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EIGENVALUE CLUSTER TRACES FOR QUANTUM GRAPHS WITH EQUAL EDGE LENGTHS

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ABSTRACT. On a finite metric graph with standard vertex conditions and equal edge lengths, the large magnitude eigenvalues of the Schrödinger operator $\Delta + q$ cluster near the eigenvalues of the Laplace operator Δ. Based on the spectral 'periodicity' of Δ, the clusters can be partitioned into a finite collection of classes. There is a class dependent formula for the cluster trace (or average eigenvalue shift) in the large magnitude limit which expresses the trace as a function of the edge integrals $\int_e q$ and data from the underlying combinatorial graph.

1. Introduction. Suppose Δ denotes the Laplace operator on a finite metric graph $\mathcal{G}$ with edges of length 1 and standard vertex conditions. The eigenvalues λ_n of Δ are known to have a simple 'periodic' structure. Let $\omega_1^2 < \omega_2^2 < \cdots < \omega_K^2$ be the distinct eigenvalues of Δ in the interval $(0, 4\pi^2]$. The positive eigenvalues of Δ are precisely the set of numbers

$$\omega_{k,m}^2 = [\omega_k + 2\pi m]^2, \quad m = 0, 1, 2, \dots ,$$

with multiplicities independent of m.

The large magnitude eigenvalues μ of a Schrödinger operator $\Delta + q$ will fall into clusters around the unperturbed eigenvalues λ_n, with cluster multiplicities depending on the class k of λ_n. A previous paper [**6**] studied the eigenvalues of $\Delta + q$ by developing characteristic function asymptotic expansions. One of the results in [**6**] was a contour integral formula for the cluster trace or average eigenvalue shift in the cluster about λ_n. This formula showed that the average eigenvalue shift in the cluster associated to $\lambda_n = \omega_{k,m}^2$ has a limit as $m \to \infty$, and the limiting

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shift is given by a linear combination of the numbers $\int_e q(x)\,dx$, the integrals of q over the graph edges. The contour integral formulation obscured the value of the coefficients, along with any geometric content they might have.

The main aim of this paper is to redevelop the cluster trace formula, making explicit the connection between the average eigenvalue shifts and the underlying combinatorial graph. This development begins in the second section with a review of quantum graphs, emphasizing graphs whose edges are all of length 1. Most of the material in this section is a reworking of results by other authors, although Theorem 2.8 seems to be new.

The third section develops some new material on eigenfunctions of graphs with edges of length one. It is well known [**14**, page 31] that the derivative of simple eigenvalues with respect to a (real) potential is the square of the normalized eigenfunction. To exploit this fact, we consider subsequential weak limits of squared magnitudes of eigenfunctions, identifying the limits with features of the underlying combinatorial graph.

The technical developments of Section 3 are used in Section 4 to describe two important properties of clusters of eigenvalues μ_j of $\Delta + q$ near λ_n. If q is differentiable, vanishes at the vertices, and satisfies $\int_e q = 0$ for each edge e, then the eigenvalues μ_j converge to λ_n as $n \to \infty$. When these conditions are relaxed, there is a general cluster trace formula for

$$\lim_{m\to\infty} \sum_j (\mu_j - \omega_{k,m}^2),$$

the limit being described in terms of the numbers $\int_e q$ and purely combinatorial data from the graph.

2. Background.

2.1. The Laplace operator for continuous graphs. A review of quantum graphs will help to establish notation and the background needed for our results, Details and additional information may be found in [**5, 12**] or in the collections [**3, 8, 11**]. Most of the material in this section is closely related to developments in [**1, 2, 9**]. For this work a graph $\mathcal{G}$ will be finite, with vertex set $\mathcal{V}$ consisting of $N_{\mathcal{V}}$ vertices and edgeset $\mathcal{E}$ having $N_{\mathcal{E}}$ edges. To avoid minor technical issues, the graph is assumed simple, and all edges have length 1. Multigraphs, graphs with loops, or graphs whose edge lengths are integer multiples of a common value can be easily incorporated by inserting additional vertices. There are no boundary vertices; that is, every vertex has degree at least 2.

The edges e_n are initially assumed to be oriented and numbered, although this is mainly for notational convenience. Consistent with the edge orientations, each edge is identified with the interval $[0,1]$. The usual metric and Lebesgue measure on intervals are extended to $\mathcal{G}$. $L^2(\mathcal{G})$ will denote the Hilbert space $\oplus_n L^2(e_n)$ with the inner product

$$\langle f, g\rangle = \int_{\mathcal{G}} f\overline{g} = \sum_{n=1}^{N} \int_0^1 f_n(x)\overline{g_n(x)}\, dx, \quad f = (f_1, f_2, \dots, f_N).$$

The Laplace operator is a self adjoint operator on $L^2(\mathcal{G})$ which acts componentwise on functions in its domain by $\Delta f = -\partial^2 f/\partial x^2$. The domain of Δ may be characterized by vertex conditions. For vertices v with degree $\deg(v) \ge 2$, pick a local indexing $e_1, \dots, e_{\deg(v)}$ for the edges incident on v. Assume the standard local coordinates, which identify e_n with $[0,1]$ so that 0 corresponds to v for each edge. The standard continuity and derivative (Kirchhoff) conditions at v are required,

(2.1)

$$y_n(0) = y_{n+1}(0), \qquad n = 1, \dots, \deg(v) - 1, \qquad \sum_{n=1}^{\deg(v)} y_n'(0) = 0.$$

Let $\mathcal{D}_{\max}$ denote the set of functions $f \in L^2(\mathcal{G})$ with f' absolutely continuous on each e_n, and $f'' \in L^2(\mathcal{G})$. The domain $\mathcal{D}(\Delta)$ is then the set of $f \in \mathcal{D}_{\max}$ satisfying the standard vertex conditions (2.1). By the classical theory of ordinary differential operators [**12**, page S123] the operator Δ is self adjoint with compact resolvent. The distinct eigenvalues of Δ are denoted $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$.

For $\lambda \in \mathbf{C}$, let $E(\lambda)$ denote the eigenspace for Δ with eigenvalue λ. For $\omega > 0$, let $\lambda = \omega^2$, and with respect to a fixed edge orientation write $y(x) \in E(\omega^2)$ as

$$y(x) = A_e \cos(\omega x) + B_e \sin(\omega x).$$

The linear map $\mathcal{T}$ taking y to

$$y_1(x) = A_e \cos([\omega + 2n\pi]x) + B_e \sin([\omega + 2n\pi]x)$$

preserves vertex values, while y_1 satisfies the same interior vertex conditions that y does. It is easy to check that the eigenvalues of Δ exhibit the following 'periodicity.'

