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ON THE RATIONALITY OF MODULI SPACES OF POINTED HYPERELLIPTIC CURVES

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ABSTRACT. Let $\mathcal{M}_{g,n}$ be the (coarse) moduli space of smooth, integral, projective curves of genus $g \geq 1$ with n marked points defined over the complex field $\mathbf{C}$. We denote by $\mathcal{H}_{g,n} \subseteq \mathcal{M}_{g,n}$ the locus of points corresponding to curves carrying a g_2^1. It is known that $\mathcal{H}_{g,n}$ is rational for $g = 1$ and $n \leq 10$, for $g = 2$ and $n \leq 12$ and for each $g \geq 3$ and $n = 0$. We prove here that the same is true for each $g \geq 3$ and $1 \leq n \leq 2g + 8$.

1. Introduction. Let $\mathcal{M}_{g,n}$ be the (coarse) moduli space of smooth, integral, projective curves of genus $g \geq 1$ with n marked points defined over the complex field $\mathbf{C}$, i.e. ordered $(n+1)$–tuples of the form $(C, p_1, \dots, p_n)$ where C is a smooth, integral, projective curves curve of genus g and $p_1, \dots, p_n \in C$ are pairwise distinct points.

We denote by $\mathcal{H}_{g,n} \subseteq \mathcal{M}_{g,n}$ the locus of points $(C, p_1, \dots, p_n)$ such that C carries a g_2^1. If $g = 1, 2$ we have $\mathcal{H}_{g,n} = \mathcal{M}_{g,n}$ whilst, if $g \geq 3$ we have strict inclusions $\mathcal{H}_{g,n} \subset \mathcal{M}_{g,n}$ and the points of $\mathcal{H}_{g,n}$ represent pointed hyperelliptic curves.

The locus $\mathcal{H}_{g,n}$ is irreducible. In [**3**] its Euler-Poincaré and orbifold Euler characteristics are computed. Moreover, $\mathcal{H}_{g,n}$ is known to be rational when $n = 0$ (see [**5, 11, 12, 13**]; see also [**8, 16, 17**] as general references), then it seems to be quite natural to inspect the case $n \geq 1$. A complete analysis in the initial case $g = 1$ can be found in [**2**], where the author proves that $\mathcal{H}_{1,n} = \mathcal{M}_{1,n}$ is rational for $n \leq 2g + 8 = 10$, and in [**4**], where the authors prove that the Kodaira dimension satisfies $\kappa(\overline{\mathcal{H}}_{1,11}) = 0$ and $\kappa(\overline{\mathcal{H}}_{1,n}) = 1$ for each $n \geq 12$ (here $\overline{\mathcal{H}}_{g,n}$ denotes the closure of $\mathcal{H}_{g,n}$ inside the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$).

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When $g = 2$, in the paper [**7**], the rationality of $\mathcal{H}_{2,n} = \mathcal{M}_{2,n}$ is proved for $n \leq 2g + 8 = 12$ as a particular case of more general rationality results for $\mathcal{M}_{g,n}$ when $2 \leq g \leq 5$. The proof is based on the existence of a particular plane model of $(C, p_1, \dots, p_n) \in \mathcal{H}_{2,n}$ which explicitly depends only upon the fixed points $p_1, \dots, p_n$. As far as the author knows, there are no results about $\kappa(\overline{\mathcal{H}}_{2,n})$ when $n \geq 13$.

In this short note we generalize this proof to $\mathcal{H}_{g,n}$ for each g as follows.

Main theorem. *For each $g \geq 1$, the locus $\mathcal{H}_{g,n} \subseteq \mathcal{M}_{g,n}$ is irreducible for each n and it is rational for each $n \leq 2g + 8$.*

Taking into account the above theorem and the complete description in case $g = 1$ it is then natural to ask the following:

Main questions. What can be said on the birational structure of $\mathcal{H}_{g,n}$ when $g \geq 2$ and $n \geq 2g + 9$? For example, is $\kappa(\overline{\mathcal{H}}_{g,n}) \geq 0$ in that range?

Analogous results and questions for $\mathcal{M}_{g,n}$ have been partially answered in the quoted paper [**7**] and also in [**1, 9, 14**].

Notation. We work over the field $\mathbf{C}$ of complex numbers. We denote by GL_3 the general linear group of 3×3 matrices with entries in $\mathbf{C}$.

Let $\mathbf{C}[x_0, \dots, x_k]$ be the ring of polynomials in the variables $x_0, \dots, x_k$ with coefficients in $\mathbf{C}$, $\mathbf{C}[x_0, \dots, x_k]_d$ the vector space of degree d forms.

