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VALUE DISTRIBUTION OF DIFFERENCES OF MEROMORPHIC FUNCTIONS

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ABSTRACT. In this paper, we investigate the value distribution of differences of meromorphic functions. Some results are proved concerning the existence of zeros of the $f^k\Delta_c f - a(z)$, $k \in \mathbf{Z}$, which can be viewed as discrete analogues of the Hayman conjecture [**10**].

1. Introduction. A function $f(z)$ is called meromorphic, if it is analytic in the complex plane $\mathbf{C}$ except at possible isolated poles. If no poles occur, then f reduces to an entire function. For $f(z)$ a meromorphic function, let $\sigma(f)$ be the order of growth, $\mu(f)$ the lower order of growth, $\lambda(f)$ the exponent of convergence of the zeros and $\lambda(1/f)$ the exponent of convergence of the poles. In what follows, we assume that the reader is familiar with the basic notation of Nevanlinna's value distribution theory [**11, 13, 18**].

Recently, there has been an increasing interest in studying difference equations in the complex plane $\mathbf{C}$. Many authors investigated the growth of solutions of complex difference equations, such as [**4, 15**], and value distribution of differences analogues of Nevanlinna's theory, such as [**2, 3, 7, 8, 14, 16**]. The forward differences are defined as

$$\Delta_c f := f(z+c) - f(z) \quad \text{and} \quad \Delta_c^n f := \Delta_c^{n-1}(\Delta_c f), \quad n \in N, \ n \geq 2,$$

where $f(z)$ is meromorphic and $c \in \mathbf{C} \setminus \{0\}$. If $c = 1$, we use the usual difference notation $\Delta_c f = \Delta f$.

Laine and Yang [**14**] first investigated the value distribution of difference products of entire functions, and obtained the following theorem.

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Theorem A. *Let $f(z)$ be a transcendental entire function of finite order, and c a non-zero complex constant. Then for $n \geq 2$, $f(z)^n f(z+c)$ assumes every non-zero value $a \in \mathbf{C}$ infinitely often.*

In this paper, we improve Theorem A to meromorphic functions and obtain the following results.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function of finite order $\sigma(f)$, $S(z) = R(z)e^{Q(z)}$, where $R(z)$ is a non-zero rational function, $Q(z)$ is a polynomial that satisfies $\deg Q(z) < \sigma(f)$. If $\lambda(1/f) < \sigma(f)$ and $\sum_{j=1}^{n} \nu_j \geq 3$, and at least one of $\nu_j \geq 2$, then $\prod_{j=1}^{n} f(z+c_j)^{\nu_j} - S(z)$ has infinitely many zeros.*

Remark 1. The restriction on the order in Theorem 1.1 cannot be deleted. This can be seen by taking $f(z) = 1/(ze^{e^z})$, $e^c = -n$ $(n \geq 2)$ and $R(z)$ is a rational function. Then $f(z)$ is of infinite order and has only one pole, while

$$f(z)^n f(z+c) - R(z) = \frac{1 - z^n(z+c)R(z)}{z^n(z+c)}$$

has finitely many zeros.

Remark 2. Obviously, if $n = 1$, then $\nu_1 \geq 3$. Theorem 1.1 is not true, if $\sum_{j=1}^{n} \nu_j = 2$. This can be seen by taking $f(z) = (1+e^z)/z$, $c = \pi i$. Then

$$f(z)f(z+c) - \frac{1}{z(z+c)} = \frac{-e^{2z}}{z(z+c)}$$

has no zeros.

Remark 3. Theorem 1.1 is not true, if $\lambda(1/f) = \sigma(f) = 1$. This can be seen by taking $f(z) = (1-e^z)/(1+e^z)$, $c = \pi i$. Then

$$f(z)^2 f(z+c) f(z+3c) - R(z) = 1 - R(z)$$

has finitely many zeros, where $R(z)$ is a rational function.