Proposition 2.1. *If $\omega > 0$, the linear map $\mathcal{T} : E(\omega^2) \to E((\omega + 2\pi)^2)$ is a vector space isomorphism.*

The spectrum of Δ is determined by algebraic structures of the discrete graph $\mathcal{G}$. If $\lambda \notin \{n^2\pi^2 \mid n = 1, 2, 3, \dots\}$, the positive eigenvalues and their multiplicities are derived from the discrete Laplacian acting on the vertex space of $\mathcal{G}$. If $\lambda \in \{n^2\pi^2 \mid n = 1, 2, 3, \dots\}$, the key roles are played by subspaces of the edge space of $\mathcal{G}$.

2.2. The case $\lambda \ne n^2\pi^2$. For a vertex $v \in \mathcal{G}$, let $u_1, \dots , u_{\deg(v)}$ denote the vertices adjacent to v. The vertex space $\mathbf{V}$ of a discrete graph consists of functions $f : \mathcal{V} \to \mathbf{C}$. Among the linear operators acting on the vertex space are the adjacency operator

$$Af(v) = \sum_{i=1}^{\deg(v)} f(u_i),$$

and the degree operator

$$D_v f(v) = \deg(v) f(v).$$

Define the operator Δ_1 by

$$\Delta_1 f(v) = f(v) - D_v^{-1} A f(v),$$

which is similar to the discrete Laplacian $I - D_v^{-1/2} A D_v^{-1/2}$ of [**7**, page 3].

Relating Δ_1 to Δ is facilitated by the following observation. If

$$-y'' = \lambda y, \quad x \in [0, 1], \ \lambda \notin \{n^2\pi^2\},$$

then

$$y(x,\lambda) = y(0)\cos(\omega x) + [y(1) - y(0)\cos(\omega)]\frac{\sin(\omega x)}{\sin(\omega)}. \tag{2.2}$$

The next lemma follows easily.

Lemma 2.2. *Fix $\lambda \in \mathbf{C}$, and consider the vector space of solutions of $-y'' = \lambda y$ on the interval $[0,1]$. The linear function taking $y(x)$ to $(y(0), y(1))$ is an isomorphism if and only if $\lambda \notin \{n^2\pi^2 \mid n = 1, 2, 3, \dots\}$.*

The following consequence for graphs is immediate.

Lemma 2.3. *Suppose the edges of $\mathcal{G}$ have length 1, and $\lambda \notin \{n^2\pi^2 \mid n = 1, 2, 3, \dots\}$. Let $y : \mathcal{G} \to \mathbf{C}$ be continuous, and satisfy $-y'' = \lambda y$ on the edges. If $y(v) = 0$ at all vertices of $\mathcal{G}$, then $y(x) = 0$ for all $x \in \mathcal{G}$.*

Theorem 2.4. *Suppose $\lambda \notin \{n^2\pi^2 \mid n = 1, 2, 3, \dots\}$ and y is an eigenfunction for Δ. If v has adjacent vertices $u_1, \dots, u_{\deg(v)}$, then*

$$\cos(\omega) y(v) = \frac{1}{\deg(v)} \sum_{i=1}^{\deg(v)} y(u_i). \tag{2.3}$$

Proof. Using local coordinates which identify each u_i with 0, (2.2) for y_i on the edges $e_i = (u_i, v)$ gives

$$y_i'(v) = -\omega\sin(\omega) y_i(u_i) + [y_i(v) - y_i(u_i)\cos(\omega)]\frac{\omega\cos(\omega)}{\sin(\omega)}. \tag{2.4}$$

Summing over i, the derivative condition at v then yields

$$\begin{aligned} 0 &= \sum_i y_i'(v) \\ &= -\omega\sin(\omega)\sum_i y_i(u_i) + \frac{\omega\cos(\omega)}{\sin(\omega)}\sum_i [y_i(v) - y_i(u_i)\cos(\omega)]. \end{aligned}$$

Using the continuity of y at v and elementary manipulations gives (2.3). $\square$

Equation (2.3) is an eigenvalue equation, with eigenvalue $\cos(\omega)$, for the linear operator $T^{-1}A$ acting on $\mathbf{V}$. The next result makes the connection between the spectra of Δ_1 and Δ explicit.

Theorem 2.5. *If $\lambda \notin \{n^2\pi^2 \mid n = 0,1,2,\dots\}$, then λ is an eigenvalue of Δ if and only if $1-\cos(\omega) = 1-\cos(\sqrt{\lambda})$ is an eigenvalue of Δ_1, with the same geometric multiplicity.*

Proof. Since $\Delta_1 = I - T^{-1}A$, it suffices to consider $T^{-1}A$. Suppose $y(x,\lambda)$ is an eigenfunction of Δ satisfying (2.1). Then Theorem 2.4 shows that the (linear) evaluation map taking $y : \mathcal{G} \to \mathbf{C}$ to $y : \mathcal{V} \to \mathbf{C}$ takes eigenfunctions to solutions of (2.3). The map is injective by Lemma 2.3.

Suppose conversely that $y : \mathcal{V} \to \mathbf{C}$ satisfies

$$T^{-1}Ay(v) = \mu y(v), \quad |\mu| < 1.$$

Pick $\lambda \in \cos^{-1}(\mu)$. By Lemma 2.2 the function $y : \mathcal{V} \to \mathbf{C}$ extends to a unique continuous function $y(x,\lambda) : \mathcal{G} \to \mathbf{C}$ satisfying $-y'' = \lambda y$ on each edge. In local coordinates identifying v with 0 for each edge $e_i = (v, u_i)$ incident on v, this extended function satisfies (2.4). Summing gives

$$\begin{aligned}\sum_i y_i'(v) &= \frac{\omega}{\sin(\omega)}[-\sin^2(\omega) - \cos^2(\omega)] \sum_i y_i(u_i) \\ &\quad + \sum_i y_i(v) \frac{\omega\cos(\omega)}{\sin(\omega)} \\ &= -\frac{\omega}{\sin(\omega)} \sum_i y_i(u_i) + \deg(v) y(v) \frac{\omega\cos(\omega)}{\sin(\omega)}.\end{aligned}$$

The vertex values satisfy (2.3), so

$$\sum_i y_i'(v) = -\frac{\omega}{\sin(\omega)} \deg(v)\cos(\omega) y(v) + \deg(v) y(v) \frac{\omega\cos(\omega)}{\sin(\omega)} = 0.$$

The extended functions, satisfying (2.1), are eigenfunctions of Δ. Since the extension map is linear, and the kernel is the zero function, this map is also injective. □

2.3. The case $\lambda = n^2\pi^2$. Turning to eigenvalues $\lambda \in \{n^2\pi^2\}$, recall [**7**, page 7] that for both Δ_1 and Δ the eigenspace for eigenvalue 0 is spanned by functions which are constant on connected components of $\mathcal{G}$. For $n \geq 1$ the eigenspaces $E(n^2\pi^2)$ for Δ have a combinatorial interpretation closely related to the cycles in $\mathcal{G}$.

