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STRUCTURE THEORY OF TENSOR PRODUCT LOCALLY H^*-ALGEBRAS

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ABSTRACT. The tensor product of two proper Hausdorff locally m-convex H^*-algebras with continuous involution, endowed with the projective tensor product topology, along with its completion, are algebras of the same type with the factors. Under appropriate conditions, a canonical orthogonal basis is provided in the completion of the tensor product algebra. Based on this, the minimal closed 2-sided ideals are determined, yielding, in turn, the *second Wedderburn structure theorem.*

0. Introduction. The theory of H^*-(Banach) algebras with the corresponding Wedderburn structure theorems have been developed by Ambrose in [**1**]. The notion of an H^*-algebra is the abstract version of characteristic properties of the algebra $L^2(G)$ of a compact group (with the convolution as ring multiplication). It is known (ibid) that an H^*-algebra lies between the group algebra of a compact group and that of a (non-compact) locally compact group.

In [**9–11, 14, 15**] we considered extensions of the results in [**1**] to locally m-convex topological algebras. Our point of view is justified by theoretical reasons with an increasing interest in topological $*$-algebras (function algebras, topological K-theory, [**7, 19, 20**]), especially, in $*$-algebras endowed with locally convex topologies generated by C^*-seminorms, applicable, for instance, even to relativistic quantum theory. See, e.g., [**2, 3, 16, 18**]. In this context, we also note that a particular locally m-convex H^*-algebra admits a locally m-convex C^*-topology (cf. [**13**, page 198, Proposition 2.5], [**4**, page 265, Proposition 2.3]; see also [**5, 6**]). Our study reveals some hidden characteristic

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properties of H^*-(Banach) algebras, as for instance, among others, the (bs) and (Pbs) properties (see Section 2 and Theorem 2.1).

In [**8**], Grove, employing results of [**1**], considered tensor products of H^*-algebras and gave, among other things, the second Wedderburn structure theorem for the resulted tensor product H^*-algebra.

Here, we consider tensor products of locally m-convex H^*-algebras, thus generalizing results of [**8**]. In particular, we prove that the tensor product of two proper Hausdorff locally m-convex H^*-algebras with continuous involution, in the projective tensor product topology, as well as its completion are algebras of the same type with the factors (Theorem 4.2). The existence of a canonical orthogonal basis (Definition 3.2) in a complete algebra, as before, is crucial for its structure. Thus, we give the framework in which such a basis exists. As a matter of fact, the basis at issue is generated by analogous ones in the factor algebras. More precisely, we give conditions so that each factor algebra is decomposed through minimal closed 2-sided ideals (second Wedderburn structure theorem). Each one of these ideals contains a family of axes, which in turn, generates a canonical orthogonal basis for the ideal. The family consisting of the union of the bases, as they vary over the factor ideals, gives a canonical orthogonal basis for the factor algebra (Proposition 3.4). The bases in the factor algebras give in turn, the required basis for the completion of the tensor product algebra (Theorem 4.4). Based on this, we determine the minimal closed 2-sided ideals in the algebra concerned (Proposition 5.1). Thus, we have at hand the "building stones" for a decomposition of the algebra in question, thus getting the analogous here second Wedderburn structure theorem, as we did for the factors (Theorem 5.5).

1. Notation and preliminaries. Throughout this paper all algebras are over the field $\mathbf{C}$ of complexes. $\mathcal{A}_l^E(S) \equiv \mathcal{A}_l(S)$ (respectively $\mathcal{A}_r^E(S) \equiv \mathcal{A}_r(S)$) denotes the left (right) annihilator of a (non-empty) subset S of an algebra E. If $\mathcal{A}_r(E) = (0)$, E is called *proper*, while E is said to be *preannihilator*, if $\mathcal{A}_l(E) = \mathcal{A}_r(E) = (0)$. If E is a topological algebra (separately continuous multiplication), $\mathcal{L}_l(E) \equiv \mathcal{L}_l$ ($\mathcal{L}_r(E) \equiv \mathcal{L}_r$, $\mathcal{L}(E) \equiv \mathcal{L}$) denotes the set of all closed left (right, 2-sided) ideals in E. A topological algebra E is *topologically simple* if (0) is the only proper closed 2-sided ideal in E. A *locally convex* (respectively *locally m-convex*) *H^*-algebra* is an algebra E equipped with a family $(p_\alpha)_{\alpha\in A}$ of *Ambrose seminorms* in the sense that p_α, $\alpha \in A$, arises from a positive semi-definite (pseudo-) inner product $\langle\ ,\ \rangle_\alpha$, such that the induced topology makes E into a locally convex (respectively locally m-convex) (topological) algebra. Moreover, the following conditions are satisfied: For any $x \in E$, there is an $x^* \in E$, such that

$$\langle xy, z\rangle_\alpha = \langle y, x^*z\rangle_\alpha \tag{1.1}$$

$$\langle yx, z\rangle_\alpha = \langle y, zx^*\rangle_\alpha \tag{1.2}$$

for any $y, z \in E$ and $\alpha \in A$. The element x^* (*not necessarily unique*) is called an *adjoint* of x. If E is proper and Hausdorff, x^* is unique and the correspondence $x \mapsto x^*$ defines on E an involution (see [**9**, page 451, Definition 1.1; page 452, Theorem 1.3]).

In what follows, a locally m-convex H^*-algebra is called, for short, a *locally H^*-algebra.*

Given a locally convex H^*-algebra $(E, (p_\alpha)_{\alpha\in A})$ the orthogonal $S^\perp$ of a non-empty subset S of E is

$$S^\perp = \{x \in E :\ \langle x, y\rangle_\alpha = 0 \text{ for every } y \in S \text{ and } \alpha \in A\}, \tag{1.3}$$

being a closed linear subspace of E. If I is in $\mathcal{L}_l$ (respectively $\mathcal{L}_r, \mathcal{L}$), then $I^\perp$ is a closed left (respectively, right, 2-sided) ideal in E [**9**, page 456, Lemma 3.2]. Two elements $x, y \in E$, are called *orthogonal* if $\langle x, y\rangle_\alpha = 0$ for every $\alpha \in A$, while $S, T \subseteq E$ are *mutually orthogonal* if their elements are pair-wise orthogonal. Besides, two elements $x, y \in E$ with $x \neq y$ are called *algebraically orthogonal* if $xy = yx = 0$. Two orthogonal and algebraically orthogonal elements of E are called *doubly orthogonal*. Besides, an element of E is said to be *H-primitive* if it cannot be expressed as the sum of two (non-zero) doubly orthogonal projections. In this respect, an *idempotent* (*projection*) of an algebra E is an element $x \in E$ with $0 \neq x = x^2$. $\mathcal{I}d(E)$ denotes the set of all projections in E. A family $(x_i)_{i\in K}$ of elements in an algebra E is called (*algebraically*) *orthogonal*, if for every $i \neq j$ in $K, x_i x_j = 0$. Moreover, a maximal family of doubly orthogonal, H-primitive, self-adjoint, projections of a locally convex H^*-algebra, which also has an involution, is called a *family of axes*. In particular, *if the underlying space is Hausdorff, then any two elements of a family of axes, being self-adjoint idempotents, are orthogonal, if and only if they are algebraically orthogonal.* If $(E_\lambda)_{\lambda\in\Lambda}$ is a family of mutually orthogonal subalgebras in a locally convex H^*-algebra, then their algebraic direct sum is called an *orthogonal direct sum* (in short, *orthodirect sum*), denoted by $\bigoplus_{\lambda\in\Lambda}{}^{\perp}E_\lambda$ (see also [**17**, page 119] and [**22**, page 46]). Besides, by the *topological orthogonal direct* of the E_λ's is meant the closure $\overline{\bigoplus_{\lambda\in\Lambda}{}^{\perp}E_\lambda}$ of their orthodirect sum.