Theorem 1.2. *Let $f(z)$ be a transcendental meromorphic function of finite order $\sigma(f)$, $S(z) = R(z)e^{Q(z)}$, where $R(z)$ is a non-zero rational* *function, $Q(z)$ is a polynomial that satisfies* $\deg Q(z) < \sigma(f)$. *If $f(z)$ has finitely many poles and $n \geq 4$ is an integer, then*

$$\overline{N}\left(r, \frac{1}{f(z)^n f(z+c) - S(z)}\right) \geq T(r,f) + O(r^{\sigma(f)-1+\varepsilon}) + S(r,f).$$

Remark 4. From the proof of Theorem 1.2, if $c = 0$, then we can assume $n \geq 1$. Using a similar proof as for Theorem 1.2, if there exists a

$$\nu_j > n + \sum_{i \neq j}^{n} \nu_i,$$

then we get

$$\overline{N}\left(r, \frac{1}{\prod_{j=1}^{n} f(z+c_j)^{\nu_j} - S(z)}\right) \geq T(r,f) + O(r^{\sigma-1+\varepsilon}) + S(r,f),$$

where c_j are distinct complex numbers.

Concerning the value distribution of derivatives of transcendental functions, there is a classical result, which can be seen in [**6**, Theorem 1.5].

Theorem B. *Let $f(z)$ be a transcendental meromorphic function in the plane with*

$$\liminf_{r \to \infty} \frac{T(r,f)}{r} = 0. \tag{1.1}$$

Then f' has infinitely many zeros.

If f satisfies the conditions of Theorem B, it follows from Hurwitz's theorem that if z_0 is a zero of f', then $\Delta_c f$ has a zero near z_0 for all sufficiently small $c \in C \backslash \{0\}$. This makes it is natural to ask whether $\Delta_c f$ must have infinitely many zeros. Bergweiler and Langley [**2, 16**] investigated the existence of zeros of $\Delta^n f$ and of $(\Delta^n f)/f$, where f is a transcendental meromorphic function of $\sigma(f) < 1$, and obtained many profound and significant results, which can be viewed as discrete analogues of Theorem B on the zeros of $f'(z)$. They obtained the following theorem.

Theorem C ([**2**, Theorem 1.4]). *Let $f(z)$ be a transcendental meromorphic function of lower order $\mu(f) < 1$. Let $c \in \mathbf{C}\backslash\{0\}$ be such that at most finitely many poles z_j, z_k of f satisfy $z_j - z_k = c$. Then $\triangle_c f$ has infinitely many zeros.*

Hayman [**10**] conjectured that if f is a non-constant transcendental meromorphic function and $n \in \mathbf{N}$, then $f^n f'$ takes every finite non-zero value infinitely often. In fact, earlier Hayman [**9**] had proved that if f is a transcendental meromorphic function and n is an integer satisfying $n \geq 3$, then $f^n f'$ takes every non-zero complex value infinitely often. Later, the case $n = 2$ was settled by Mues [**17**]. Bergweiler and Eremenko [**1**] proved the last case $n = 1$, if f is a transcendental meromorphic function of finite order, then ff' takes every non-zero complex value infinitely often. Similar as to the Hayman conjecture, it is natural to ask the following question.

Question 1. If f is a transcendental meromorphic function, can we get the counterpart results to $f^k f' - a(z)$, when f' is replaced with $\Delta_c f$, where k is an integer?

In this paper, we investigate the zero distribution of $f^k \Delta_c f - a(z)$, where $k = 0$ or $k \in \mathbf{N}$. We first consider the case when $k = 0$ and $a(z)(\neq 0)$ is a constant. Obviously, if f is a transcendental entire function of order $\sigma(f) < 1$, then $\Delta_c f - a$ has infinitely many zeros. From the function $f(z) = e^z + az$, we know that if $\sigma(f) = 1$, $\lambda(f) = 1$ and $c = 1$, then $\Delta f - a = (e-1)e^z$ has no zeros. Then we give the following two results to see what conditions $f(z)$ should satisfy. Then $\Delta f - a$ has infinitely many zeros.

