ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 2, 2012

ON THE INDECOMPOSABLE TORSION-FREE ABELIAN GROUPS OF RANK TWO

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ABSTRACT. Let G be an abelian group and H a subgroup of G. Then G is called nil modulo H if $G * G \subset H$ for every ring $(G, *)$ on G. The square subgroup of G, denoted by $\square G$, is defined to be the intersection of all subgroups H where G is nil modulo H. In this paper we show that the square subgroup of a non-homogeneous and indecomposable torsion-free group G of rank two is a pure subgroup of G and that $G/\square G$ is a nil group.

1. Introduction. All groups considered in this paper are abelian, with addition as the group operation. Given an abelian group G, we call R a ring over G if the group G is isomorphic to the additive group of R. In this situation we write $R = (G, *)$, where $*$ denotes the ring multiplication. This multiplication is not assumed to be associative. Every group may be turned into a ring in a trivial way, by setting all products equal to zero; such a ring is called a zero-ring. If this is the only multiplication over G then G is said to be a nil group. For example, every divisible torsion group is a nil group, and these are the only torsion nil groups (see [**5**, Theorem 120.3]).

A generalization of the notion of nil groups was considered by Feigelstock [**4**]. Suppose that H is a subgroup of G; G is said to be nil modulo H if $G * G \subseteq H$ for every ring $(G, *)$ on G. Clearly, G is a nil group if and only if G is nil modulo $\{0\}$. Following [**7**], the square subgroup of an abelian group G, denoted by $\square G$, is the intersection of all subgroups H of G such that G is nil modulo H. The importance of the square subgroup lies in the fact that it gives an appropriate criterion to decide when a group is nil or not. In fact, an abelian group G is nil exactly if $\square G = 0$. For the first time the square subgroup was studied by Stratton and Webb [**7**]. The basic question about the square subgroup is

2010 AMS *Mathematics subject classification.* Primary 20K15.
Keywords and phrases. Square subgroup, nil modulo a subgroup, indecomposable, rank two.
The second author is the corresponding author.
Received by the editors on April 2, 2008, and in revised form on September 12, 2009.

DOI:10.1216/RMJ-2012-42-2-425

whether $G/\Box G$ is a nil group, and if this is not true in general then under what conditions is it true?

In this paper we show that the square subgroup of any non-homogenous and indecomposable torsion-free group G of rank two is a pure subgroup of G and that $G/\Box G$ is a nil group. For this, we study $\Box G$ by classifying G according to the cardinality of the type set of G.

2. Notation and preliminaries. Let G be a torsion-free abelian group. Given a prime p, the p-height of x, denoted by $h_p^G(x)$, is the largest integer k such that p^k divides x in G; if no such maximal integer exists, we set $h_p^G(x) = \infty$. Now let $p_1, p_2, \dots, p_n, \dots$ be the sequence of all primes in the order of magnitude. Then the sequence

$$\chi_G(x) = (h_{p_1}^G(x), h_{p_2}^G(x), \dots, h_{p_n}^G(x), \dots),$$

is said to be the height-sequence of x. We omit the subscript G if no ambiguity arises. For any two height-sequences $\chi = (k_1, k_2, \dots, k_n, \dots)$ and $\mu = (l_1, l_2, \dots, l_n, \dots)$ we set $\chi \geq \mu$ if $k_n \geq l_n$ for all n. Moreover, χ and μ will be considered equivalent if $\sum_n |k_n - l_n|$ is finite (if we set $\infty - \infty = 0$). An equivalence class of height-sequences is called a type. If $\chi(x)$ belongs to the type $\mathbf{t}$, then we say that x is of type $\mathbf{t}$. The type set of G is the partially ordered set of types, i.e.,

$$T(G) = \{t(x) \mid 0 \neq x \in G\}.$$

A torsion-free group G in which all the non-zero elements are of the same type $\mathbf{t}$ is called homogeneous. For example, every rank one group G is homogeneous. We use the symbol $t(G)$ to the type set of a rank one group G, which is indeed the type of any non-zero element of G. Finally, we write $\langle x \rangle_*$ for the pure subgroup of G generated by x. (A good reference for basic facts about type and undefined concepts is [**5**, page 109].)