2. Hereditary Ambrose algebras. A closed (left) ideal I of a locally convex H^*-algebra E is called *orthocomplemented* if $E = I\oplus^{\perp}I^{\perp}$ ($I^{\perp}$ is called *the orthocomplement* of I in E; see comments after (1.3)). E is called an *orthocomplemented algebra*, if every closed (left) ideal I is orthocomplemented in E. A locally H^*-algebra whose every closed left (right) ideal is a left (right) orthocomplemented algebra is named a *hereditary left* (*right*) *orthocomplemented algebra* (see [**12**, page 3728, Definition 3.4; see also page 3727, Theorem 3.1] as well as [**9**, page 457, (3.3)]). A locally convex H^*-algebra E *has the* (H)*-property* (*on the left*), if the following condition holds:

$$(H)\qquad \begin{array}{l}\textit{Every closed left ideal } I \textit{ in } E, \textit{ with } I\subseteq Ex, \textit{ for some}\\ x\in Id\,(E), \textit{ has an orthocomplement in } Ex. \textit{ Namely,}\\ Ex = I\oplus^{\perp}I^{\perp}.\end{array}$$

A proper Hausdorff complete locally H^*-algebra with continuous involution, having the (H)-property is called an *Ambrose algebra*. A (bs) *Ambrose algebra* is an Ambrose algebra $(E,(p_\alpha)_{\alpha\in A})$, satisfying the condition:

$$(bs)\qquad \begin{array}{l}\textit{There exists a non-zero (self-adjoint) element } \omega\in E\\ \textit{of the form } \omega = h^*h,\ h\in E, \textit{ such that } \sup_{\alpha\in A} p_\alpha(\omega) < +\infty\end{array}$$

(see [**14**, page 65, Lemma 1.1]). Of course, *every strong spectrally bounded $*$-algebra* (viz. a locally m-convex $*$-algebra $(E,(p_\alpha)_{\alpha\in A})$ with $\sup_{\alpha\in A} p_\alpha(x) < +\infty$, for every $x\in E$; see [**20**, page 488, (3.63)]) *has the* (bs)*-property.*

The next result gives rise to the class of *hereditary Ambrose algebras* (cf. Definition 2.3). In what follows $\mathbf{C}_0[x]$ stands for the $\mathbf{C}$-algebra of polynomials in x without constant term. For a (bs) Ambrose algebra E

and x a (non-zero) self-adjoint element of it, $\overline{\mathbf{C}_0[x]}$ is self-adjoint and hence a proper Hausdorff complete locally H^*-algebra with continuous involution (see [**11**, page 145, proof of Theorem 3.3] and the footnote in [**14**, page 69]).

Theorem 2.1. *Let $(E, (p_\alpha)_{\alpha \in A})$ be an orthocomplemented (bs) Ambrose algebra satisfying the condition*:

(Pbs) *Every subalgebra of E of the form $\overline{C_0[x]}$, with x a self-adjoint element in E, has the (bs)-property.*

Then E is the topological orthodirect sum of the minimal closed 2-sided ideals E_i generated by Ex_i, $i \in K$ (where $(x_i)_{i\in K}$ is a family of axes in E). Each one of the E_i's is a topologically simple, proper, Hausdorff, complete, locally H^*-algebra with continuous involution. In particular, E_i, $i \in K$ has the (H)-property if

$$\mathcal{L}_l(I) \subseteq \mathcal{L}_l(E_i), \quad I \in \mathcal{L}_l(E_i),\ i \in K. \tag{2.1}$$

Thus the E_i's are orthocomplemented Ambrose algebras.

Proof. E has a family of axes, say $(x_i)_{i\in K}$, such that

$$E = \overline{\bigoplus_{i\in K}{}^{\perp} \mathcal{RL}(Ex_i)} \equiv \overline{\bigoplus_{i\in K}{}^{\perp} E_i}. \tag{2.2}$$

Namely, E is the topological orthodirect sum of the minimal closed 2-sided ideals $\mathcal{RL}(Ex_i) \equiv E_i$, generated by Ex_i, $i \in K$.

$(*)$ [(2.2) above is called a *canonical analysis* of E (with respect to a family of axes $(x_i)_{i\in K}$), while the E_i's are called *canonical factors* of E].

Moreover, each E_i has all the properties, stated in the first part of the theorem, but (bs) (see [**9**, page 457, Lemma 3.4] and [**11**, page 143, comments after Lemma 2.4; page 144, Theorem 3.1; and page 148, Theorem 3.8]; cf. also the Note after this proof).

Now, suppose that (2.1) is fulfilled; E, as proper, is left preannihilator (see [**9**, page 452, Theorem 1.2]); hence, by Corollary 3.2 in [**12**, page 3728], *every closed* 2*-sided ideal in* E *is a* (*left*) *complemented algebra* (this is independent of (2.1)). Therefore, by (2.1) and [**12**, page 3727, Theorem 3.1; see also its proof] every I in $\mathcal{L}_l(E_i), i \in K$ is a left (ortho)complemented algebra, as well. So, for J in $\mathcal{L}_l(E_i)$ with $J \subseteq E_i x$, for some $x \in Id(E_i)$, we get $E_i x = J \oplus^{\perp} J^{\perp}$, $E_i x$ being a closed left ideal in E_i. Namely, $E_i, i \in K$ has the (H)-property and this completes the proof. Notice that, by abuse of notation, we put here $J^p = J^{\perp}$, where $J^p = J^{\perp} \cap E_i x$ (see [**12**, page 3727, Theorem 3.1 and its proof]). Namely, "p" denotes the relative orthocomplementor on $E_i x$. □

Note. Concerning the previous proof, we note that Theorem 3.8 in [**11**] still holds by employing the (Pbs) condition, in place of (PH). In this context, the same as before, remains true for Theorems 3.3, 4.5 and Corollary 3.9 therein. On the other hand, the following results in [**14**], viz. Theorems 1.2, 2.1 and 2.4 with the Scholium after it, along with Propositions 2.3, 2.9 and Corollaries 2.2, 2.5, 2.7 and 2.8 are again valid, under the condition (Pbs).