Theorem 1.3. *Let $f(z)$ be a meromorphic function of order $1 \leq \sigma(f) < \infty$, and let $f(z)$ have infinitely many zeros with $\lambda(f) < 1$, $\Delta_c f \not\equiv 0$, a a non-zero constant. If $f(z)$ has finitely many poles, then $\Delta_c f - a$ has infinitely many zeros.*

Corollary 1.4. *Let $f(z)$ be an entire function of order $1 \leq \sigma(f) < \infty$, and let $f(z)$ have infinitely many zeros with $\lambda(f) < 1$, $\Delta_c f \not\equiv 0$, a a non-zero constant. Then $\Delta_c f - a$ has infinitely many zeros.*

Remark 5. Theorem 1.3 is not true, if $f(z)$ has infinitely many poles and $\lambda(f) = 1$. This can be seen by $f(z) = 1/(e^z + 1) + R(z)$, where $c = 2\pi i$ and $R(z)$ is a rational function. Then $\Delta_c f - a = R(z+c) - R(z) - a$ has finitely many zeros.

We next proceed to proving a result for the case when $k = 0$ and $a(z) = 0$. In fact, Theorem C investigated this case. If f is a transcendental entire function of order $\sigma(f) < 1$, then each difference $\Delta_c^n f$ obviously has infinitely many zeros. In this paper, we prove the following theorem, when f is an entire function of order $\sigma(f) \geq 1$, which is an improvement of Theorem 3 in [**3**].

Theorem 1.5. *Let f be an entire function of order $1 \leq \sigma(f) < \infty$, and let f have infinitely many zeros with the exponent of convergence of zeros $\lambda(f) = \lambda < 1$, $c \in \mathbf{C} \setminus \{0\}$, $\Delta_c f \not\equiv 0$. Then $\Delta_c f$ has infinitely many zeros and infinitely many fixed points.*

Remark 6. The restriction on infinitely many zeros cannot be deleted, which can be seen by the function $f(z) = ze^z$, and $c = 2\pi i$. Then $f(z + c) - f(z) = ce^z$ has no zeros, while it has infinitely many fixed points. The condition $\lambda(f) < 1$ cannot be replaced by $\lambda(f) = 1$, which can be seen by $f(z) = e^z + (1/2)z^2 - (1/2)z + 1$. Then $f(z+1) - f(z) = (e-1)e^z + z$ has no fixed points, while it has infinitely many zeros.

Finally, we will give two theorems to the case of $k \in \mathbf{N}$ and $a(z)$ is a small function to $f(z)$.

Theorem 1.6. *Let $f(z)$ be a transcendental meromorphic function of finite order, $\Delta_c f \not\equiv 0$, c a non-zero constant, $n \geq 2$ an integer, $R(z)$ a rational function. If $f(z)$ has finitely many poles, then $f(z)^n \Delta_c f - R(z)$ has infinitely many zeros.*

The method of proof Theorem 1.6 is similar as the proof of Theorem 1.1.

Regarding to the meromorphic function of infinitely many poles, we investigate the function of order $\sigma(f) < 1$, and get the following theorem.

Theorem 1.7. *Let $f(z)$ be a transcendental meromorphic function of $\sigma(f) < 1$, c a non-zero constant, and a set $B = \{b_j\}$ consists of all poles of $f(z)$ such that $b_j + kc \notin B(k = 1, \dots, n)$ at most finitely many exceptions. Then $f(z)^n \Delta_c f - a$ has infinitely many zeros.*

Remark 7. Some ideas of this paper are from [**2, 3, 14**].

2. Some lemmas. Recently, Halburd-Korhonen [**7**] and Chiang-Feng [**4**] investigated the value distribution theory of difference expressions and obtained two similar results which can be viewed the difference analogue of the logarithmic derivative lemma. In what follows, we mostly refer to [**4**, Corollary 2.6]:

Lemma 2.1. *Let $f(z)$ be a meromorphic function of finite order σ, and let η_1, η_2 be two distinct complex constants. Then for any $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\sigma-1+\varepsilon}). \tag{2.1}$$

As for the original formulation of the classical Clunie lemma, see [**5**]. The following version ([**13**, Theorem 2.4.2]), slightly more general than the original one, has been reinvented several times in the literature:

Lemma 2.2. *Let $f(z)$ be a transcendental meromorphic solution of*

$$f^n A(z,f) = B(z,f),$$

where $A(z,f)$, $B(z,f)$ are differential polynomials in f and its derivatives with small meromorphic coefficients a_λ, $\lambda \in I$ in the sense of $m(r, a_\lambda) = S(r,f)$ for all $\lambda \in I$. If the total degree of $B(z,f)$ as a polynomial in f and its derivatives is $\leq n$, then $m(r, A(z,f)) = S(r,f)$.