Recall [**4**, pages 51–58] the construction of the edge space and cycle subspace of $\mathcal{G}$. The edge space $\mathbf{E}$ is the complex vector space of functions $f : \mathcal{E} \to \mathbf{C}$. The standard basis is the set of functions $\{f_e\}$ with $f_e(e) = 1$ while $f_e(e_1) = 0$ for edges $e_1 \neq e$. Suppose $(v_0, \dots, v_{K-1})$ is an ordered K-tuple of distinct vertices of $\mathcal{G}$ such that (v_k, v_{k+1}) is an edge of $\mathcal{G}$ for $k = 0, \dots, K-2$, as is (v_{K-1}, v_0). The cycle $(v_0, \dots, v_{K-1})$ is the subgraph of $\mathcal{G}$ with these vertices and edges. The oriented cycle $(v_0, \dots, v_{K-1})$ orients the edges of the cycle from v_k to v_{k+1} for $k = 0, \dots, K-2$ and from v_{K-1} to v_0. Cycles are the simple closed curves of a graph.

Given an orientation for the edges e of $\mathcal{G}$, and an oriented cycle γ, we may define a function $f_\gamma : \mathcal{E} \to \mathbf{C}$ by taking $f(e) = 0$ if e is not an edge of the cycle, while if e is an edge of the cycle, $f(e) = 1$ (respectively -1) if the graph orientation of e agrees (respectively disagrees) with the cycle orientation of e. The cycle subspace $\mathbf{Z}_0$ of the (oriented) edge space is the span of the images of the cycles. If $\mathcal{G}$ is connected, the cycle subspace has dimension $N_E - N_V + 1$.

When $\lambda = n^2\pi^2$ for $n \geq 1$, eigenfunctions of Δ have special vanishing properties.

Lemma 2.6. *Suppose $\psi \in E(n^2\pi^2)$ for $n = 1, 2, 3, \dots$.*

If $\mathcal{G}$ is connected, and $\psi(v) = 0$ for some vertex v, then ψ vanishes at all vertices.

The eigenspaces $E(4n^2\pi^2)$ have an eigenfunction vanishing at no vertices.

The eigenspaces $E((2n-1)^2\pi^2)$ have an eigenfunction vanishing at no vertices if and only if $\mathcal{G}$ is bipartite.

Proof. Suppose $\psi(v) = 0$ for some vertex v. On each incident edge, $\psi(x) = B\sin(n\pi x)$, so at all adjacent vertices w one finds $\psi(w) = B\sin(n\pi) = 0$, which suffices for the first claim.

For the second claim the desired eigenfunction is simply $\cos(2n\pi x)$ in local coordinates on each edge.

Suppose $\mathcal{G}$ is bipartite, with the two classes of vertices labeled 0 and 1. Pick local coordinates on each edge consistent with the vertex class labels, and define the eigenfunction to be $\cos([2n-1]\pi x)$.

Suppose conversely that for some eigenvalue $(2n-1)^2\pi^2$ there is an eigenfunction ψ vanishing at no vertex. Without loss of generality, assume ψ is real valued. In local coordinates for an edge,

$$\psi(x) = a\cos([2n-1]\pi x) + b\sin([2n-1]\pi x), \quad a \neq 0.$$

If w and v are adjacent vertices, then $\psi(w) = -\psi(v)$, showing that vertices can be labeled by the sign of ψ, so $\mathcal{G}$ is bipartite. $\square$

Let $E_0(n^2\pi^2)$ denote the subspace of $E(n^2\pi^2)$ consisting of those eigenfunctions of Δ vanishing at the vertices. Fix an orientation for the edges of $\mathcal{G}$ and edge coordinates $x : e \to [0,1]$ consistent with the orientation. For each $\psi \in E_0(n^2\pi^2)$ and each edge e, the restriction of ψ to e has the form $\psi_e(x) = a_e\sin(2\pi nx)$. Thus, there is a linear map J_0 from $E_0(n^2\pi^2)$ to the edge space of $\mathcal{G}$ taking ψ to the function given by $f_\psi(e) = a_e$. If ψ is in the null space of J_0, then ψ_e is the zero function for each edge e, so J_0 is injective. First we treat the cases when n is even.

Theorem 2.7. *For $n = 1,2,3,\ldots$, the linear map $J_0 : E_0(4n^2\pi^2) \to \mathbf{E}$ has range equal to the cycle space $\mathbf{Z}_0$.*

Proof. Suppose γ is an oriented cycle and, for the edges of the cycle, let $t : e \to [0,1]$ be an edge coordinate consistent with the cycle orientation. An eigenfunction ψ of Δ may be defined as 0 on edges not in the cycle, while for the edges of the cycle $\psi_e(t) = \sin(2n\pi t)$. For e in the cycle, if the cycle orientation of e agrees with the graph orientation, then $a_e = 1$. If the cycle orientation of e disagrees with the graph orientation, then $t = 1 - x$ and $\psi_e = -\sin(2n\pi x)$, so $a_e = -1$. Thus, each oriented cycle is in the range of J_0.

It suffices to prove the result for a connected graph. Suppose $\mathbf{Z}_0$ has dimension M. Pick a spanning tree $\mathcal{T}$ for $\mathcal{G}$. There are [**4**, p. 53] M edges e_j of $\mathcal{G}$ not in $\mathcal{T}$, and a basis of cycles $\gamma_1, \dots, \gamma_M$ such that each γ_j contains an edge $e_j \notin \mathcal{T}$, with e_j not contained in any other γ_i. Fix $n \in \{1, 2, 3, \dots\}$, and construct eigenfunctions ψ_j as above for each γ_j.

Now suppose that $\psi \in E_0(4n^2\pi^2)$. After subtracting a linear combination $\sum a_j\psi_j$, we may assume that ψ vanishes on all edges e_j not in the spanning tree. For every boundary vertex v of $\mathcal{T}$, ψ vanishes identically on all but one edge of $\mathcal{G}$ incident on v, and by the vertex conditions it then vanishes on all edges incident on v. Continuing away from the boundary of the spanning tree, we see that ψ is the 0 function, so the functions ψ_j are a basis for $E_0(4n^2\pi^2)$. □

Let $I_0 : \mathbf{Z}_0 \to E_0(4n^2\pi^2)$ be J_0^{-1}.