Remarks. By Theorem 3.9 in [**9**, page 458], *the algebras* E *and* E_i, $i \in K$, *as in Theorem* 2.1, *are actually dual* (viz. $\mathcal{A}_l(\mathcal{A}_r(I)) = I$ for all $I \in \mathcal{L}_l(E)$ and $\mathcal{A}_r(\mathcal{A}_l(J)) = J$ for all $J \in \mathcal{L}_r(E)$, respectively for the E_i's) (a fortiori *annihilator algebras*, namely preannihilator algebras where the right (respectively left) annihilator of any proper closed left (right) ideal, is non-zero). Besides, (2.1) *holds for any commutative topologically semiprime, annihilator algebra* (see [**12**, page 3726, Theorem 3.12 and page 3728, comments after Theorem 3.1]). Moreover, (2.1) *is satisfied, if* $I \in \mathcal{L}_l(E_i)$ *is* $*$*-closed* (*viz.* $I^* \subseteq I$; *actually,* $I^* = I$) *and the algebra* E_i *is left orthocomplemented*: Indeed, for $I \in \mathcal{L}_l(E_i)$, $i \in K$, we get $E_i = I \oplus^{\perp} I^{\perp}$. If $x \in I^{\perp}$, $\langle x, y \rangle_\alpha = 0$ for every $y \in I$, $\alpha \in A$. Since I is $*$-closed, $yy^* \in I$ and hence $\langle x, yy^* \rangle_\alpha = 0$. Namely, $\langle xy, y \rangle_\alpha = 0$ for every $y \in I$, $\alpha \in A$. Thus $xy \in I^{\perp}$. Moreover, $xy \in I$, and hence $xy = 0$ for every $y \in I$. Namely, $I^{\perp} \subseteq \mathcal{A}_l^{E_i}(I)$. Thus $E_i \subseteq I + \mathcal{A}_l^{E_i}(I)$ and hence $E_i = I + \mathcal{A}_l^{E_i}(I)$ from which we get $\mathcal{L}_l(I) \subseteq \mathcal{L}_l(E_i)$ (see also [**12**, page 3728]). On the other hand, if (2.1) is satisfied, E_i is left complemented (see the proof of Theorem 2.1). See also Proposition 2.5 below and the comment preceding it. Finally, (2.1) *always implies that* E_i *is left complemented* in the sense of [**12**].

Corollary 2.2. *Theorem* 2.1 *holds for any orthocomplemented strong spectrally bounded Ambrose algebra that satisfies* (2.1) *and has the* (Pbs)*-property.*

In view of Theorem 2.1, we set the next definition, in the context of which canonical orthogonal bases appear (see Definition 3.2 and Proposition 3.4 below).

Definition 2.3. Let $(E,(p_\alpha)_{\alpha\in A})$ be an orthocomplemented (bs) Ambrose algebra. Then, (i) E is called a *pre-hereditary Ambrose algebra*, if it satisfies the (Pbs)-property.

(ii) E is said to be a *hereditary Ambrose algebra*, if every closed subalgebra has the (bs) property and every canonical factor E_i, $i \in K$ (with respect to a family of axes $(x_i)_{i\in K}$), satisfies (2.1).

It is obvious that *a hereditary Ambrose algebra is pre-hereditary.* Moreover, *each E_i has the* (Pbs)*-property*: Indeed, for any non-zero self-adjoint element x in E_i, $i \in K$, the subalgebra $\overline{\mathbf{C}_0[x]}^{E_i} = E_i \bigcap \overline{\mathbf{C}_0[x]}$ is a closed subalgebra in E; hence, by definition, it has the (bs)-property.

Scholium (of terminology). In Definition 2.3 two types of hereditary properties appear. The (a priori) property (bs) and the (a posteriori) property of orthocomplementation (see also Theorem 2.1). Heredity of orthocomplementation is given with respect to the minimal closed 2-sided ideals as in Theorem 2.1. This is exactly what one really needs to state a structural theorem for certain tensor product Ambrose algebras (see Theorem 5.5 below). Apart from this, types of hereditary complemented algebras have already appeared in [**12**, Section 3] (see also Section 1 above). In that case, heredity of complementation applies to closed left (right) ideals of a left (right) complemented topological algebra (cf. also Proposition 2.7 below).

For the commutative case, *every orthocomplemented* (bs) *Ambrose algebra satisfies* (2.1) *and hence it is a hereditary complemented Ambrose algebra* (here, heredity of complementation is taken in the sense of [**12**]), as the following corollary shows (see also its proof).

Corollary 2.4. *Every commutative pre-hereditary Ambrose algebra E is the topological orthodirect sum of minimal closed* (2*-sided*) *ideals,* *say E_i, $i \in K$, being algebras of the same type with E (having not necessarily the (bs)-property). Moreover, the factors E_i, $i \in K$, are topologically simple.*

Proof. By Theorem 2.9 in [**12**, page 3725],

$$I^{\perp} = \mathcal{A}_l(I)(= \mathcal{A}_r(I)), I \in \mathcal{L}(E_i), \quad i \in K.$$

Thus (2.1) is fulfilled. The assertion now follows from Theorem 2.1. $\square$

In the rest of this section, the results are still valid when interchanging left by right. We refer now to some hereditary properties related with substructures of a certain locally H^*-algebra.

Proposition 2.5. *Let $(E, (p_\alpha)_{\alpha \in A})$ be a Hausdorff left orthocomplemented locally H^*-algebra. Then every $I \in \mathcal{L}_l$, which is $*$-closed, is an algebra of the same type with E.*

In particular, I has the (H)-property, if Iw with $w \in Id(I)$ is $*$-closed.

Proof. If $x \in I^{\perp}$, then for any $y, z \in I$, $\alpha \in A$,

$$\langle xy, z\rangle_\alpha = \langle x, zy^*\rangle_\alpha = 0.$$

Thus $xy = 0$ for every $y \in I$, and hence $x \in \mathcal{A}_l(I)$. Therefore, $E = I \oplus^{\perp} I^{\perp} \subseteq I + \mathcal{A}_l(I)$ from which we get in turn, $E = I + \mathcal{A}_l(I)$ and $\mathcal{L}_l(I) \subseteq \mathcal{L}_l$ (see also [**12**, page 3728, comments preceding Corollary 3.2]). The first part of the assertion follows now from Theorem 3.1 in [**12**, page 3727] and Lemma 1.4 in [**9**, page 453].