The following result is very important in the proof of Theorem 1.7, and the first part proof which can be seen in [**2**, Lemma 2.1]. We give the proof to complete the proof of Lemma 2.3.

Lemma 2.3. *Let f be a transcendental meromorphic function satisfying* (1.1)*, then $\Delta_c f$ is transcendental. Furthermore, if f satisfies the condition of Theorem* 1.7*, then $f(z)^n \Delta_c f$ is transcendental.*

Proof. Assume that $\Delta_c f$ is a rational function. Then there exists a rational function $R_1(z)$ such that

$$f(z+c) = f(z) + R_1(z), \quad f(z-c) = f(z) - R_1(z-c). \tag{2.2}$$

If f has infinitely many poles, we take r_0 large enough that $R_1(z)$ has neither zeros nor poles in $|z| > r_0$. Suppose that z_0 is a pole of f with $|z_0| > r_0$, then (2.2) shows that either $z_0 + kc$ or $z_0 - kc$ $(k = 0, 1, \dots)$ are the poles of f, depending on the sign of $\operatorname{Re} z_0$, which is a contradiction to (1.1).

If f has finitely many poles, then there exists a rational function R_2 such that $h = f - R_2$ is transcendental entire, and it can be assumed that R_1 is a polynomial by considering h in place of f. There exists a polynomial P such that

$$P(z+c) - P(z) = R_1(z);$$

then we get $R_1 \equiv 0$ by considering $f - P$ in place of f. So we get $f(z+c) \equiv f(z)$. Then f is a periodic function, which means that $\sigma(f) \geq 1$ contradicting (1.1). Thus, we have proved that $\Delta_c f$ is transcendental.

From Theorem C, we know that $\Delta_c f$ must have infinitely many zeros. If $f(z)^n \Delta_c f$ is a rational function, assume that $f(z)^n \Delta_c f = R(z)$, so f must have infinitely many poles. We take a pole z_1 of f with $|z_1| = r_1$ large enough that R has no poles in $|z_1| > r_1$, so we know $z_1 \pm kc$ $(k = 0, 1, \dots)$ must be the poles of f, which is a contradiction to the condition.

Lemma 2.4 [**2**]. *Let f be a transcendental meromorphic function of lower order $\mu(f) < 1$. Then there exists an arbitrarily large R with the* *following properties. First,*

$$T(32R, f') < R^{\mu(f)}.$$

Second, there exists a set $J_R \subseteq [(R/2), R]$ of linear measure $(1 - o(1))(R/2)$ such that, for $r \in J_R$,

$$f(z+1) - f(z) \sim f'(z) \quad on \quad |z| = r,$$

where if $\phi(z)$ and $\psi(z)$ are two functions, then $\phi(z) \sim \psi(z)$ means that $\phi(z)/\psi(z)$ tends to unity as $|z| = r$.

Following Hayman [**12**, pages 75–76], we define an ε-set E to be a countable union of discs not containing the origin, and subtending angles at the origin whose sum is finite. If E is an ε-set, then the set of $r \geq 1$ for which the circle $S(0, r)$ meets E has finite logarithmic measure.

Lemma 2.5 [**2**]. *Let $g(z)$ be a transcendental meromorphic function of order $\sigma(f) < 1$, $h > 0$. Then there exists an ε-set E such that*

$$\frac{g'(z+c)}{g(z+c)} \longrightarrow 0 \quad and \quad \frac{g(z+c)}{g(z)} \longrightarrow 1 \quad as \quad z \to \infty \ in\ \mathbf{C} \setminus E,$$

uniformly in c for $|c| \leq h$. Further, E may be chosen so that for large $z \notin E$, the function g has no zeros or poles in $|\zeta - z| \leq h$.

Lemma 2.6 ([**18**, Theorem 1.51]). *Let $f_j(z)$ $(j = 1, \ldots, n)$ $(n \geq 2)$ be meromorphic functions, $g_j(z)$ $(j = 1, \ldots, n)$ entire functions, and satisfy*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$,

(ii) it when $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ is not constant,

(iii) *when $1 \leq j \leq n$, $1 \leq h < k \leq n$,*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}, \quad r \to \infty, r \notin E,$$

where $E \subset (1, \infty)$ is of finite linear measure.