The projective plane will be denoted by $\mathbf{P}^2_{\mathbf{C}}$: we set

$$E_0 := [1, 0, 0], \qquad E_1 := [0, 1, 0], \qquad E_2 := [0, 0, 1].$$

If V is a vector space then $\mathbf{P}(V)$ is the associated projective space. A curve C is a projective scheme of dimension 1. We denote isomorphisms by $\cong$ and birational equivalences by $\approx$.

For other definitions, results and notation we always refer to [**10**].

2. The proof of the main theorem. Let $\mu_n\colon \mathcal{M}_{g,n} \to \mathcal{M}_{g,0}$ be the natural forgetful morphism. Inside $\mathcal{M}_{g,0}$ there is the hyperelliptic locus $\mathcal{H}_{g,0}$, i.e., the locus of points representing smooth, integral, hyperelliptic curves, which is a closed irreducible subscheme of dimension $2g-1$, and we define $\mathcal{H}_{g,n} := \mu_n^{-1}(\mathcal{H}_{g,0})$.

Thanks to Lemma 3.1 of [**6**] we know that $\mathcal{H}_{g,n}$ is irreducible and its dimension is $2g+n-1$. The locus $\mathcal{H}_{g,n}$ is a coarse moduli space for smooth and connected n-pointed curves of genus g carrying a g_2^1, hence hyperelliptic when $g \geq 2$.

In this section we will prove the main theorem stated in the introduction. As explained there, the cases $g = 1, 2$ or g arbitrary and $n = 0$ have already been known. Thus, from now on, we restrict to $g \geq 3$ and $n \geq 1$.

2.1. The case $n = 1$. Let

$$V_1 := x_0^2 \mathbf{C}[x_1, x_2]_g \oplus x_0 \mathbf{C}[x_1, x_2]_{g+1} \oplus \mathbf{C} x_2^{g+2} \subseteq \mathbf{C}[x_1, x_2, x_3]_{g+2}.$$

Lemma 2.1.1. *The general element of $D \in \mathbf{P}(V_1)$ is an integral curve carrying a g-fold ordinary point at E_0 as unique singularity.*

Proof. It is an immediate application of the standard Bertini's theorem (see for instance [**10**, Theorem II.8.18 and Remark II.8.18.1]). □

Let $D \in \mathbf{P}(V_1)$ be general; then D has geometric genus g, by the genus formula (see [**10**, Example V.3.9.2]) and the line $r := \{x_0 = 0\}$ cut out on D the divisor $(g+2)E_1$. The lines through E_0 cut out on D a g_2^1 (which is unique by [**10**, Proposition IV.5.3]), hence the desingularization $\pi: C \to D$ is a hyperelliptic curve of genus g which is naturally pointed by the point p_1 corresponding to the non–singular point E_1. Then we have a rational map

$$h_1: \mathbf{P}(V_1) \dashrightarrow \mathcal{H}_{g,1}.$$

Let $\Delta := (g+2)p_1$: Since the tangent line at E_1 does not pass through E_0, then p_1 is not a Weierstrass point of C, then $h^0\big(C, \omega_C(-\Delta)\big) = 0$ and, by construction, π is the morphism associated to a fixed basis of $H^0\big(C, \mathcal{O}_C(\Delta)\big)$.

Assume that $D' \in h_1^{-1}(C', p_1')$ is another pointed curve. The existence of an isomorphism $\Psi: C \to C'$ mapping p_1 to p_1' induces a projectivity ψ from $|\Delta|^{\check{}} \cong \mathbf{P}^2_{\mathbf{C}}$ to $|\Delta'|^{\check{}} \cong \mathbf{P}^2_{\mathbf{C}}$ (here $\Delta' := (g+2)p_1'$) such that $\psi(D) = D'$ and fixing E_1, hence r. Moreover, it also must obviously fix the unique singular point E_0.