Proposition 2.1. *Let G be a torsion-free group of finite rank. Then the length of every chain in $T(G)$ is at most equal to the rank of G.*

Proof. See [**3**, Proposition 1]. □

We recall that a torsion-free group G is said to be completely decomposable if it is a direct sum of rank one groups.

Proposition 2.2. *Let H be a pure subgroup of a torsion-free group G such that*

(a) *G/H is completely decomposable and homogeneous of type $\mathbf{t}$;*

(b) *all elements in $G \setminus H$ are of type $\mathbf{t}$.*

Then H is a summand of G.

Proof. See [**5**, Proposition 86.5]. $\square$

Theorem 2.3. *A torsion-free ring of rank one is either a zero ring or isomorphic to a subring of the rational number field. A torsion-free group of rank one is not a nil group if and only if its type is idempotent.*

Proof. See [**5**, Theorem 121.1]. $\square$

Lemma 2.4. *Let G be a subgroup of $\mathbf{Q}$. If $1/b, 1/d \in G$ with $(b, d) = 1$, then $1/bd \in G$.*

Proof. Obvious. $\square$

Proposition 2.5. *Let A, B be subgroups of $\mathbf{Q}$ such that $1 \in A \cap B$. Suppose there exists a non-zero integer n such that $nA \leq B$. If m is the least such positive integer, then the following statements hold:*

(a) *Let p be a prime number such that $\alpha = h_p^A(1) < \beta = h_p^B(1)$. Then $1/(p^{k-\alpha})(mA) \leq B$ for all $k \leq \beta$. Furthermore, p does not divide m.*

(b) *If $B \leq A$, then $mA = B$ and $1/m \in A$.*

(c) *Let d be a positive integer such that d divides m and $1/d \in B$. If $B^2 = B$ then $d = 1$.*

Proof. (a) Let $k \leq \beta$. Then $k - \alpha \leq \beta$, so $1/p^{k-\alpha} \in B$, and, if $1/p^e \in A$ then $e \leq \alpha$; hence, $k + e - \alpha \leq k \leq \beta$, therefore $(1/p^{k-\alpha}) \cdot (1/p^e) = (1/p^{k+e-\alpha}) \in B$, which implies $(1/p^{k-\alpha})(m(1/p^e)) \in B$. Also if $1/r \in A$ and $(r,p) = 1$, then $m/r \in B$, and since $1/p^{k-\alpha} \in B$, Lemma 2.4 yields $1/p^{k-\alpha}(m/r) \in B$. Consequently, $1/p^{k-\alpha}(mA) \leq B$ as required.

Let $k = \alpha + 1$ in $1/p^{k-\alpha}(mA) \leq B$. Then $(1/p)(mA) \leq B$. Now if p divided m then m/p would be an integer, so $(m/p)(A) \leq B$ which contradicts the hypothesis that m is the least positive integer with $mA \leq B$. Therefore p does not divide m.

(b) The hypothesis implies that A/mA is cyclic of order m, for if A/mA were of order d, d a proper divisor of m, then $dA \leq mA \leq B$, which contradicts the minimality of m. Furthermore, B/mA is cyclic of order s, where s is a divisor of m. Suppose that $m = sr$. Then

$$B/mA = r(A/mA) = rA/mA,$$

and consequently $B = rA$. Now the minimality of m implies that $m = r$; hence, $B = mA$, and since $1 \in B$, we have $1/m \in A$.

