{g(z)}{z}\right|, \quad z \in U, \tag{14}$$

then the condition

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > 0, \quad z \in U \tag{15}$$

implies that $F = A(f) \in S^*$.

Proof. Differentiating equality $F = A(f)$ twice, we obtain

$$F'(z) + zF''(z) = f'(z).$$

This can be rewritten using the notations $p(z) = (zF'(z))/(F(z))$, $P(z) = (F(z))/(g(z))$ in the following way

$$P(z)(zp'(z) + p^2(z)) = \frac{zf'(z)}{g(z)}, \quad z \in U.$$

The conditions of the theorem imply that

$$\operatorname{Re} P(z)(zp'(z) + p^2(z)) > 0, \quad z \in U. \tag{16}$$

First we prove the inequality $\operatorname{Re} P(z) > 0$, $z \in U$.

According to Theorems 4 and 5, inequalities (14) and (15) imply (10). From (10) and (14), the inequality $\operatorname{Re} P(z) > 0$, $z \in U$, follows.

We are now in a position to prove $\operatorname{Re} p(z) > 0$, $z \in U$.

If $\operatorname{Re} p(z) > 0$, $z \in U$ is not true, then according to Lemma 2 there are two real numbers $s, t \in \mathbf{R}$ and a point $z_0 \in U$, such that $p(z_0) = is$ and $z_0 p'(z_0) = t \leq -(1/2)(s^2 + 1)$. Thus,

$$P(z_0)(z_0 p'(z_0) + p^2(z_0)) = P(z_0)(t - s^2)$$

and $\mathrm{Re}\, P(z_0) > 0$ implies that

$$\mathrm{Re}\,[P(z_0)(z_0 p'(z_0) + p^2(z_0))] \le 0.$$

This inequality contradicts (16); hence, we deduce $\mathrm{Re}\, p(z) = \mathrm{Re}\,(zF'(z))/(F(z)) > 0$, $z \in U$. □

Remark 1. Theorems 2 and 3 show that condition (1) implies inequality (14); thus, Theorem 6 is an improvement of Theorem 1, but the conditions of Theorem 6

$$\mathrm{Re}\,\frac{g(z)}{z} > \frac{100}{83}\left|\mathrm{Im}\,\frac{g(z)}{z}\right|, \quad z \in U$$
$$\mathrm{Re}\,\frac{zf'(z)}{g(z)} > 0, \quad z \in U$$

do not imply that f is a close-to-convex function.

We get that a subclass of C is mapped by the Alexander operator to S^*, supplying the conditions of Theorem 6.

The following result is also an improvement of Theorem 1 in spite of the fact that we have supplied the initial conditions of Theorem 6. The new result claims that a subclass of C is mapped in S^* by the operator A.

Corollary 2. *If $f, g \in \mathcal{A}$ and*

$$Re\,\frac{g(z)}{z} > \frac{100}{83}\left|\mathrm{Im}\,\frac{g(z)}{z}\right|, \quad z \in U$$

and

$$\mathrm{Re}\,\frac{zg'(z)}{g(z)} > 0, \quad z \in U,$$

then the condition

$$Re\,\frac{zf'(z)}{g(z)} > 0, \quad z \in U$$

implies that $F = A(f) \in S^*$.

The following open question has been brought up in [**4**]: if we replace condition (1) in Theorem 1 by the weaker condition

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > \left| \operatorname{Im} \frac{zg'(z)}{g(z)} \right|, \quad z \in U,$$

will the theorem remain valid or not? Theorem 3, Theorem 4, Theorem 5 and Theorem 6 imply the following result regarding this question:

Corollary 3. *Let A be the operator of Alexander, and let $g \in \mathcal{A}$ be a function which satisfies the condition:*

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > 2.273 \left| \operatorname{Im} \frac{zg'(z)}{g(z)} \right|, \quad z \in U.$$

If $f \in \mathcal{A}$ satisfies

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in U,$$

then $F = A(f) \in S^*$.

REFERENCES

1. D.J. Hallenbeck and T.H. MacGregor, *Linear problems and convexity techniques in geometric function theory*, Pitman Advanced Publishing Program, Boston, 1984.

2. S.S. Miller and P.T. Mocanu, *Differential subordinations theory and applications*, Marcel Dekker, New York, 2000.

3. R. Szász, *A counter-example concerning starlike functions*, Stud. Univ. Babeş-Bolyai Math. **52** (2007), 171–172.

4. ———, *An improvement of a criterion for starlikeness*, Math. Pannonica **20** (2009), 69–77.

5. Robert Szász, *The sharp version of a criterion for starlikeness related to the operator of Alexander*, Ann. Polon. Math. **94** (2008), 1–14.

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