To prove the (H)-property for I, consider a closed left ideal J in I with $J \subseteq Iw$ for some $w \in Id(I)$. It is easily seen that $Iw \in \mathcal{L}_l(I)$. By assumption, Iw is $*$-closed so that, by applying similar reasoning as above, we get $\mathcal{L}_l(Iw) \subseteq \mathcal{L}_l(I)$. Therefore, Iw is a left orthocomplemented algebra (see [**12**, page 3727, Theorem 3.1]) and hence $Iw = J \oplus^{\perp} J^{\perp}$ (here $J^{\perp}$ denotes the relative orthocomplement induced from I) and this completes the proof. $\square$

Based on Proposition 2.5, we provide *another proof* concerning the orthocomplementation of the factors E_i as in Theorem 2.1 (see also (2.2)):

Let z be an element in E_i. Since $E_i = \langle Ex_iE \rangle$ (viz. the linear span of Ex_iE), $z = \sum_{j=1}^{n} y_j x_i w_j$ and hence $z^* = \sum_{j=1}^{n} w_j{}^* x_i y_j{}^*$. Thus E_i is $*$-closed and, therefore, by Proposition 2.5, it is an orthocomplemented algebra.

Corollary 2.6. *Every Hausdorff left orthocomplemented locally H^*-algebra E whose every closed left ideal I is $*$-closed is a hereditary left orthocomplemented algebra.*

Proposition 2.7. *Every hereditary left orthocomplemented locally H^*-algebra E along with every closed left ideal I in E have the (H)-property.*

Proof. Let J be a closed left ideal in E with $J \subseteq Ex$ for some $x \in Id(E)$. Since $Ex \in \mathcal{L}_l$ and J is a (closed) ideal in Ex, we get $Ex = J \oplus^{\perp} J^{\perp}$. Namely, E has the (H)-property. Now, let K be a closed left ideal in I with $K \subseteq Ix$ for some $x \in Id(I)$. It is easily seen that $Ix \in \mathcal{L}_l(I)$. So, since $EIx \subseteq Ix$, we get $Ix \in \mathcal{L}_l$, as well. Hence, $Ix = K \oplus^{\perp} K^{\perp}$ ($\perp$ denotes the relative orthocomplementor) and this terminates the proof. $\square$

3. Canonical orthogonal bases. For completeness sake, we state the following result taken from [**10**, page 1184, Lemma 4.7]. In this context, we still note that, the *continuity of the involution of an Ambrose algebra*, therein, *is not needed* in the following lemma.

Lemma 3.1. *Let $(E, (p_\alpha)_{\alpha \in A})$ be a topologically simple Ambrose algebra and $(x_i)_{i \in K}$ a family of axes in E. Moreover, let $(E_{ij})_{(i,j) \in K^2}$ be the family of linear subspaces defined by $E_{ij} := x_i E x_j, (i,j) \in K^2$. Then there exists a family $(x_{ij}) \in \prod_{(i,j) \in K^2} E_{ij}$, such that*

1) $x_{ii} = x_i$, $i \in K$.
2) $x_{ij} x_{jk} = x_{ik}$, $i, j, k \in K$.
3) $x_{ij} x_{kl} = 0$, $i, j, k, l \in K$ *with* $j \neq k$.
4) $x_{ij}^* = x_{ji}$, $i, j \in K$.
5) $p_\alpha(x_{ij}) = \text{const.} \equiv \sqrt{t_\alpha}$ *for any* $i, j \in K$ *with* $\alpha \in A$.

The x_{ij}'s, as before, are called *matrix units* [**10**, page 1185]. As concerns 5), we note that $t_\alpha \geq 1$, $\alpha \in A$ [**10**, page 1185; see also the comments after Lemma 4.7]. Besides, it is easily checked that *the family* $(x_{ij})_{(i,j)\in K^2}$ *is linearly independent.*

Lemma 3.1 supplies all the properties of a canonical orthogonal basis in a locally H^*-algebra, in the sense of the next.

Definition 3.2. Let $(E,(p_\alpha)_{\alpha\in A})$ be a locally H^*-algebra and $(x_i)_{i\in K}$ a family of axes in E. A family $(x_{ij})_{(i,j)\in K^2} \in \prod\limits_{(i,j)\in K^2} x_iEx_j (\equiv E_{ij})$ is called a *canonical orthogonal basis* for E, if the x_{ij}'s satisfy properties 1)–5) of Lemma 3.1, and every $z \in E$ has the form

$$z = \sum_{i,j} \lambda_{ij} x_{ij} \equiv \lim_\delta z_\delta,$$

such that $z_\delta = \sum\limits_{i,j} \lambda_{ij} x_{ij}$ (finite sum; dependent on δ), $\lambda_{ij} \in \mathbf{C}$, $\delta \in \Delta$.

The next result justifies the above definition.

Proposition 3.3. *Every topologically simple pre-hereditary Ambrose algebra* $(E,(p_\alpha)_{\alpha\in A})$ *has a canonical orthogonal basis.*

Proof. By Theorem 3.1 in [**11**, page 144], E has a family of axes, say $(x_i)_{i\in K}$, which gives a family $(x_{ij})_{(i,j)\in K^2}$, as in Lemma 3.1 (see also the comments following it). Besides,

$$E = \overline{\bigoplus_{i,j}{}^{\perp} x_iEx_j} \equiv \overline{\bigoplus_{i,j}{}^{\perp} E_{ij}}.$$

[**11**, page 145, Theorem 3.3]. Thus, for $z \in E$, $z = \lim\limits_\delta z_\delta$ with $z_\delta \in \oplus^\perp x_iEx_j$. Therefore, $z_\delta = \sum x_i w_\delta x_j$ (finite sum). So, since E_{ij} is one-dimensional (see [**10**, page 1183, Lemma 4.5]), we get $x_i w_\delta x_j = \lambda_{ij}^\delta x_{ij}$, with $\lambda_{ij}^\delta \in \mathbf{C}$ uniquely determined. $\square$

Proposition 3.4. *Let* $(E,(p_\alpha)_{\alpha\in A})$ *be a hereditary Ambrose algebra. Then, each canonical factor* $E_i, i \in K$ *(see* $(*)$ *in the proof of*

Theorem 2.1) *has a canonical orthogonal basis, say* $(x^i_{kl})_{(k,l)\in L_i^2}$ *(here* L_i *depends on* i*). Moreover, the family*

$$\mathcal{B} \equiv \{(x^i_{kl})_{(k,l)\in L_i^2}\}_{i\in K}$$

is a canonical orthogonal basis for E.