(c) Suppose m' is an integer such that $m = dm'$. Then

$$m'A = (mA)\left(\frac{1}{d}\right) \leq B\left(\frac{1}{d}\right) \leq B^2 = B.$$

So if $d > 1$ then $m' < m$, which contradicts the minimality of m, hence $d = 1$. □

Proposition 2.6. *Let A, B be subgroups of $\mathbf{Q}$ satisfying $1 \in B \leq A$ and $B^2 = B$. Suppose that $mA \leq B$ and m is the least positive integer with this property. Then $1/m + B$ generates A/B as a cyclic group.*

Proof. Our hypotheses together with Proposition 2.5 (b) imply that $mA = B$, $1/m \in A$ and A/B is cyclic of order m. Thus it is sufficient to show that $1/m + B$ has order m in A/B. Suppose not, i.e., $1/m + B$ has order $d < m$. Then $d/m \in B$; hence, $dA = (d/m)(mA) \leq B^2 = B$, contradicting the minimality of m. □

We recall some definitions and results from [**2**]. Let x, y be independent elements of a torsion-free group G of rank two. Each element z of G has a unique representation $z = ux + vy$, where u, v are rational numbers. Let

$$U_0 = \{u_0 \in \mathbf{Q} : u_0 x \in G\}, \quad U = \{u \in \mathbf{Q} : ux + vy \in G \text{ for some } v \in \mathbf{Q}\},$$
$$V_0 = \{v_0 \in \mathbf{Q} : v_0 y \in G\}, \quad V = \{v \in \mathbf{Q} : ux + vy \in G \text{ for some } u \in \mathbf{Q}\}.$$

Then U_0 and V_0 are subgroups of U and V, respectively. U, U_0, V, V_0 are called the groups of rank one belonging to the independent set $\{x, y\}$.

Theorem 2.7. *Let G be a torsion-free abelian group of rank two. If U, U_0, V, V_0 are the groups of rank one belonging to $\{x, y\}$, then $U/U_0 \cong V/V_0$.*

Proof. See [**2**, page 107]. □

Proposition 2.8. *Let G be a torsion-free group of rank two and x, y independent elements of G. Assume U, U_0, V, V_0 are the rank one groups belonging to $\{x, y\}$. Then $G/\langle y\rangle_* \cong U$ and $G/\langle x\rangle_* \cong V$.*

Proof. Clearly $V_0 y = \langle y\rangle_*$ is the kernel of the epimorphism $\varphi : G \to U$ defined by $\varphi(ux + vy) = u$ for any $ux + vy \in G$, and thus the first assertion follows. The second is obtained similarly. □

Lemma 2.9. *Let G be a torsion-free group of rank two. Let x, y be independent elements of G, and U, U_0, V, V_0 rank one groups belonging to $\{x, y\}$. Suppose that $U_0^2 = U_0$ and there exists an integer m such that $mU = U_0$. Then the multiplication*

$$x^2 = m^2 x, \qquad xy = yx = y^2 = 0,$$

yields a ring on G such that $G^2 = U_0 x$.

Proof. Let $g_1 = u_1 x + v_1 y$ and $g_2 = u_2 x + v_2 y$ be arbitrary in G. Then $u_1, u_2 \in U$ and $g_1 g_2 = m^2 u_1 u_2 x$. Also, $m^2 u_1 u_2 = (mu_1)(mu_2) \in (mU)^2 = U_0^2 = U_0$; hence, $m^2 u_1 u_2 x \in U_0 x \subseteq G$. Thus, the product actually lies in G, which yields a ring structure on G such that $G^2 \le U_0 x$. Now in view of $U_0^2 = U_0$ and $mU = U_0$ we have $(mU)^2 = U_0^2 = U_0$; hence, any $u_0 \in U_0$ may be written in the form $u_0 = (mu_1)(mu_2)$ for some $u_1, u_2 \in U$. By definition of U there exist elements $u_1x + v_1y$ and $u_2x + v_2y$ in G such that $(u_1x + v_1y)(u_2x + v_2y) = u_0x$, which yields $U_0x \leq G^2$. Consequently, $G^2 = U_0x$ as required. □

Theorem 2.10. *Let G be a torsion-free group of rank two. If G is non-nil, then $T(G)$ contains a unique minimal member and at most three elements.*

Proof. See [**6**, Theorem 3.3]. □

Remark 2.11. According to the proof of Theorem 2.10, the following possibilities are realized for $T(G)$:

(a) one type-in this case the type must be idempotent;

(b) two types-one minimal the other maximal;

(c) three types-one minimal and two maximal, in this case at least one of the maximal types is idempotent.