We conclude that ψ is represented by a matrix in the solvable group

$$G_1 := \left\{ \begin{pmatrix} a_{0,0} & 0 & 0 \\ 0 & a_{1,1} & a_{1,2} \\ 0 & 0 & a_{2,2} \end{pmatrix} \right\} \subseteq \mathrm{GL}_3.$$

Conversely, each projectivity $\psi: \mathbf{P}^2_{\mathbf{C}} \to \mathbf{P}^2_{\mathbf{C}}$ represented by a matrix in G_1 maps V_1 into itself. It follows that the fibers of h_1 are orbits in V_1 with respect to the action of G_1; hence,

$$\dim(\mathrm{im}(h_1)) \geq \dim(\mathbf{P}(V_1)) - \dim(G_1) = 2g = \dim(\mathcal{H}_{g,1}):$$

we conclude that h_1 is dominant, since $\mathcal{H}_{g,1}$ is irreducible, i.e.,

Proposition 2.1.2. *There is a birational equivalence $V_1/G_1 \approx \mathcal{H}_{g,1}$.*

2.2. The case $n = 2, 3$. In this case we consider

$$V_n := x_0^2\mathbf{C}[x_1, x_2]_g \oplus x_0\mathbf{C}[x_1, x_2]_{g+1} \oplus \mathbf{C}x_2^{g+3-n}x_1(x_1 - x_2)^{n-2} \\ \subseteq \mathbf{C}[x_1, x_2, x_3]_{g+2}.$$

Again (see Lemma 2.1.1),

Lemma 2.2.1. *The general element of $D \in \mathbf{P}(V_n)$ is an integral curve carrying a g-fold ordinary point at E_0 as unique singularity.*

Each general $D \in \mathbf{P}(V_n)$ has geometric genus g and the line $r := \{x_0 = 0\}$ cut out on D the divisor $(g+3-n)E_1 + E_2 + (n-2)E$ where $E := [0, 1, 1]$. Again the lines through E_0 cut out on D a g_2^1; hence, the desingularization $\pi: C \to D$ is a hyperelliptic curve of genus g which is naturally n-pointed by the points p_1, p_2 and possibly p_3 corresponding to the non-singular point E_1, E_2 and E. Thus, we have a rational map

$$h_n: \mathbf{P}(V_n) \dashrightarrow \mathcal{H}_{g,n}.$$

Let

$$G_2 := \left\{ \begin{pmatrix} a_{0,0} & 0 & 0 \\ 0 & a_{1,1} & 0 \\ 0 & 0 & a_{2,2} \end{pmatrix} \right\} \subseteq \mathrm{GL}_3,$$
$$G_3 := \left\{ \begin{pmatrix} a_{0,0} & 0 & 0 \\ 0 & a_{1,1} & 0 \\ 0 & 0 & a_{1,1} \end{pmatrix} \right\} \subseteq \mathrm{GL}_3,$$

respectively.

An argument similar to the one for 1-pointed curves allows us to prove the following:

Proposition 2.2.3. *There is a birational equivalence $V_n/G_n \approx \mathcal{H}_{g,n}$, $n = 2, 3$.*

2.3. The case $4 \leq n \leq 2g+8$. Let D be an integral curve singular only at E_0 and having equation in

$$V := x_0^2\mathbf{C}[x_1, x_2]_g \oplus x_0\mathbf{C}[x_1, x_2]_{g+1} \oplus x_1^{g-1}x_2(x_1 - x_2)\mathcal{C}[x_1, x_2]_1 \\ \subseteq \mathbf{C}[x_1, x_2, x_3]_{g+2}.$$

As in the previous cases $n = 1, 2, 3$, we have:

Lemma 2.3.1. *The general element of $D \in \mathbf{P}(V)$ is an integral curve carrying a g-fold ordinary point at E_0 as a unique singularity.*

Again the lines through E_0 cut out on each general $D \in \mathbf{P}(V)$ a g^1_2, and the desingularization $\pi\colon C \to D$ is a hyperelliptic curve of genus g which is naturally 4-pointed by the points p_1, p_2, p_3, p_4 corresponding to the non-singular point $E_1, E_2, E := [0, 1, 1], U$ where U is the remaining intersection of D with the line $r := \{x_0 = 0\}$.

If we set

$$G := \left\{ \begin{pmatrix} a_{0,0} & 0 & 0 \\ 0 & a_{1,1} & 0 \\ 0 & 0 & a_{1,1} \end{pmatrix} \right\} \subseteq \mathrm{GL}_3,$$

we can argue as in the cases $n = 1, 2, 3$, obtaining:

Proposition 2.3.2. *There is a birational equivalence $V/G \approx \mathcal{H}_{g,4}$.*

Now take $n \geq 5$, and consider the incidence variety

$$X_n := \{(D, A_5, \dots, A_n) \in \mathbf{P}(V) \times (\mathbf{P}^2_{\mathbf{C}})^{n-4} \mid A_i \in D,\ i = 5, \dots, n\}.$$

The scheme X_n is fibered on $(\mathbf{P}^2_{\mathbf{C}})^{n-4}$ with fiber $\mathbf{P}^{2g+8-n}_{\mathbf{C}}$. Taking into account the above description of the curves corresponding to the polynomials in V, the points of X_n represent n-pointed hyperelliptic curves of genus g when $4 \leq n \leq 2g+8$. The action on X_n of the image of G via the natural quotient map $\mathrm{GL}_3 \to \mathrm{PGL}_3$ is equivalent to the action of $\mathbf{C}^*$ given by $[x_1, x_2, x_3] \mapsto [x_1, \alpha x_2, \alpha x_3]$.