Then $f_j(z) \equiv 0$ $(j = 1, \ldots, n)$.

3. Proof of Theorem 1.1. We first consider that all $c_j = 0$, $\sum_{j=1}^{n} \nu_j$ =m. Since $S(z) = R(z)e^{Q(z)}$ and $\deg Q(z) < \sigma(f)$, then we get $T(r, S(z)) = O(r^{\sigma-1+\varepsilon})$. Let $G = f(z)^m - S(z)$. Applying the proof of the second main theorem for three small target functions in [**11**, Theorem 2.5] with slight modifications, we get

$$\begin{aligned} mT(r,f) + S(r,f) &= T(r,G) \\ &\le \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G+S(z)}\right) + \overline{N}\left(r, \frac{1}{G}\right) \\ &\quad + O(r^{\sigma-1+\varepsilon}) + S(r,G) \\ &\le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{G}\right) \\ &\quad + O(r^{\sigma-1+\varepsilon}) + S(r,f) \\ &\le 2T(r,f) + \overline{N}\left(r, \frac{1}{G}\right) + O(r^{\sigma-1+\varepsilon}) + S(r,f). \end{aligned}$$

Since $m \ge 3$, we conclude that G must have infinitely many zeros.

Assume now that at least one $c_j \ne 0$. Suppose on contrary to the assertion that $\prod_{j=1}^{n} f(z+c_j)^{\nu_j} - S(z)$ has finitely many zeros and $\lambda(1/f) < \sigma(f) < \infty$. Then we get

$$D(z) =: \prod_{j=1}^{n} f(z+c_j)^{\nu_j} - S(z) = \frac{A(z)}{B(z)} e^{P(z)}, \tag{3.1}$$

where $A(z)$ is the canonical product formed with the zeros of $D(z)$, a polynomial, $B(z)$ is the canonical product formed with the poles of $D(z)$ with $\lambda(B) = \sigma(B) = \lambda(1/f) < \sigma(f) - \varepsilon$ and $P(z)$ is a non-constant polynomial. Differentiating (3.1) and eliminating $e^{P(z)}$, we obtain

$$\prod_{j=1}^{n} f(z+c_j)^{\nu_j} F(z,f) = H(z)e^{Q(z)}, \tag{3.2}$$

where

$$F(z,f) = \frac{A'(z)B(z) - A(z)B'(z)}{B^2(z)} + \frac{A(z)P'(z)}{B(z)} - \frac{A(z)}{B(z)} \sum_{j=1}^{n} \nu_j \frac{f'(z+c_j)}{f(z+c_j)}, \tag{3.3}$$

and

$$H(z) = \left(\frac{A'(z)B(z) - A(z)B'(z)}{B^2(z)} + \frac{A(z)P'(z)}{B(z)} \right) R(z) \tag{3.4}$$
$$- \frac{A(z)}{B(z)} \Big(R'(z) + R(z)Q'(z) \Big).$$

We affirm that $F(z, f)$ cannot vanish identically. Indeed, if $F(z, f) \equiv 0$, then by (3.2) and (3.4) we get

$$\frac{A'(z)}{A(z)} - \frac{B'(z)}{B(z)} - \frac{R'(z)}{R(z)} + P'(z) - Q'(z) \equiv 0.$$

So, we get

$$\frac{A(z)}{B(z)R(z)} = ae^{Q(z)-P(z)}, \tag{3.5}$$

where a is a non-zero constant. Since $A(z)$ is a polynomial and $R(z)$ is a rational function, $B(z)$ must have finitely many zeros. Then we may write $B(z) = h^*(z)e^{h(z)}$, where $h^*(z)$ is a polynomial. So we know that $P(z) - Q(z) - h(z)$ must be a constant. By (3.1), we have

$$\prod_{j=1}^{n} f(z + c_j)^{\nu_j} = M(z)e^{Q(z)}, \tag{3.6}$$

where $M(z)$ is a rational function. Since $\lambda(1/f) < \sigma(f)$, then we may assume, without loss of generality, that $\lambda[1/(f(z + c_1))] < \sigma(f)$ and $\nu_1 \geq 2$. From (3.6), we get that there exists an $\varepsilon > 0$ such that

$$\sum_{j=1}^{n} \nu_j m\left(r, f(z + c_1)\right) \tag{3.7}$$
$$\leq \sum_{i=2}^{n} \nu_i m\left(\frac{f(z + c_1)}{f(z + c_i)} \right) + O(r^{\sigma(f)-\varepsilon}) + O(\log r)$$
$$\leq O(r^{\sigma(f)-1+\varepsilon}) + O(r^{\sigma(f)-\varepsilon}) + O(\log r).$$