Since we are dealing with non-homogeneous groups, we only consider (b) and (c) in Sections 3 and 4 respectively.

3. Type set of cardinality equal to two. In this section we prove that if $T(G)$ has two elements, then the square subgroup of G is pure and $G/\square G$ is a nil group. To do this, we need a lemma.

Lemma 3.1. *Let G be an indecomposable torsion-free group of rank two and $T(G) = \{t_1, t_2\}$ with $t_1 < t_2$. If $\{x, y\}$ is an independent set such that $t(x) = t_1$, $t(y) = t_2$, then all non-trivial rings on G satisfy*

$$x^2 = by, \qquad xy = yx = y^2 = 0,$$

for some rational number b.

Proof. See [**1**, Lemma 3]. □

Theorem 3.2. *Let G be an indecomposable torsion-free group of rank two. If $T(G) = \{t_1, t_2\}$ with $t_1 < t_2$, then the square subgroup of G is pure and $G/\square G$ is a nil group.*

Proof. If G is a nil group, then we are done. Let R be a non-zero ring over G and $\{x, y\}$ a subset of G such that $t(x) = t_1$ and $t(y) = t_2$. Then, by Lemma 3.1,

$$x^2 = by, \qquad xy = yx = y^2 = 0, \quad b(\neq 0) \in \mathbf{Q}.$$

Let U, U_0, V, V_0 be rank one groups belonging to $\{x, y\}$ and $w = ux+vy$, $w' = u'x + v'y$ be arbitrary elements of G. Then $ww' = buu'y$, which means $ww' \in \langle y\rangle_*$, hence $G^2 \subseteq \langle y\rangle_*$. This happens for all rings; therefore,

$$\Box G \subseteq \langle y\rangle_*. \tag{3.1}$$

Also from $ww' = buu'y$ and $b \neq 0$ we deduce $bU^2 \leq V_0$; hence,

$$t(U^2) \leq t(V_0),$$

so there exists a least positive integer m such that

$$mU^2 \leq V_0, \qquad mU^2 \leq U^2 \cap V_0 \leq U^2. \tag{3.2}$$

On the other hand, Proposition 2.5 (b) implies

$$mU^2 = U^2 \cap V_0, \qquad 1/m \in U^2. \tag{3.3}$$

Now let $\chi_{V_0}(1) = (n_1, n_2, \dots, n_i, \dots)$ and $\chi_U(1) = (m_1, m_2, \dots, m_i, \dots)$ be the height sequences of 1 in V_0 and U, respectively. Then

$$\chi_{U^2}(1) = (2m_1, 2m_2, \dots, 2m_i, \dots).$$

We prove $(1/p_i^{\alpha_i})y \in \Box G$ for all α_i such that $0 \leq \alpha_i \leq n_i$ ($i = 1, 2, 3, \dots$). To do this, we consider two cases for each fixed i: $n_i \leq 2m_i$ or $2m_i < n_i$. First, suppose that $n_i \leq 2m_i$. Then we define a multiplication over G by

$$x^2 = my, \qquad xy = yx = y^2 = 0.$$

Let $w = ux + vy$ and $w' = u'x + v'y$ be arbitrary elements of G, so $ww' = muu'y$. By (3.2), $muu' \in V_0$, so the product actually lies in G, which yields a ring structure on G. Since $n_i \leq 2m_i$, we get

$1/p_i^{\alpha_i} \in U^2 \cap V_0$ and, in view of (3.3), $1/p_i^{\alpha_i} \in mU^2$. Consequently, $1/p_i^{\alpha_i} = mu_1u_2$ for some $u_1, u_2 \in U$. On the other hand, there exist $v_1, v_2 \in V$ such that $z = u_1x + v_1y$ and $z' = u_2x + v_2y$ belong to G, so $zz' = u_1u_2x^2 = mu_1u_2y = 1/(p_i^{\alpha_i})y$. That is, $1/(p_i^{\alpha_i})y \in \Box G$.