Again, we have, as in the previous cases, the following:

Proposition 2.3.3. *There is a birational equivalence $X_n/\mathbf{C}^* \approx \mathcal{H}_{g,n}$, $5 \leq n \leq 2g+8$.*

We are now able to give the following

Proof of the main theorem. In view of Propositions 2.1.2, 2.2.2, 2.3.2 and 2.3.3 it suffices to prove that the quotients V_n/G_n, $n = 1, 2, 3$, V/G and $X_n/\mathbf{C}^*$ are rational.

In the first case V_n is a linear representation of a solvable and connected algebraic group G_n. Thus the quotient on the right is rational by [**15**] or [**18**]. The same argument holds for the quotient V/G.

On the other hand, if $5 \leq n \leq 2g+8$, we have checked above that $\mathcal{H}_{g,n} \approx X_n/\mathbf{C}^*$ where X_n is a $\mathbf{C}^*$-linearized projective bundle over the rational base $(\mathbf{P}^2_{\mathbf{C}})^{n-4}$ with typical fiber $\mathbf{P}^{2g+8-n}_{\mathbf{C}}$ over an open and dense subset $\mathcal{U} \subseteq (\mathbf{P}^2_{\mathbf{C}})^{n-4}$.

The scheme X_n is contained in the $\mathbf{C}^*$-equivariant trivial projective bundle $\mathbf{P}(V) \times \mathcal{U}$. The subspace

$$W := x_0^2\mathbf{C}[x_1, x_2]_g \oplus x_0\mathbf{C}[x_1, x_2]_{g+1} \oplus x_1^g x_2(x_1 - x_2)\mathbf{C} \subseteq V$$

is G-invariant; hence, $L := \mathbf{P}(W) \times \mathcal{U} \subseteq \mathbf{P}(V) \times \mathcal{U}$ is a $\mathbf{C}^*$-invariant unisecant $L \subseteq \mathbf{P}(V) \times \mathcal{U}$ (i.e., a divisor intersecting the general fiber in a hyperplane). It follows that the scheme $L \cap X_n$ is then a $\mathbf{C}^*$-invariant unisecant on X_n; hence, X_n is $\mathbf{C}^*$-equivariantly birational to the vector bundle $\mathbf{C}^{2g+8-n} \times (\mathbf{P}^2_{\mathbf{C}})^{n-4}$ for $4 \le n \le 2g+8$.

Obviously, the action of $\mathbf{C}^*$ on $(\mathbf{P}^2_{\mathbf{C}})^{n-4}$ is almost free (i.e., the stabilizer in $\mathbf{C}^*$ of the general point of $(\mathbf{P}^2_{\mathbf{C}})^{n-4}$ is trivial). Thus, by the results of Section 4 of [**8**], $\mathcal{H}_{g,n} \approx X_n/\mathbf{C}^* \approx \mathbf{C}^{2g+8-n} \times (\mathbf{P}^2_{\mathbf{C}})^{n-4}/\mathbf{C}^*$ is a vector bundle over the base $(\mathbf{P}^2_{\mathbf{C}})^{n-4}/\mathbf{C}^*$ with $(2g+8-n)$-dimensional fiber.

Thus it suffices to prove that the quotient $(\mathbf{P}^2_{\mathbf{C}})^{n-4}/\mathbf{C}^*$ is rational. This is more or less trivial: indeed, $\mathbf{P}^2_{\mathbf{C}}$ is the quotient of $\mathbf{C}^3$ modulo the standard diagonal action of $\mathbf{C}^*$; hence, $(\mathbf{P}^2_{\mathbf{C}})^{n-4}/\mathbf{C}^* \approx \mathbf{C}^{3(n-4)}/T$ where $T = (\mathbf{C}^*)^{n-4} \times \mathbf{C}^*$ acts linearly. Since T is solvable and connected being a torus, the rationality of $(\mathbf{P}^2_{\mathbf{C}})^{n-4}/\mathbf{C}^*$ again follows from [**15**] or [**18**]. $\square$

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