Proof. Each canonical factor E_i, $i \in K$ of E is a topologically simple orthocomplemented (bs) Ambrose algebra (see Theorem 2.1 and Definition 2.3). Moreover, E_i, $i \in K$ has the (Pbs)-property, as well (see the comments after Definition 2.3). Namely, E_i, $i \in K$ is, finally, a pre-hereditary Ambrose algebra. Therefore, by Proposition 3.3, each E_i has a canonical orthogonal basis, as in the statement. Obviously, the elements in $\mathcal{B}$ satisfy properties 1)–5) of Lemma 3.1. Besides, for y in $\underset{i\in K}{\oplus}{}^{\perp} E_i$, $y = \sum_i y_i$, $y_i \in E_i$. Hence $y = \sum_i \sum_{k,l} \lambda^i_{kl} x^i_{kl} \equiv \sum_{i,k,l} \lambda^i_{kl} x^i_{kl}$. Namely, $\mathcal{B}$ spans $\underset{i\in K}{\oplus}{}^{\perp} E_i$. Since $(x^i_{kl})_{(k,l)\in L_i^2}$ is a basis for E_i, the λ^i_{kl}'s are uniquely determined. Now, for $z \in E$, $z = \lim\limits_{\delta} z_\delta$, with $(z_\delta)_{\delta\in\Delta}$ a net in $\underset{i\in K}{\oplus}{}^{\perp} E_i$. Thus, by the preceding argument, $z_\delta = \sum_{i,k,l} \lambda^i_{kl} x^i_{kl}$ (finite sum) and this completes the proof. $\square$

4. Tensor products of locally H^*-algebras. In what follows E' denotes the topological dual of a topological vector space E. By considering locally H^*-algebras, we set the next definition (see also [**21**, pages 364, 375]).

Definition 4.1. Let $(E, (p_\alpha)_{\alpha\in A})$, $(F, (q_\beta)_{\beta\in B})$ be locally H^*-algebras. A topology τ on $E \otimes F$ is said to be *compatible* on $E \otimes F$, if the following conditions are satisfied:

1) $E \underset{\tau}{\otimes} F$ is a locally H^*-algebra.

2) The canonical bilinear map $\varphi : E \times F \to E \otimes F$ is separately continuous. Namely, if $(r_\gamma)_{\gamma\in\Gamma}$ is a family of submultiplicative seminorms defining τ, then for any $\gamma \in \Gamma$, there exists $(\alpha, \beta) \in A \times B$ with $r_\gamma(x \otimes y) \leq p_\alpha(x) q_\beta(y)$ for any $x \otimes y \in E \otimes F$.

3) For any equicontinuous subsets $S \subseteq E'$ and $T \subset F'$ the set $S \otimes T = \{x' \otimes y' : (x', y') \in S \times T\}$ is an equicontinuous subset of $(E \underset{\tau}{\otimes} F)'$.

Let $(E, (p_\alpha)_{\alpha\in A})$, $(F, (q_\beta)_{\beta\in B})$ be locally convex spaces. Then the relation

$$(4.1) \qquad (p_\alpha \otimes q_\beta)(z) \equiv r_{\alpha,\beta}(z) = \inf \sum_{i=1}^{n} p_\alpha(x_i) q_\beta(y_i)$$

where the infimum is taken over all expressions of z in the form $z = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F$, defines a family $(r_{\alpha,\beta})_{(\alpha,\beta)\in A\times B}$ of seminorms on $E\otimes F$, making it a locally convex space. The locally convex topology defined on $E \otimes F$ via (4.1) is the *projective tensorial topology* π. We denote by $E \underset{\pi}{\otimes} F$ the respective locally convex space and by $E \widehat{\underset{\pi}{\otimes}} F$ its completion (see [**21**, page 365, Lemma 2.1, see also page 366]). Now, restricting ourselves to tensor products of locally H^*-algebras, we get the following result. In its proof, we use the notion of a "positive element" in the sense that, an element x in a $*$-algebra E is *positive* (we write $x \geq 0$ or $0 \leq x$), if it has the form $x = \sum_{i=1}^n x_i x_i{}^*$, $x_i \in E$, $1 \leq i \leq n$, and $n \in \mathbf{N}$. We denote by E^+ *the set of positive elements* in E (a *convex cone*) with $0 \in E^+$. A *pre-ordering* in E, denoted by $\geq$ or yet $\leq$, is defined by setting $x \geq y$ (or $y \leq x$) if and only if $x - y \in E^+$. Besides, $x \in E^+$ is *strictly positive* (we write $x > 0$ or $0 < x$) if $x \neq 0$.

Theorem 4.2. *Let $(E,(p_\alpha)_{\alpha\in A}), (F,(q_\beta)_{\beta\in B})$ be proper Hausdorff locally H^*-algebras with continuous involution. Then $E \underset{\pi}{\otimes} F$ and $E \widehat{\underset{\pi}{\otimes}} F$ are algebras of the same type as the given ones.*

Proof. Each seminorm $r_{\alpha,\beta}, (\alpha, \beta) \in A \times B$ (see (4.1)) is submultiplicative (see [**21**, page 15, Corollary 3.1 and page 377, Lemma 3.2]). Thus $E \underset{\pi}{\otimes} F$ is, in particular, a locally m-convex algebra. If $\langle\ ,\ \rangle_\alpha$, $\langle\ ,\ \rangle_\beta$, $(\alpha, \beta) \in A \times B$ are the quasi-inner products corresponding to p_α, q_β respectively, then (4.1) is written in the form

$$r_{\alpha,\beta}(z) = \inf \sum_{i=i}^{n} \langle x_i, x_i\rangle_\alpha^{1/2} \langle y_i, y_i\rangle_\beta^{1/2}.$$

Moreover, the linear space $E \otimes F$ is equipped with a family of quasi-inner products defined by the relation

$$(4.2) \qquad (\langle\ ,\ \rangle_\alpha \otimes \langle\ ,\ \rangle_\beta)(z, z') \equiv \langle z, z'\rangle_{\alpha,\beta} = \sum_{i,j=1}^{n,m} \langle x_i, {x_j}'\rangle_\alpha \langle y_i, {y_j}'\rangle_\beta,$$

for any $z = \sum_{i=1}^n x_i \otimes y_i$, $z' = \sum_{j=1}^m {x_j}' \otimes {y_j}'$ in $E \otimes F$ and $(\alpha, \beta) \in A \times B$. In particular, denote by $\|\cdot\|_{\alpha,\beta}$ the quasi-norm defined by $\langle\ ,\ \rangle_{\alpha,\beta}$, which actually, coincides with the greatest (projective) cross norm, see also (4.1),

$$\|z\|_\gamma^{\alpha,\beta} := \inf \sum_{i=1}^n p_\alpha(x_i) q_\beta(y_i), \quad z \in E \otimes F.$$