In the other case, i.e., $2m_i < n_i$, by Proposition 2.5 (a), p_i does not divide m. By (3.3), $1/m \in U^2$; hence, $1/m = 1/m'm''$ where $1/m'$, $1/m'' \in U$. If $m_i = \infty$, then $n_i = \infty$ and so $2m_i = n_i$, contrary to $2m_i < n_i$; thus, $m_i < \infty$. Now since $1/p_i^{m_i} \in U$ and p_i does not divide m, we have $(p_i, m') = (p_i, m'') = 1$, hence by Lemma 2.4,

$$\frac{1}{p_i^{m_i}m'}, \qquad \frac{1}{p_i^{m_i}m''} \in U. \tag{3.4}$$

Define another multiplication over G by

$$x^2 = \frac{m}{p_i^{\alpha_i - 2m_i}}y, \qquad xy = yx = y^2 = 0.$$

Since $2m_i < n_i$, (3.2) and Proposition 2.5 (a) imply

$$(mU^2)\frac{1}{p_i^{\alpha_i - 2m_i}} \leq V_0;$$

thus, the product lies in G, which yields a ring structure on G. By (3.4) there exist $v_1, v_2 \in V$ such that

$$z = \frac{1}{p_i^{m_i}m'}x + v_1y \in G, \qquad z' = \frac{1}{p_i^{m_i}m''}x + v_2y \in G,$$

and since $m'm'' = m$, we have

$$zz' = \frac{m}{p_i^{\alpha_i}m'm''}y = \frac{1}{p_i^{\alpha_i}}y.$$

Consequently, in this case $1/(p_i^{\alpha_i})y \in \Box G$. Therefore, $\langle y \rangle_* \subseteq \Box G$, and by (3.1), $\langle y \rangle_* = \Box G$, which means $\Box G$ is a pure subgroup of G.

Now we are going to prove that $G/\Box G$ is a nil group. Let $w \in G$. Then $w = ux + vy$ for some $u \in U$ and $v \in V$; hence, $wx = ux^2 = uby$ which implies $bu \in V_0$ for all $u \in U$ and so $bU \leq V_0$. It follows that $t(U) \leq t(V_0)$.

Now if $G/\Box G$ were non-nil, then $t(U)$ would be idempotent, so $h_p^U(1) = 0$ or ∞ for almost all prime numbers p. We prove $t(U) = t(U_0)$. For this we note that if $h_p^U(1) = 0$ then since $U_0 \leq U$ we have $h_p^{U_0}(1) = 0$; hence, we suppose $h_p^U(1) = \infty$. Then in view of $t(U) \leq t(V_0)$, we have $h_p^{V_0}(1) = \infty$. Now if $1/p^n \in U$ for some integer n, then there is an $a/b \in V$ such that

$$w = \frac{1}{p^n}x + \frac{a}{b}y \in G.$$

Let $b = b'p^m$ where $(b', p) = 1$. Then $1/p^m \in V_0$ and $b'w = (b'/p^n)x + (a/p^m)y$, which yields $(b'/p^n)x = b'w - a((1/p^m)y) \in G$; hence, $(1/p^n) \in U_0$. Therefore, if $h_p^U(1) = \infty$, then $h_p^{U_0}(1) = \infty$. We conclude that $t(U) = t(U_0)$ and consequently Proposition 2.2 implies that $\langle y\rangle_*$ is a direct summand of G, contrary to the hypothesis that G is indecomposable. Therefore, $G/\Box G$ is a nil group. □

4. Type set of cardinality greater than or equal to three. In this section we prove that the square subgroup of G is pure and $G/\Box G$ is a nil group in the remaining case, i.e., $T(G)$ with three elements such that either one or both of the maximal types are idempotent. We need the following proposition in our arguments.