A combinatorial construction of $E_0((2n-1)^2\pi^2)$ is also available. Let $e \simeq v$ indicate that e is incident on v, and let $\mathbf{Z}_1$ denote the subspace of the edge space consisting of functions $f : \mathcal{E} \to \mathbf{C}$ with

$$\sum_{e \simeq v} f(e) = 0, \quad v \in \mathcal{V}.$$

Theorem 2.8. *For $n = 1, 2, 3, \dots$, the linear map I_1 taking $f \in \mathbf{Z}_1$ to $g(x) \in E_0((2n-1)^2\pi^2)$ defined by*

$$g_e(x) = f(e)\sin((2n-1)\pi x),$$

is an isomorphism. The subspace $\mathbf{Z}_1$ has an integral basis.

Proof. Since $\sin((2n-1)\pi x) = \sin((2n-1)\pi(1-x))$, the edge orientation does not affect the definition, so g is well defined. Clearly $g(x)$ satisfies the eigenvalue equation and vanishes at each vertex. The condition $\sum_{e\simeq v} f(e) = 0$ for all $v \in \mathcal{V}$ gives the derivative condition, so $g \in E_0((2n-1)^2\pi^2)$. Moreover, the map is one to one.

Suppose $\psi(x) \in E_0((2n-1)^2\pi^2)$. Then $\psi_e(x) = a_e \sin((2n-1)\pi x)$ on each edge, and the derivative condition (2.1) gives

$$\sum_{e \simeq v} a_e = 0, \quad v \in \mathcal{V}.$$

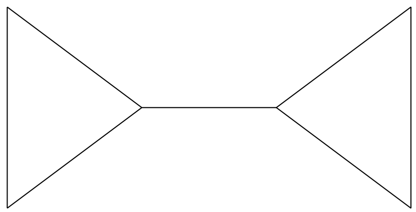

FIGURE 2.1. Bowtie graph.

Thus the injection J_0 maps $E_0((2n-1)^2\pi^2)$ to $\mathbf{Z}_1$. This establishes the isomorphism.

To see that $\mathbf{Z}_1$ has an integral basis, let e_n be a numbering of the edges of $\mathcal{G}$, and let $x_n = f(e_n)$. The set of functions $\mathbf{Z}_1$ is then given by the set of $x_1, \dots, x_{N_\mathcal{E}}$ satisfying the $N_\mathcal{V}$ equations

$$\sum_{e_n \simeq v} x_n = 0.$$

This is a system of linear homogeneous equations whose coefficient matrix consists of ones and zeros. Reduction by Gaussian elimination shows that the set of solutions has a rational basis, and so an integral basis. □

2.1. An example. A simple example will help illustrate Theorems 2.5, 2.7 and 2.8, as well as material remaining to be considered. The example is the bowtie graph in Figure 2.1. This graph has two independent cycles (triangles) of length 3.

Each eigenspace $E(4n^2\pi^2)$ has an orthonormal basis ϕ_1, ϕ_2, ϕ_3, where $\phi_1 = \sqrt{2/7}\cos(2n\pi x)$ on each edge, $\phi_2 = \sqrt{2/3}\sin(2n\pi x)$ on the left triangle, while vanishing on the other edges, and $\phi_3 = \sqrt{2/3}\sin(2n\pi x)$ on the right triangle, while vanishing on the other edges. The sum of the squares of the magnitudes of an orthonormal basis for each eigenspace will soon be a concern. In this case, the function $|\phi_1(x)|^2 + |\phi_2(x)|^2 + |\phi_3(x)|^2$ is nonconstant, being strictly smaller for almost every point on the central edge than at corresponding points on the other edges. Let $T_1(x)$ denote the function which is the constant ε on the leftmost triangle, and 0 on all other edges. The operator $\Delta + T_1$ will still have ϕ_2 and ϕ_3 as eigenfunctions, but with eigenvalues $4n^2\pi^2$ and $4n^2\pi^2 + \varepsilon$.

For $n = 1, 2, 3, \ldots$, each eigenspace $E((2n-1)^2\pi^2)$ is one dimensional. An eigenfunction ϕ can be constructed by letting $\phi(x) = 2\sin((2n-1)\pi x)$ on the middle edge. The function ϕ then continues as $-\sin((2n-1)\pi x)$ on the edges adjacent to the middle edge, and $\sin((2n-1)\pi x)$ on the remaining two edges. Notice that, on each edge, the function ϕ is an integer multiple of $\sin(\pi x)$. Also, $|\phi_1(x)|^2$ gives 4 times as much weight to the points of the central edge compared to corresponding points in the triangles.

3. Eigenfunctions of Δ and the shift map $\mathcal{T}$. This section provides additional information about the eigenfunctions of Δ. The shift map $\mathcal{T}$ discussed in Proposition 2.1 plays an important role. The cases $\omega = n\pi$ and $\omega \neq n\pi$ are treated separately.

3.1. The case $\omega = n\pi$. As shown above, the eigenspaces $E(n^2\pi^2)$ are closely linked to subspaces of the edge space of $\mathcal{G}$. Define an inner product on the edge space, and its subspaces $\mathbf{Z}_0$ and $\mathbf{Z}_1$ by

$$\langle f, g\rangle_E = \frac{1}{2}\sum_{e\in\mathcal{E}} f(e)\overline{g(e)}.$$

Lemma 3.1. *For $n = 1, 2, 3, \ldots$, the maps*

$$I_0 : \mathbf{Z}_0 \longrightarrow E_0(4n^2\pi^2), \qquad I_1 : \mathbf{Z}_1 \longrightarrow E_0((2n-1)^2\pi^2)$$

and

$$\mathcal{T} : E(n^2\pi^2) \longrightarrow E([n\pi + 2\pi]^2)$$

preserve inner products.

Proof. These results are elementary edgewise computations. Treating I_0 and I_1 first, suppose $y, z \in E_0(n^2\pi^2)$. For $x \in e$,

$$y(x) = A_e \sin(n\pi x), \qquad z(x) = B_e \sin(n\pi x).$$

Then

$$\int_e y\overline{z} = A_e\overline{B_e}\int_0^1 \sin^2(n\pi x) = \frac{1}{2}A_e\overline{B_e}.$$

For the shift $\mathcal{T}$, suppose $y, z \in E(n^2\pi^2)$. For $x \in e$ write

$$y(x) = \alpha e^{in\pi x} + \beta e^{-in\pi x}, \qquad z(x) = \gamma e^{in\pi x} + \delta e^{-in\pi x},$$

and

$$\mathcal{T}y = \alpha e^{i[n\pi+2\pi]x} + \beta e^{-i[n\pi+2\pi]x},$$
$$\mathcal{T}z = \gamma e^{i[n\pi+2\pi]x} + \delta e^{-i[n\pi+2\pi]x}.$$