Thus

$$\begin{aligned} \|z\|_{\alpha,\beta} &= \left(\sum_{i=1}^n p_\alpha(x_i)^2 q_\beta(y_i)^2\right)^{1/2} \equiv \|z\|_\gamma^{\alpha,\beta} \\ &= \inf \sum_{i=1}^n p_\alpha(x_i) q_\beta(y_i) = r_{\alpha,\beta}(z), \end{aligned}$$

and hence $r_{\alpha,\beta}(z) = \langle z, z\rangle_{\alpha,\beta}^{1/2}$ for every $z \in E \otimes F$. Namely, the seminorm $r_{\alpha,\beta}$ arises from the quasi-inner product $\langle\ ,\ \rangle_{\alpha,\beta}$. Routine computations show that (1.1) and (1.2) are fulfilled and hence $E \underset{\pi}{\otimes} F$ is a locally H^*-algebra. By Lemma 2.2 in [**21**, page 369], $E \underset{\pi}{\otimes} F$ is a Hausdorff space. Now, if $z \in E \otimes F$ with $z \neq 0$, then $z = \sum_{i=1}^n x_i \otimes y_i$, where $(x_i), (y_i), 1 \leq i \leq n$ are families of linearly independent elements in E and F respectively [**21**, page 361, Lemma 1.2]. Thus $x_i \neq 0$ and $y_i \neq 0$ for every $1 \leq i \leq n$ and hence, by the "properness" of E and F, $x_i x_i^* \neq 0$ and $y_i y_i^* \neq 0$, $1 \leq i \leq n$ (see [**9**, page 452, Theorem 1.3]), while the conjugate correspondence $*$ in E and F, respectively, is actually an involution [**9**, page 452, Theorem 1.3]. Now, by (1.1) and (1.2), we see that $\sum_{i=1}^n x_i^* \otimes y_i^*$ is an adjoint of $z = \sum_{i=1}^n x_i \otimes y_i$. Thus, by defining $z^* = (\sum_{i=1}^n x_i \otimes y_i)^* := \sum_{i=1}^n x_i^* \otimes y_i^*$, we get $zz^* = \sum_{i=1}^n x_i x_i^* \otimes y_i y_i^*$. Besides, for every $1 \leq i \leq n$, $0 < x_i x_i^* + y_i y_i^* \leq \sum_{i=1}^n (x_i \otimes y_i)(x_i \otimes y_i)^* = zz^*$. Namely, zz^* is strictly positive, which means that $zz^* \neq 0$. This assures that $E \underset{\pi}{\otimes} F$ is proper [**9**, page 452, Theorem 1.3]. Now, by the continuity of the involution in E and F, we get $(p_\alpha \otimes q_\beta)(z^*) = (p_\alpha \otimes q_\beta)(z)$, $z \in E \underset{\pi}{\otimes} F$. Therefore, $E \underset{\pi}{\otimes} F$ has a continuous involution too. The rest of the assertion now follows from [**9**, page 453, Theorem 1.5 and the comment following it]. □

Theorem 4.3. *Let* $(E, (p_\alpha)_{\alpha \in A})$, $(F, (q_\beta)_{\beta \in B})$ *be proper (Hausdorff) locally* H^**-algebras. If* $(x_{kl})_{(k,l)\in\Lambda^2}$, $(y_{mn})_{(m,n)\in L^2}$ *are canonical orthogonal bases for* E *and* F *respectively, then the family*

$$\mathcal{B} = \{x_{kl} \otimes y_{mn},\ (k,l,m,n) \in \Lambda^2 \times L^2\}$$

is a canonical orthogonal basis for $E \widehat{\underset{\pi}{\otimes}} F$.

Proof. It is easily seen that $\mathcal{B}$ spans $E \underset{\pi}{\otimes} F$. If $\sum_{k,l,m,n} \lambda_{klmn}(x_{kl} \otimes y_{mn}) = 0$, then $\sum_{m,n}(\sum_{k,l} \lambda_{klmn} x_{kl}) \otimes y_{mn} = 0$. Since $(y_{mn})_{(m,n)\in L^2}$ is linearly independent, $\sum_{k,l} \lambda_{klmn} x_{kl} = 0$ for every m, n (see for instance [**21**, page 361, Lemma 1.3]). So that $\lambda_{klmn} = 0$ for every k, l, m, n over the finite sums. Namely, $\mathcal{B}$ is linearly independent. Moreover, properties 1)–5) are satisfied by the elements in $\mathcal{B}$, as follows by the respective ones in the factors. Now, let z be an element in $E \widehat{\underset{\pi}{\otimes}} F = \overline{E \underset{\pi}{\otimes} F}$. Then $z = \lim_\delta z_\delta$ with $(z_\delta)_{\delta\in\Delta}$ a net in $E \underset{\pi}{\otimes} F$. Since $\mathcal{B}$ spans

$$E \otimes_\pi F, \quad z_\delta = \sum_{k,l,m,n} \xi_{klmn} x_{kl} \otimes y_{mn}$$

(finite sum; dependent upon δ), $\xi_{klmn} \in \mathbf{C}$. Thus $\mathcal{B}$ is, finally, a canonical orthogonal basis for $E \widehat{\underset{\pi}{\otimes}} F$. □

Now, based on Proposition 3.4 and Theorem 4.3, we get the following.

Theorem 4.4. *Let* $(E, (p_\alpha)_{\alpha\in A})$, $(F, (q_\beta)_{\beta\in B})$ *be hereditary Ambrose algebras and* $((x^i_{kl})_{(k,l)\in L_i^2})_{i\in K}$, $((y^j_{mn})_{(m,n)\in\Lambda_j^2})_{j\in M}$ *canonical orthogonal bases in* E *and* F, *respectively. Then, the family*

$$\{(x^i_{kl} \otimes y^j_{mn})_{(k,l,m,n)\in L_i^2\times\Lambda_j^2}\}_{(i,j)\in K\times M}$$

is a canonical orthogonal basis for $E \widehat{\underset{\pi}{\otimes}} F$.

Notation. In the framework of Theorem 4.4, any $z \in E \widehat{\underset{\pi}{\otimes}} F$ has the form

$$z = \sum_{k,l,m,n} \lambda^{ij}_{klmn} x^i_{kl} \otimes y^j_{mn} \equiv \lim_\delta z_\delta,$$

such that

$$z_\delta = \sum_{k,l,m,n} \lambda^{ij}_{klmn} x^i_{kl} \otimes y^j_{mn} \text{ (finite sum)}, \ \lambda^{ij}_{klmn} \in \mathbf{C}.$$

5. A Wedderburn-type structure theorem. Let $(E, (p_\alpha)_{\alpha \in A})$, $(F, (q_\beta)_{\beta \in B})$ be hereditary Ambrose algebras and $(x_i)_{i \in K}$, $(y_j)_{j \in M}$ families of axes in E and F respectively. Then, by Theorem 2.1 (see also its proof, as well as Definition 2.3),

$$E = \overline{\bigoplus_i{}^{\perp} \mathcal{RL}(Ex_i)} \equiv \overline{\bigoplus_i{}^{\perp} E_i} \text{ and } F = \overline{\bigoplus_j{}^{\perp} \mathcal{RL}(Fy_j)} \equiv \overline{\bigoplus_j{}^{\perp} F_j}$$

where E_i (respectively F_j) are minimal closed 2-sided ideals in E (respectively F), each one of which is a topologically simple, proper, Hausdorff, complete (bs), locally H^*-algebra with continuous involution, having the (H)-property. In this framework we state the next result that describes the minimal closed 2-sided ideals in $E \widehat{\otimes}_\pi F$ (with respect to canonical orthogonal bases of the factors).