Proposition 4.1. *Let G be a torsion-free group of rank two and $T(G) = \{t_0, t_1, t_2\}$ with $t_0 < t_1$ and $t_0 < t_2$. Let $x, y \in G$ be such that $t(x) = t_1$ and $t(y) = t_2$. If t_1, t_2 are incomparable, then any ring on G satisfies*

$$x^2 = ax, \qquad y^2 = by, \qquad xy = yx = 0,$$

for some $a, b \in \mathbf{Q}$.

Proof. Let $z \in G$ with $t(z) = t_0$. Then $z \notin G(t_1)$. Since $t(x^2) \geq t(x) = t_1$, both x^2 and x belong to $G(t_1)$ which is a rank one subgroup of G, so they are dependent, that is, $x^2 = ax$ for some $a \in \mathbf{Q}$. Similarly, $y^2 = by$ for some $b \in \mathbf{Q}$.

On the other hand, $t(yx) \geq t(x)$, so yx and x belong to $G(t_1)$; therefore, $yx = ex$ for some $e \in \mathbf{Q}$ and similarly $yx = fy$ for some $f \in \mathbf{Q}$. Now if $yx \neq 0$ then $t(x) = t(xy) = t(y)$, contrary to our hypothesis; therefore, $yx = 0$. By the same reasoning, $xy = 0$. □

Lemma 4.2. *Let G be a torsion-free group of rank two and $T(G) = \{t_0, t_1, t_2\}$ with $t_1^2 = t_1$, $t_2^2 \neq t_2$, $t_0 < t_1$, $t_0 < t_2$. Then $\Box G$ is a pure subgroup of G and $G/\Box G$ is a nil group.*

Proof. First, we observe that if G is a nil group then $\Box G = 0$, so we are done. Note that our hypotheses ensure that $t_1 \neq t_2$ and, in view of Proposition 2.1, t_1 and t_2 are incomparable. Now let $x, y \in G$ be such that $t(x) = t_1$, $t(y) = t_2$ and let U, U_0, V, V_0 be rank one groups belonging to $\{x, y\}$; we may assume that $U_0^2 = U_0$. Now let R be an arbitrary non-trivial ring on G. Then by Proposition 4.1

$$x^2 = ax, \qquad xy = yx = 0, \qquad y^2 = by,$$

for some $a, b \in \mathbf{Q}$. If $b \neq 0$, then $t(y) = t(y^2) \geq t^2(y)$, which implies that $t(y)$ is idempotent, a contradiction to our hypothesis, so $y^2 = 0$. Furthermore, since R is nontrivial, a is non-zero. Now pick $z, z' \in G$. Then $z = ux + vy$ and $z' = u'x + v'y$ for some $u, v, u', v' \in \mathbf{Q}$; hence, $zz' = auu'x$, which implies $G^2 \subseteq \langle x \rangle_*$. But since R is arbitrary, we have $\Box G \subseteq \langle x \rangle_*$. Now suppose $u \in U$. Then there exists $v \in \mathbf{Q}$ such that $ux + vy \in G$, so $(ux + vy)x = aux$; hence, $au \in U_0$ for all $u \in U$. Thus $aU \leq U_0 \leq U$. It follows that there is a positive integer k such that $kU \leq U_0$. If m is the least such positive integer, Proposition 2.5 (b) yields $mU = U_0$. We may now apply Lemma 2.9 to construct a ring on G satisfying $G^2 = U_0 x$; thus, $\langle x \rangle_* \subseteq \Box G$ and consequently $\Box G = \langle x \rangle_*$.