Then

$$\int_0^1 \mathcal{T}y(x)\overline{\mathcal{T}z(x)} = \alpha\overline{\gamma} + \beta\overline{\delta} + \alpha\overline{\delta}\int_0^1 e^{i[2n\pi+4\pi]x}\,dx + \beta\overline{\gamma}\int_0^1 e^{-i[2n\pi+4\pi]x}\,dx = \alpha\overline{\gamma} + \beta\overline{\delta} = \int_0^1 y(x)\overline{z(x)}. \qquad \square$$

Lemma 3.2. *For a fixed n, suppose $y, z \in E(n^2\pi^2)$. Then the sequence $\mathcal{T}^m y\overline{\mathcal{T}^m z}$ converges weakly in $L^2(\mathcal{G})$ to a function constant on each edge.*

Proof. For each edge e write

$$y_e(x) = A_e\sin(n\pi x + \delta_e),$$
$$\mathcal{T}^m y_e(x) = A_e\sin([n\pi + 2m\pi]x + \delta_e)$$

and

$$z_e(x) = B_e\sin(n\pi x + \gamma_e),$$
$$\mathcal{T}^m z_e(x) = B_e\sin([n\pi + 2m\pi]x + \gamma_e).$$

By an elementary formula,

$$\mathcal{T}^m y_e(x)\overline{\mathcal{T}^m z_e(x)} = A_e\overline{B_e}\frac{\cos(\delta_e - \gamma_e)}{2} - A_e\overline{B_e}\frac{\cos([2n\pi + 4m\pi]x + \delta_e + \gamma_e)}{2}. \tag{3.1}$$

If $q \in L^2(\mathcal{G})$, then $q_e \in L^2[0,1]$. Expanding q_e in a Fourier series,

$$q_e(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(2\pi kx) + b_k \sin(2\pi kx), \tag{3.2}$$

one computes that

$$\lim_{m\to\infty} \int_0^1 \mathcal{T}^m y_e \overline{\mathcal{T}^m z_e} q_e(x) = \int_0^1 \frac{1}{2} A_e \overline{B_e} \cos(\delta_e - \gamma_e) q_e(x). \tag{3.3}$$ □

3.2. $\omega \ne n\pi$**.** Next we turn to the eigenvalues $\lambda \notin \{n^2\pi^2\}$. As noted in (2.2), the eigenfunctions of Δ are identified through their vertex values with eigenvectors of Δ_1 on the vertex space $\mathbf{V}$. The vertex space is given the inner product

$$\langle f, g\rangle_{\mathbf{V}} = \frac{1}{2} \sum_{v\in\mathcal{V}} \deg(v) f(v) \overline{g(v)}.$$

For eigenfunctions y and z with the same eigenvalue ω^2, the operator $\mathcal{T}^m$ acts by

$$\mathcal{T}^m y(x,\lambda) = y(0)\cos([\omega + 2m\pi]x) + \frac{y(1) - y(0)\cos(\omega)}{\sin(\omega)} \sin([\omega+2m\pi]x). \tag{3.4}$$

The following limiting behavior is observed.

Lemma 3.3. *Suppose y and z are eigenfunctions for Δ with a common eigenvalue $\omega^2 \notin \{n^2\pi^2\}$. Then, for each edge e, the sequence $\mathcal{T}^m y_e(x,\lambda)\overline{\mathcal{T}^m z_e(x,\lambda)}$ has the weak limit*

$$\frac{1}{2} y(0)\overline{z(0)} + \frac{1}{2\sin^2(\omega)} [y(1) - y(0)\cos(\omega)][\overline{z(1)} - \overline{z(0)}\cos(\omega)] \tag{3.5}$$

and

$$\lim_{m\to\infty} \int_{\mathcal{G}} \mathcal{T}^m y \overline{\mathcal{T}^m z} = \frac{1}{2}\sum_v deg(v) y(v)\overline{z(v)}. \tag{3.6}$$

The sequence of operators $\mathcal{T}^m : E(\omega^2) \to E([\omega + 2m\pi]^2)$ is bounded, with a strictly positive lower bound.

Proof. For $\theta \in \mathbf{R}$ one has

$$[A\cos(\theta) + B\sin(\theta)][C\cos(\theta) + D\sin(\theta)] \\ = \frac{AC+BD}{2} + \frac{AC-BD}{2}\cos(2\theta) + \frac{BC+AD}{2}\sin(2\theta).$$

Thus

$$\begin{aligned} \mathcal{T}^m y \overline{\mathcal{T}^m z} &= y(0)\overline{z(0)}\frac{1+\cos([2\omega+4m\pi]x)}{2} \\ &\quad + y(0)[\overline{z(1)} - \overline{z(0)}\cos(\omega)]\frac{\sin([2\omega+4m\pi]x)}{2\sin(\omega)} \\ &\quad + [y(1) - y(0)\cos(\omega)]\overline{z(0)}\frac{\sin([2\omega+4m\pi]x)}{2\sin(\omega)} \\ &\quad + [y(1) - y(0)\cos(\omega)] \\ &\quad \times [\overline{z(1)} - \overline{z(0)}\cos(\omega)]\frac{1-\cos([2\omega+4m\pi]x)}{2\sin^2(\omega)}. \end{aligned} \tag{3.7}$$

A Fourier series computation establishes (3.5). In particular, the weak limit of $|\mathcal{T}^m y_e(x,\lambda)|^2$ is

$$\frac{|y(0)|^2}{2} + \frac{|y(1) - y(0)\cos(\omega)|^2}{2\sin^2(\omega)}. \tag{3.8}$$

Since $y(1)\overline{y(0)} + \overline{y(1)}y(0) \le |y(0)|^2 + |y(1)|^2$ and $\omega \ne n\pi$, the expression in (3.8) is a positive definite form of the vertex values of e.