Proposition 5.1. *Let $(E, (p_\alpha)_{\alpha \in A})$, $(F, (q_\beta)_{\beta \in B})$ be hereditary Ambrose algebras and $(x_i)_{i \in K}$, $(y_j)_{j \in M}$ families of axes in E and F, respectively. For a fixed pair (i,j) in $K \times M$, consider the canonical orthogonal bases $(x^i_{kl})_{(k,l) \in L^2_i}$ and $(y^j_{mn})_{(m,n) \in \Lambda^2_j}$ of the factors E_i, F_j, respectively. Denote by I the closed linear span of the set $S \equiv \{x^i_{kl} \otimes y^j_{mn}, (k,l) \in L^2_i, (m,n) \in \Lambda^2_j\}$ in $E \widehat{\otimes}_\pi F$. Then I is a (closed) 2-sided ideal identical with $E_i \widehat{\otimes}_\pi F_j$. Moreover, the closed 2-sided ideals $E_i \widehat{\otimes}_\pi F_j$, $(i,j) \in K \times M$ are minimal and mutually orthogonal.*

Proof. Applying Theorem 4.3 for the proper locally H^*-algebras E_i, F_j, we get that S spans $E_i \otimes F_j$, while S is a canonical orthogonal basis for $E_i \widehat{\otimes}_\pi F_j$. Thus $\langle S \rangle = E_i \otimes F_j$ and $I = \overline{\langle S \rangle} = \overline{E_i \otimes_\pi F_j} = E_i \widehat{\otimes}_\pi F_j$. It is easily seen that $I = E_i \widehat{\otimes}_\pi F_j$ is a (closed) 2-sided ideal. Let $z = \sum_{\rho=1}^n x_\rho \otimes y_\rho$, $z' = \sum_{\mu=1}^m x'_\mu \otimes y'_\mu$ be elements in $E_i \otimes F_j$, $E_{i_0} \otimes F_{j_0}$, respectively, with $i \neq i_0$ and $j \neq j_0$. Then (see also (4.2)),

$$\langle z, z' \rangle_{\alpha,\beta} = \sum_{i,j=1}^{n,m} \langle x_\rho, x'_\mu \rangle_\alpha \langle y_\rho, y'_\mu \rangle_\beta, \quad (\alpha, \beta) \in A \times B. \tag{5.1}$$

Besides, E_i, E_{i_0} (respectively F_j, F_{j_0}) are orthogonal (see [**11**, page 148, Lemma 3.7]). Thus (5.1) assures that $E_i \otimes F_j$, $E_{i_0} \otimes F_{j_0}$ are orthogonal in $E \otimes F$. Now, by taking limits, we get that $E_i \widehat{\underset{\pi}{\otimes}} F_j, E_{i_0} \widehat{\underset{\pi}{\otimes}} F_{j_0}$ are orthogonal in $E \widehat{\underset{\pi}{\otimes}} F$.

We complete the proof by showing that the closed ideal $E_i \widehat{\underset{\pi}{\otimes}} F_j$, $(i, j) \in K \times M$, is minimal: Our argument follows [**8**, page 77, Theorem 2.2]. So, let J be a nonzero closed 2-sided ideal of $E \widehat{\underset{\pi}{\otimes}} F$, contained in $E_i \widehat{\underset{\pi}{\otimes}} F_j$. Then, there exists an element $0 \neq z \in J$, with

$$z = \sum_{k,l,m,n} \lambda^{ij}_{klmn} x^i_{kl} \otimes y^j_{mn} \equiv \lim_{\delta} z_\delta,$$

such that

$$z_\delta = \sum_{k,l,m,n} \lambda^{ij}_{klmn} (x^i_{kl} \otimes y^j_{mn}) \text{ (finite sum)}, \ \lambda^{ij}_{klmn} \in \mathbf{C}$$

with some $\lambda^{ij}_{klmn} \neq 0$. Since $(x^i_{kl})_{(k,l) \in L^2_i}$ and $(y^j_{mn})_{(m,n) \in \Lambda^2_j}$ are canonical orthogonal bases (see Definition 3.2), we easily get

$$(x^i_{kk} \otimes y^j_{mm}) z (x^i_{ll} \otimes y^j_{nn}) = \lambda^{ij}_{klmn} (x^i_{kl} \otimes y^j_{mn}) \in J,$$

so that $x^i_{kl} \otimes y^j_{mn} \in J$. Now, for an arbitrary element $x^i_{pq} \otimes y^j_{rs}$ in the canonical basis of $E_i \widehat{\underset{\pi}{\otimes}} F_j$ (see above), we obtain

$$(x^i_{pk} \otimes y^j_{rm})(x^i_{kl} \otimes y^j_{mn})(x^i_{lq} \otimes y^j_{ns}) = x^i_{pq} \otimes y^j_{rs}.$$

The previous relations and the fact that J is a 2-sided ideal lead to $x^i_{pq} \otimes y^j_{rs} \in J$. Thus, J contains the elements of the closed linear span of $E_i \widehat{\underset{\pi}{\otimes}} F_j$ hence, $J = E_i \widehat{\underset{\pi}{\otimes}} F_j$. □

Our next task is to give a "decomposition" of $E \widehat{\underset{\pi}{\otimes}} F$ through the minimal closed ideals $E_i \widehat{\underset{\pi}{\otimes}} F_j$ provided by Proposition 5.1. For this, we use *the next result*, which actually, *holds without the assumption of continuity of the involution.*

Proposition 5.2. *Let* $(E,(p_\alpha)_{\alpha\in A})$, $(F,(q_\beta)_{\beta\in B})$ *be Ambrose algebras. If* $(x_i)_{i\in K}$, $(y_j)_{j\in M}$ *are families of axes in* E *and* F, *respectively, then* $\mathcal{J}\equiv(x_i\otimes y_j)_{(i,j)\in K\times M}$ *is a family of axes in* $E\underset{\pi}{\otimes}F$.

Proof. It is easily checked that the elements $x_i\otimes y_j$, $(i,j)\in K\times M$ are doubly orthogonal, self-adjoint projections. We show that $x_i\otimes y_i$, $(i,j)\in K\times M$ are also H-primitive. By Lemma 4.3 in [**10**, page 1183], $x_iEx_i\cong\mathbf{C}$, $i\in K$ and $y_jFy_j\cong\mathbf{C}$, $j\in M$ within isomorphisms of topological algebras. Moreover, for each $(i,j)\in K\times M$

$$(x_i\otimes y_j)(E\underset{\pi}{\otimes}F)(x_i\otimes y_j)\cong(x_iEx_i)\otimes(y_jFy_j)$$