Therefore, $G/\Box G = G/\langle x \rangle_*$ and, by Proposition 2.8, $G/\Box G \cong V$. On the other hand, Theorem 2.7 yields $U/U_0 \cong V/V_0$ and since $mU = U_0$, we have $t(V) = t(V_0)$. Now if $G/\Box G$ were not a nil group then by Theorem 2.3, $t(G/\Box G) = t(V)$ would be idempotent; hence, in view of $t(V) = t(V_0)$ we conclude that $t(V_0) = t_2$ is idempotent, contrary to assumption. Therefore, $G/\Box G$ is a nil group. $\square$

Lemma 4.3. *Let G be a torsion-free group of rank two and $T(G) = \{t_0, t_1, t_2\}$ with $t_0 < t_1$, $t_0 < t_2$, $t_1^2 = t_1$, $t_2^2 = t_2$ and t_1, t_2 incomparable. If G is not a nil group, then $\Box G = G$.*

Proof. Let $x, y \in G$ be such that $t(x) = t_1$, $t(y) = t_2$, and let U, U_0, V, V_0 be rank one groups belonging to $\{x, y\}$. Also, suppose that

$U_0^2 = U_0$ and $V_0^2 = V_0$. Now, if R is a non-trivial ring on G, then by Proposition 4.1,

$$x^2 = ex, \qquad xy = yx = 0, \qquad y^2 = ry,$$

for some $e, r \in \mathbf{Q}$. We may assume, without loss of generality, that $e \neq 0$. For any $u \in U$ there exists a $v \in \mathbf{Q}$ such that $ux + vy \in G$; hence, $(ux + vy)x = eux$ which implies that $eu \in U_0$, and since u is arbitrary, we have $eU \leq U_0$. Consequently, there is an integer n such that $nU \leq U_0$. Choosing m to be the least such integer we have $mU = U_0$. Now Lemma 2.9 allows us to construct a ring R on G satisfying $G^2 = U_0 x$, so $U_0 x \leq \square G$. Since $mU = U_0$ and $U/U_0 \cong V/V_0$, we have $mV = V_0$, and applying Lemma 2.9, we deduce that $V_0 y \leq \square G$. Consequently,

$$U_0 x \oplus V_0 y \leq \square G. \tag{4.1}$$

Also,

$$\begin{aligned} &1 \in U_0 \leq U, \qquad U_0^2 = U_0, \qquad mU \leq U_0,\\ &1 \in V_0 \leq V, \qquad V_0^2 = V_0, \qquad mV \leq V_0, \end{aligned}$$

where in both cases m is the least positive integer with the given property. From Proposition 2.6 we deduce that $1/m + U_0$ generates U/U_0 and $1/m + V_0$ generates V/V_0. Furthermore, since $U/U_0 \cong V/V_0$ are cyclic groups of order m, the isomorphism $\phi : U/U_0 \to V/V_0$ must be defined by $\phi(\beta/m + U_0) = k\beta/m + V_0$, where k is a fixed integer coprime with m and β an integer such that $0 \leq \beta < m$. This leads to the following construction for G :

$$G = \{(\beta/m + u_0)x + (k\beta/m + v_0)y \mid 0 \leq \beta < m,\ u_0 \in U_0,\ v_0 \in V_0\},$$

and we shall denote an arbitrary element of G as

$$\beta(1/mx + k/my) + u_0 x + v_0 y. \tag{4.2}$$

In particular, we set

$$g = \frac{1}{m}x + \frac{k}{m}y \in G. \tag{4.3}$$

Now define a multiplication $(G,*)$ over G as follows:

$$x*y=y*x=0, \qquad x*x=kmx, \qquad y*y=my.$$

Let

$$y_1=u_1x+v_1y, \qquad y_2=u_2x+v_2y, \qquad y_3=u_3x+v_3y,$$

be arbitrary elements of G. Then for $(G,*)$ to be a ring we must show:

(i) $y_1*y_2\in G$;

(ii) $y_1*(y_2+y_3)=y_1*y_2+y_1*y_3$, $(y_1+y_2)*y_3=y_1*y_3+y_2*y_3$.