Turning to the inner products,

$$
\begin{aligned}
\lim_{m\to\infty}\int_{\mathcal{G}} \mathcal{T}^m y\overline{\mathcal{T}^m z} &= \frac{1}{2\sin^2(\omega)}\sum_e \sin^2(\omega)y(0)\overline{z(0)} \\
&\quad + [y(1)-y(0)\cos(\omega)][\overline{z(1)}-\overline{z(0)}\cos(\omega)] \\
&= \frac{1}{2\sin^2(\omega)}\sum_e y(0)\overline{z(0)}+y(1)\overline{z(1)} \\
&\quad - [y(1)\overline{z(0)}+y(0)\overline{z(1)}]\cos(\omega) \\
&= \frac{1}{2\sin^2(\omega)}\sum_v \deg(v)y(v)\overline{z(v)} \\
&\quad - \cos(\omega)\overline{z(v)}\sum_{u\sim v} y(u).
\end{aligned}
\tag{3.9}
$$

When the vertex values $y(v)$ and $z(v)$ come from eigenvectors of Δ_1, (2.3) shows that

$$
\begin{aligned}
\lim_{m\to\infty}\int_{\mathcal{G}} \mathcal{T}^m y\overline{\mathcal{T}^m z} &= \frac{1}{2\sin^2(\omega)}\sum_v \deg(v)y(v)\overline{z(v)} \\
&\quad - \cos^2(\omega)\deg(v)y(v)\overline{z(v)} \\
&= \frac{1}{2}\sum_v \deg(v)y(v)\overline{z(v)}.
\end{aligned}
\tag{3.10}
$$

To see that the sequence of operators $\mathcal{T}^m : E(\omega^2)\to E([\omega+2m\pi]^2)$. is bounded, simply choose an orthonormal basis for $E(\omega^2)$ and apply (3.6) to the basis elements. If there were no strictly positive lower bound, then there would be a subsequence m_k and functions y_{m_k} with $\|y_{m_k}\| = 1$, but $\|T^{m_k}y_{m_k}\|\to 0$. Since the unit sphere in $E(\omega^2)$ is compact, we may pass to a subsequence and conclude there is a unit vector y which is the limit of y_{m_k}. Since T^m is a bounded sequence of operators,

$$T^{m_k}y = T^{m_k}y_{m_k} + T^{m_k}(y-y_{m_k}) \longrightarrow 0,$$

contradicting (3.6). $\square$

3.3. Some integrals. More information about the weak convergence results from Lemmas 3.2 and 3.3 will be required. A general observation about orthonormal functions will be helpful.

Lemma 3.4. *Suppose $\phi_1,\dots,\phi_M$ is an orthonormal basis of continuous functions for a subspace S of $L^2(\mathcal{G})$. Then the function $\sum_{j=1}^M |\phi_j(x)|^2$ is independent of the choice of orthonormal basis for S.*

Proof. Let ι be a subinterval of an edge e of $\mathcal{G}$, and let P_ι denote the orthogonal projection from $L^2(\mathcal{G})$ to the subspace of functions with support in ι. Let P_S denote the orthogonal projection onto S. Computing the trace,

$$\operatorname{tr}(P_S P_\iota P_S) = \sum_m \int_{\mathcal{G}} (P_\iota \phi_m)\overline{\phi_m} = \int_\iota \sum_m |\phi_m(x)|^2,$$

and this number is independent of the basis. The result follows since ι is any subinterval of any edge, and the functions ϕ_m are continuous. $\square$

Theorem 3.5. *Assume that $\mathcal{G}$ is connected, $q \in L^2(\mathcal{G})$, and $\phi_1,\dots,\phi_M$ is an orthonormal basis of eigenfunctions for $E([\omega + 2m\pi]^2)$.*

In case $\omega = 2\pi$, pick an orthonormal basis $\{\psi_j\}$ for the cycle space $\mathbf{Z}_0$, with values $\psi_j(e)$ on the edge e. Then

$$\lim_{m\to\infty} \int_{\mathcal{G}} q(x) \sum_{j=1}^M |\phi_j(x)|^2 = \frac{1}{2}\sum_e \left(\frac{2}{N_{\mathcal{E}}} + \sum_{j=1}^{M-1} |\psi_j(e)|^2\right) \int_0^1 q_e(x). \tag{3.11}$$

In case $\omega = \pi$, pick an orthonormal basis $\{\psi_j\}$ for the space $\mathbf{Z}_1$, with values $\psi_j(e)$ on the edge e. Then

$$\lim_{m\to\infty} \int_{\mathcal{G}} q(x) \sum_{j=1}^M |\phi_j(x)|^2 = \frac{1}{2}\sum_e \sum_{j=1}^{M} |\psi_j(e)|^2 \int_0^1 q_e(x) \tag{3.12}$$

if $\mathcal{G}$ is not bipartite, and (3.11) *holds if $\mathcal{G}$ is bipartite.*

In the case $\omega \notin \{n\pi\}$, *pick an orthonormal basis* $\{\psi_j\}$ *for the eigenspace of* Δ_1 *with eigenvalue* $1 - \cos(\omega)$. *Then*

$$\begin{aligned}(3.13)\qquad \lim_{m\to\infty} \int_{\mathcal{G}} q(x) \sum_{j=1}^{M} |\phi_j(x)|^2 &= \frac{1}{2\sin^2(\omega)} \\ &+ \sum_e \sum_j \Big(|\psi_j(0)|^2 + |\psi_j(1)|^2 \\ &\qquad - 2\Re(\psi_j(0)\overline{\psi_j(1)}) \cos(\omega)\Big) \int_0^1 q_e(x).\end{aligned}$$

Proof. Suppose $\omega = 2\pi$. An orthonormal basis for $E([2(m+1)\pi]^2)$ may be constructed from an orthonormal basis of $\mathbf{Z}_0$ using I_0 (see Theorem 2.7 and the following definition) as in Lemma 3.1, together with the function which is $\sqrt{2}/\sqrt{N_{\mathcal{E}}}\cos(2(m+1)\pi x)$ on each edge. Equation (3.11) can be verified by using (3.3). Using I_1 as in Lemma 3.1, a similar argument applies if $\omega = \pi$, except that the function $\sqrt{2}/\sqrt{N_{\mathcal{E}}}\cos((2m+1)\pi x)$ is absent if $\mathcal{G}$ is not bipartite.

Suppose $\omega \notin \{n\pi\}$. Pick an orthonormal basis $\{\psi_j\}$ for the eigenspace of Δ_1 with eigenvalue $1 - \cos(\omega)$. For each m, use formula (3.4) to construct a basis $\{\phi_j(m)\}$ for $E([\omega + 2m\pi]^2)$. This basis need not be orthonormal. Notice that $\mathcal{T}\phi_j(m) = \phi_j(m+1)$. Using the Kronecker δ notation, (3.6) gives

$$(3.14)\qquad \lim_{m\to\infty} \int_{\mathcal{G}} \phi_i \overline{\phi_j} = \delta_{ij}.$$

From these ordered bases $\{\phi_j(m)\}$ the Gram-Schmidt process will produce an orthonormal basis $\{\Phi_j(m)\}$. By virtue of (3.14) we find

$$\lim_{m\to\infty} \int_{\mathcal{G}} q(x) \sum_{j=1}^{M} |\phi_j(x)|^2 = \lim_{m\to\infty} \int_{\mathcal{G}} q(x) \sum_{j=1}^{M} |\Phi_j(x)|^2,$$

and then (3.5) finishes the proof. □

The expressions in Theorem 3.5 simplify if $\int_e q_e$ has the same value for each edge. The case $\omega \notin \{n\pi\}$ is highlighted in the next result. Here $E_1(\nu)$ denotes the eigenspace of Δ_1 with eigenvalue $\nu = 1 - \cos(\omega)$.