via a topological algebraic isomorphism. Therefore, $(x_i\otimes y_j)(E\underset{\pi}{\otimes}F)(x_i\otimes y_j)\cong\mathbf{C}\otimes\mathbf{C}\cong\mathbf{C}$ within isomorphisms of topological algebras. Thus $(x_i\otimes y_j)(E\underset{\pi}{\otimes}F)(x_i\otimes y_j)$ is a division algebra (with a unit element $x_i\otimes y_j$) and hence, $(E\underset{\pi}{\otimes}F)(x_i\otimes y_j)$ is a minimal (closed) ideal (see also [**22**, page 46, Corollary 2.1.9]). Now, by Lemma 3.3 in [**10**, page 1181], $x_i\otimes y_j$ is H-primitive. Finally, we show that $\mathcal{J}$ is a maximal family (with respect to the properties of its elements). Namely, a family of axes. Suppose that $\mathcal{J}$ is contained in a family, say $\mathcal{S}$, of doubly orthogonal H-primitive, self-adjoint projections. Let $w\otimes z$ be an element in $\mathcal{S}$, but not in $\mathcal{J}$; then $(x_i\otimes y_j)(w\otimes z)=0=(w\otimes z)(x_i\otimes y_j)$ and $\langle x_i\otimes y_j,w\otimes z\rangle_{\alpha,\beta}=0$ for all $(i,j)\in K\times M$. Thus $x_iw\otimes y_jz=0=wx_i\otimes zy_j$ and $\langle x_i,w\rangle_\alpha\langle y_j,z\rangle_\beta=0$ for all i,j. If $x_{i_0}w\neq0$ for some $i_0\in K$, then from $x_{i_0}w\otimes y_jz=0$ for all $j\in M$, we get $y_jz=0$ for all j's and thus $zy_j=0=y_jz$ and $\langle z,y_j\rangle=0$ for all j's. Namely, z is (doubly) orthogonal to every y_j, but this contradicts the maximality of (y_j). □

In this context, the following lemma has an interest per se. We still note that the same result gives one of the characteristic properties for left modular complemented H-algebras studied in [**15**].

Lemma 5.3. *Let* $(E,(p_\alpha)_{\alpha\in A})$ *be an Ambrose algebra having the* (Pbs)*-property. If* I *is a left ideal of* E *that contains a family of axes, say* $(x_i)_{i\in K}$, *of* E, *then* $I^\perp=(0)$. *In particular, if* E *is also orthocomplemented, then* I *is dense in* E.

Proof. Suppose $I^{\perp} \neq (0)$. Then, there exists some $0 \neq y \in I^{\perp}$ with $0 \neq x \equiv y^* y \in I^{\perp}$ (see [**9**, page 452, Theorem 1.3; page 456, Lemma 3.2]). Consider the subalgebra $F \equiv \overline{\mathbf{C}_0[x]}$. Since $x^n = x^{n-1}x \in I^{\perp}$ for all $n \in \mathbf{N}$, we get

$$F \subseteq I^{\perp}. \tag{5.2}$$

The continuity of the involution implies that F is self-adjoint, and hence it is a proper Hausdorff complete locally H^*-algebra with continuous involution (see also [**9**, page 453, Lemma 1.4]). By assumption, F has the (bs)-property and thus, by [**11**, page 144, Theorem 2.7], it has a (nonzero) self adjoint projection, say y, which by (5.2), belongs to $I^{\perp}$. By [**10**, page 1182, Theorem 3.4], $y = \sum_{j=1}^{n} y_j$ with $y_j, 1 \leqq j \leqq n$ (nonzero) doubly orthogonal (H)-primitive self-adjoint projections in E. Therefore, $y_j = y_j y \in I^{\perp}$ for all j's. Among the y_j's, we pick (without any loss of generality) y_1. We prove that the latter element is doubly orthogonal to the x_i's. Indeed, since $wx_i \in I^{\perp}$ for all $w \in E$ and $zy_1 \in I^{\perp}$ for all $z \in E$, we get $\langle wx_i, zy_1 \rangle_\alpha = 0$ for every $w, z \in E$, $\alpha \in A$, $i \in K$. In particular, $\langle x_i, y_1 \rangle_\alpha = 0$, for all $\alpha \in A$, $i \in K$. But, in a Hausdorff locally convex H^*-algebra with an involution, the last equality is equivalent to $y_1 x_i = x_i y_1 = 0$, $i \in K$ (see Section 1). Namely, the elements $y_1, x_i, i \in K$ are doubly orthogonal, and this finally, contradicts the maximality of $(x_i)_{i \in K}$. Therefore, we get the first part of the assertion. Now, assume that E is orthocomplemented. By the above argument, $\overline{I}^{\perp} = (0)$, and hence $\overline{I} = E$. $\square$

As an immediate consequence, we get the following result. In this context, we still note that the (bs)-property in the definition of a pre-hereditary Ambrose algebra (cf. Definition 2.3) is not needed in the proof. Thus, we have

Corollary 5.4. *In any pre-hereditary Ambrose algebra E, every left ideal, containing a family of axes in E, is dense in it.*

Theorem 5.5 (2nd Wedderburn structure theorem). *Let $(E, (p_\alpha)_{\alpha \in A})$, $(F, (q_\beta)_{\beta \in B})$ be hereditary Ambrose algebras and $(x_i)_{i \in K}$, $(y_j)_{j \in M}$ families of axes in E and F, respectively. Then $E \widehat{\underset{\pi}{\otimes}} F$ is the topological* *orthodirect sum of its minimal closed* 2*-sided ideals* $E_i \widehat{\underset{\pi}{\otimes}} F_j$, $(i, j) \in K \times M$.

Proof. We first note that the minimal closed 2-ideals, as in the statement, are mutually orthogonal (see Proposition 5.1). Fix $(i, j) \in K \times M$, and take the ideal $E_i \widehat{\underset{\pi}{\otimes}} F_j$. By Proposition 3.4, the canonical factors E_i, F_j have canonical bases $(x^i_{kl})_{(k,l)\in L^2_i}$, $(y^j_{mn})_{(m,n)\in \Lambda^2_j}$, respectively. Since E_i, F_j are proper Hausdorff locally H^*-algebras (see [**9**, page 457, Lemma 3.4]), we get that

$$(x^i_{kl} \otimes y^j_{mn})_{(k,l,m,n)\in L^2_i \times \Lambda^2_j} \tag{5.3}$$

is a canonical orthogonal basis for $E_i \widehat{\underset{\pi}{\otimes}} F_j$ (Theorem 4.3). On the other hand, due to Theorem 4.4, the respective canonical basis for $E \widehat{\underset{\pi}{\otimes}} F$ contains members from (5.3), when i, j run over K and M. This implies that the family $(E_i \widehat{\underset{\pi}{\otimes}} F_j)_{(i,j)\in K\times M}$ spans $E \widehat{\underset{\pi}{\otimes}} F$. So, if $z \in E \widehat{\underset{\pi}{\otimes}} F$, $z = \lim z_\delta$ with $z_\delta \in \underset{i,j}{\oplus}{}^{\perp} E_i \widehat{\underset{\pi}{\otimes}} F_j$ (cf. also the notation after Theorem 4.4) and hence $E \widehat{\underset{\pi}{\otimes}} F = \overline{\underset{i,j}{\oplus}{}^{\perp} E_i \widehat{\underset{\pi}{\otimes}} F_j}$, that completes the proof. $\square$

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