To do this, in view of (4.2) suppose that

$$y_1=\frac{\beta_1+m\alpha_1}{m}x+\frac{\beta_1k+m\alpha_2}{m}y, \qquad y_2=\frac{\beta_2+m\gamma_1}{m}x+\frac{\beta_2k+m\gamma_2}{m}y,$$

where $0\le\beta_1<m$, $0\le\beta_2<m$, $\alpha_1,\gamma_1\in U_0$ and $\alpha_2,\gamma_2\in V_0$. So we have

$$\begin{aligned} y_1*y_2&=\frac{k(\beta_1+m\alpha_1)(\beta_2+m\gamma_1)}{m}x+\frac{(\beta_2k+m\gamma_2)(\beta_1k+m\alpha_2)}{m}y\\ &=\frac{k\beta_1\beta_2+mu_0}{m}x+\frac{k^2\beta_1\beta_2+mv_0}{m}y \end{aligned}$$

for some $u_0\in U_0$ and $v_0\in V_0$. Hence,

$$\begin{aligned} y_1*y_2&=\frac{k\beta_1\beta_2}{m}x+\frac{k^2\beta_1\beta_2}{m}y+u_0x+v_0y\\ &=k\beta_1\beta_2(\frac{1}{m}x+\frac{k}{m}y)+u_0x+v_0y. \end{aligned}$$

By (4.3), $(1/m)x+(k/m)y\in G$; therefore $y_1*y_2\in G$. Also,

$$\begin{aligned} y_1*(y_2+y_3)&=(u_1x+v_1y)*((u_2+u_3)x+(v_2+v_3)y)\\ &=kmu_1(u_2+u_3)x+mv_1(v_2+v_3)y\\ &=(kmu_1u_2x+mv_1v_2y)+(kmu_1u_3x+mv_1v_3y)\\ &=y_1*y_2+y_1*y_3 \end{aligned}$$

and in a similar way $(y_1+y_2)*y_3=y_1*y_3+y_2*y_3$; therefore, $(G,*)$ is a ring over G. By (4.3), $g=(1/m)x+(k/m)y\in G$ with

$0 < k < m$ and $(k, m) = 1$, so there exist integers a, b such that $ak + bm = 1$, and since $g^2 = (k/m)x + (k^2/m)y = kg$, we have $ag^2 = akg = (1 - bm)g = g - bmg$; therefore,

$$g = ag^2 + bmg = ag^2 + b(x + ky). \tag{4.4}$$

Now we take any $w \in G$. Then by (4.2) and (4.3) we have

$$w = (\beta/m + u_0)x + (\beta k/m + v_0)y = \beta g + u_0 x + v_0 y,$$

and by (4.4),

$$w = a\beta g^2 + b\beta(x + ky) + u_0 x + v_0 y = a\beta g^2 + (b\beta + u_0)x + (b\beta k + v_0)y.$$

But the fact that $g^2 \in \Box G$ together with (4.1) imply that $w \in \Box G$; hence, $\Box G = G$. $\square$

Theorem 4.4. *Let G be a torsion-free group of rank two. If $T(G)$ has cardinality greater than or equal to three then $\Box G$ is pure and $G/\Box G$ is a nil group.*

Proof. If $T(G)$ has cardinality greater than three, then by Theorem 2.10, G is a nil group. Hence, we are done. Suppose that $T(G)$ has cardinality equal to three and G is a non-nil group. Then by Remark 2.11, we have two cases. First, suppose that $T(G)$ contains one minimal type and two maximal ones, and precisely one of them is idempotent. Then by Lemma 4.2, $\Box G$ is pure and $G/\Box G$ is a nil group. In the other case, $T(G)$ contains one minimal type and two idempotent maximal types, so by Lemma 4.3, $\Box G = G$. Consequently, $\Box G$ is pure and $G/\Box G$ is the trivial nil group. $\square$

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