Corollary 3.6. *Suppose $\omega \notin \{n\pi\}$, and $\nu = 1 - \cos(\omega)$. Under the hypotheses of Theorem 3.5, suppose that $\int_e q_e$ is the same for all edges. Then*

$$\lim_{m\to\infty} \int_{\mathcal{G}} q(x) \sum_{j=1}^{M} |\phi_j(x)|^2 = \dim(E_1(\nu)) \int_0^1 q_e(x). \tag{3.15}$$

Proof. In case $\omega \notin \{n\pi\}$, computations as in (3.9) and (3.10) lead to

$$\lim_{m\to\infty} \int_{\mathcal{G}} q(x) \sum_{j=1}^{M} |\phi_j(x)|^2 = \int_0^1 q_e(x) \frac{1}{2} \sum_{v\in\mathcal{V}} \sum_j \deg(v) |\psi_j(v)|^2.$$

The additional simplification follows from the observation that

$$\frac{1}{2} \sum_{v\in\mathcal{V}} \sum_j \deg(v) |\psi_j(v)|^2$$

is the trace, in the vertex space, of the orthogonal projection onto $E_1(\nu)$. $\square$

Lemma 3.7. *Suppose $y_m, z_m \in E([\omega+2m\pi]^2)$, with $\|y_m\| = \|z_m\| = 1$, and with $0 < \omega \le 2\pi$. For each edge e of $\mathcal{G}$, assume that $q_e(x)$ is in $C^k[0,1]$ with $q_e^{(j)}(0) = q_e^{(j)}(1) = 0$ for $j = 0, \dots, k-1$, and that $\int_e q_e(x) = 0$. Then, for m sufficiently large,*

$$\Big| \int_0^1 y_m \overline{z_m} q_e(x) \Big| \le C m^{-k},$$

the estimates hold uniformly for all y_m and z_m.

Proof. The easiest case is $\omega = n\pi$ for $n = 1, 2$, where $y_m = \mathcal{T}^m y$ and $z_m = \mathcal{T}^m z$ for some y and z of norm 1. In the notation of Lemma 3.2,

$$|A_e|, |B_e| \le \sqrt{2},$$

since

$$\int_0^1 \sin^2(n\pi x + \delta_e) = 1/2.$$

Equation (3.1) gives

$$\Big| \int_0^1 \mathcal{T}^m y \overline{\mathcal{T}^m} z q_e(x) \Big| \le C(|a_{(2n+4m)\pi}| + |b_{(2n+4m)\pi}|),$$

and the well-known estimates on the Fourier coefficients of q_e obtained by integration by parts gives the result.

Suppose $0 < \omega \le 2\pi$, with $\omega \notin \{n\pi\}$. Again write $y_m = \mathcal{T}^m y$ and $z_m = \mathcal{T}^m z$ for some y and z in $E(\omega)$. By Lemma 3.3, since $\|y_m\| = 1$, $\|y\| \le C$ independent of m or the function y_m. By Lemma 2.2, there is a similar bound on the values $y(v)$ for each vertex v. The same remarks apply to z.

Equation (3.7) gives

$$\Big| \int_0^1 \mathcal{T}^m y \overline{\mathcal{T}^m} z q_e(x) \Big| \le C(|A_{(2n+4m)\pi}| + |B_{(2n+4m)\pi}|),$$

where A_k and B_k are the Fourier coefficients of $\sin(2\omega x)q(x)$ or $\cos(2\omega x)q(x)$. Again the integration by parts estimate for Fourier coefficients gives the result. □

4. Perturbations of the graph Laplacian. In this section perturbation theory for self-adjoint operators on a Hilbert space as developed in [**10**] will be combined with the results of the previous sections to draw some conclusions about the eigenvalues of operators $\Delta + q$ on $L^2(\mathcal{G})$, where $q : \mathcal{G} \to \mathbf{C}$ is bounded and measurable. Since Δ is self adjoint with compact resolvent $R_0(\lambda)$, the operator $\Delta + q$ also has compact resolvent $R(\lambda)$.

Let $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the sequence of distinct eigenvalues of Δ. Suppose that λ_n has multiplicity m_n. By virtue of Proposition 2.1, there is a $C_1 > 0$ such that $|\lambda_{n\pm 1} - \lambda_n| \ge C_1 n$ for $n = 1, 2, 3, \dots$. The eigenvalues μ of $\Delta + q$ cluster about the eigenvalues λ_n in the following sense. There is a constant C_2 such that for n sufficiently large the circle Γ_n of radius C_2 centered at λ_n will be in the resolvent set of $\Delta + q$ and contain exactly m_n eigenvalues μ, counted with algebraic multiplicity. Moreover, there is a constant C_3 such that every eigenvalue μ with $|\mu| \ge C_3$ is contained inside some Γ_n. Some restrictions on q will shrink the radii of these eigenvalue clusters as n increases.

Theorem 4.1. *For $k = 1$ or $k = 2$, assume that for each edge e of $\mathcal{G}$ the function $q_e(x)$ is in $C^k[0,1]$ with $q_e^{(j)}(0) = q_e^{(j)}(1) = 0$ for $j = 0, \dots, k-1$, and that $\int_e q_e(x) = 0$. Then, for n large enough, the eigenvalues μ of $\Delta + q$ contained in $|\lambda - \lambda_n| < C_2$ also satisfy*

$$|\mu - \lambda_n| = O(n^{-k/2}).$$

Proof. Improved estimates for the resolvent set of $\Delta + q$ are obtained via the usual resolvent expansion

$$\begin{aligned} R(\lambda) &= (\Delta + q - \lambda I)^{-1} \\ &= R_0(\lambda)(I + qR_0(\lambda))^{-1} \\ &= R_0(\lambda)\Big(I + \sum_{k=1}^{\infty}[-qR_0(\lambda)]^k\Big). \end{aligned} \tag{4.1}$$

Let P_1 denote the orthogonal projection onto the eigenspace $E(\lambda_n)$ of Δ, and let $P_2 = I - P_1$. Write

$$R_0(\lambda)qR_0(\lambda) = (P_1 + P_2)R_0(\lambda)qR_0(\lambda)(P_1 + P_2).$$

For a bounded operator A on a Hilbert space

$$\|A\| = \sup |\langle A\phi, \psi\rangle|, \qquad \|\phi\| = \|\psi\| = 1.$$

For $0 < |\lambda - \lambda_n| < C_2$ and n large, the Cauchy-Schwarz inequality gives

$$\begin{aligned} |\langle P_2 R_0(\lambda) q R_0(\lambda) P_2 \phi, \psi\rangle| &= |\langle q R_0(\lambda) P_2 \phi, R_0(\overline{\lambda}) P_2 \psi\rangle| \\ &\le \|q\|_\infty \frac{1}{C_1^2 n^2}, \end{aligned}$$

and

$$\begin{aligned} |\langle P_1 R_0(\lambda) q R_0(\lambda) P_2 \phi, \psi\rangle| &= |\langle q R_0(\lambda) P_2 \phi, R_0(\overline{\lambda}) P_1 \psi\rangle| \\ &\le \|q\|_\infty \frac{1}{C_1 n |\lambda - \lambda_n|}. \end{aligned}$$

This last estimate also applies if P_1 and P_2 switch positions.

The remaining term is

$$|\langle P_1 R_0(\lambda) q R_0(\lambda) P_1 \phi, \psi \rangle| = |\langle q R_0(\lambda) P_1 \phi, R_0(\overline{\lambda}) P_1 \psi \rangle|.$$

Since $y = P_1\phi$ and $z = P_1\psi$ are in $E(\lambda_n)$, Lemma 3.7 yields

$$\|P_1 R_0(\lambda) q R_0(\lambda) P_1\| = \frac{1}{|\lambda - \lambda_n|^2} \left| \int_{\mathcal{G}} q(x) y \overline{z} \right| \le \frac{C_3}{n^k |\lambda - \lambda_n|^2}.$$

These estimates show we may choose $C_4 > 0$ such that $\|q R_0(\lambda) q R_0(\lambda)\| < 1$ if $C_4 n^{-k/2} \le |\lambda - \lambda_n| \le C_2$. In this region the series (4.1) for $R(\lambda)$ converges in operator norm, so the region is in the resolvent set for $\Delta + q$. □

Under weaker hypotheses it is still possible to obtain an asymptotic description of the sum of the eigenvalues in a cluster. For $n = 0, 1, 2, \ldots$, let Γ_n be a positively oriented circle of positive radius enclosing λ_n, with the corresponding closed disks being pairwise disjoint. Γ_n will be chosen to lie in the resolvent set of $\Delta + q$, so that for n sufficiently large exactly m_n eigenvalues μ of $\Delta + q$, counted with algebraic multiplicity, are enclosed. The regular distribution of the eigenvalues λ_n permits us to select Γ_n with radius r_n satisfying $C_1 n \le r_n \le C_2 n$ for some positive constants C_1, C_2.

Recall [**10**, pages 74–81] that the part of the operator $\Delta + q$ associated to the contour Γ_n is

$$T_n = \frac{-1}{2\pi i} \int_{\Gamma_n} (\Delta + q) R(\lambda)\, d\lambda = \frac{-1}{2\pi i} \int_{\Gamma_n} \lambda R(\lambda)\, d\lambda.$$

Operator T_n is degenerate and so has a trace [**10**, pages 523–525]. If μ_j are the eigenvalues of $\Delta + q$ inside Γ_n, then

$$\sum_{\mu_j \in \Gamma_n} (\mu_j - \lambda_n) = \operatorname{tr} \frac{-1}{2\pi i} \int_{\Gamma_n} (\lambda - \lambda_n) R(\lambda)\, d\lambda.$$

These eigenvalue shifts have a limit if we restrict to a subsequence $\lambda_n = [\omega + 2m\pi]^2$.

Theorem 4.2. *Suppose q is a bounded measurable function on $\mathcal{G}$, and let $\{\phi_j(m)\}$ be an orthonormal basis for $E(\lambda_n)$ with $\lambda_n = [\omega + 2m\pi]^2$. Then*

$$\lim_{m\to\infty} \sum_{\mu_j \in \Gamma_n} (\mu_j - \lambda_n) = \lim_{m\to\infty} \sum_j \int_{\mathcal{G}} q(x)|\phi_j|^2(x),$$

with the right hand side evaluated in Theorem 3.5.

Proof. From the resolvent expansion (4.1),

$$R(\lambda) = R_0(\lambda) - R_0(\lambda)q(x)R_0(\lambda) + S(\lambda),$$

and

$$\sum_{\mu_j \in \Gamma_n} (\mu_j - \lambda_n) = \operatorname{tr} \frac{-1}{2\pi i} \int_{\Gamma_n} (\lambda - \lambda_n) \times (R_0(\lambda) - R_0(\lambda)q(x)R_0(\lambda) + S(\lambda))\, d\lambda.$$

Notice first that

$$\frac{-1}{2\pi i} \int_{\Gamma_n} (\lambda - \lambda_n) R_0(\lambda)\, d\lambda = 0.$$

Looking at $S(\lambda)$, we have

$$\left\| \sum_{k=2}^{\infty} (-1)^k [R_0(\lambda) q]^k \right\| = O(n^{-2}), \quad \lambda \in \Gamma_n,$$

and if $\|\cdot\|_1$ denotes the trace norm [**10**, page 521], [**13**, page 330], then elementary estimates give

$$\|R_0(\lambda)\|_1 = O(n^{-1/2}), \quad \lambda \in \Gamma_n.$$

Since $\|AB\|_1 \le \|A\| \|B\|_1$, it follows that $\|S(\lambda)\|_1 = O(n^{-5/2})$ for $\lambda \in \Gamma_n$. Thus,

$$\lim_{n\to\infty} \operatorname{tr} \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - \lambda_n) S(\lambda)\, d\lambda = 0,$$

and

$$\lim_{n\to\infty} \operatorname{tr}(T_n - \lambda_n) - \operatorname{tr} \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - \lambda_n) R_0(\lambda) q R_0(\lambda)\, d\lambda = 0.$$

Let ϕ_k be an orthonormal basis of eigenfunctions for Δ. Then

$$t(n) = \operatorname{tr} \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - \lambda_n) R_0(\lambda) q R_0(\lambda)\, d\lambda$$
$$= \langle \sum_k \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - \lambda_n) R_0(\lambda) q R_0(\lambda)\ d\lambda\ \phi_k, \phi_k \rangle.$$

The eigenfunctions with eigenvalue $\lambda_j \neq \lambda_n$ contribute 0, so the sum can be restricted to those eigenfunctions ϕ_j with eigenvalue λ_n.

$$t(n) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1}{\lambda - \lambda_n}\, d\lambda \sum_j \langle q\phi_j, \phi_j \rangle = \sum_j \int_{\mathcal{G}} q(x)|\phi_j|^2(x).$$

To complete the proof, restrict the traces to come from a subsequence $\lambda_n = [\omega + 2m\pi]^2$, and use Theorem 3.5. $\square$

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