So

$$T(r, f(z + c_1)) \leq O(r^{\sigma(f)-1+\varepsilon}) + O(r^{\sigma(f)-\varepsilon}) + O(\log r),$$

which contradicts the assumption that $f(z)$ is transcendental of order $\sigma(f)$; thus, we prove $F(z,f) \not\equiv 0$.

By the logarithmic derivative lemma and (3.3), we get

$$m(r, F(z,f)) = O(r^{\sigma(f)-1+\varepsilon}) + O(r^{\sigma(B)+\varepsilon}) + O(\log r).$$

By Lemma 2.2 and applying Lemma 2.1 to (3.2), we get

$$m(r, f(z+c_1)F(z,f)) = O(r^{\sigma(f)-1+\varepsilon}) + O(r^{\sigma(B)+\varepsilon}) + O(\log r).$$

From(3.2) and (3.3), we know that the poles of $F(z,f)$ may only be located at the zeros or the poles of $f(z+c_j)$, the zeros of $B(z)$ or the poles of rational functions. Assuming that $\nu_1 \geq 2$, we will show that $\lambda(f(z+c_1)) < \sigma(f)$. If z_0 is a zero of $f(z+c_1)$ and also a zero of $B(z)$, then we get $\lambda(f(z+c_1)) < \lambda(B) < \sigma(f)$. If z_0 is a zero of $f(z+c_1)$ but not a zero of $B(z)$, from (3.3), we know that the pole of $F(z,f)$ must be simple; then, from comparing the exponent of convergence of the zeros of two sides of (3.2), we get a contradiction.

Thus,

$$N(r, F(z,f)) = O(r^{\lambda(f)+\varepsilon}) + O(r^{\lambda(1/f)+\varepsilon}) + O(r^{\sigma(B)+\varepsilon}) + O(\log r),$$

and

$$\begin{aligned} &N(r, f(z+c_1)F(z,f)) \\ &\qquad = O(r^{\lambda(f)+\varepsilon}) + O(r^{\lambda(1/f)+\varepsilon}) + O(r^{\sigma(B)+\varepsilon}) + O(\log r). \end{aligned}$$

Hence

$$T(r, F(z,f)) = O(r^{\lambda(f)+\varepsilon}) + O(r^{\lambda(\frac{1}{f})+\varepsilon}) + O(r^{\sigma(B)+\varepsilon}) + O(\log r),$$

and

$$\begin{aligned} &T(r, f(z+c_1)F(z,f)) \\ &\qquad = O(r^{\lambda(f)+\varepsilon}) + O(r^{\lambda(1/f)+\varepsilon}) + O(r^{\sigma(B)+\varepsilon}) + O(\log r). \end{aligned}$$

Therefore,

$$T(r, f(z+c_1)) = O(r^{\lambda(f)+\varepsilon}) + O(r^{\lambda(1/f)+\varepsilon}) + O(r^{\sigma(B)+\varepsilon}) + O(\log r),$$

which is a contradiction to $f(z)$ being transcendental of order $\sigma(f)$, Thus, we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.2. Since $f(z)$ is a transcendental meromorphic function of finite order σ, from [**4**, Theorem 2.1], we know that the following relation

$$T(r, f(z+c)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r) \tag{4.1}$$

holds. Let $G(z) = f(z)^n f(z+c) - S(z)$. Obviously, from (4.1), we can get

$$T(r, G(z)) \leq (n+1)T(r, f) + S(r, f).$$

On the other hand, we can get

$$\begin{aligned} T(r, G(z)) &\geq T(r, f(z)^n f(z+c)) + S(r, f) \\ &\geq T(r, f(z)^n) - T(r, f(z+c)) + S(r, f) \\ &\geq (n-1)T(r, f(z)) + S(r, f). \end{aligned} \tag{4.2}$$

Applying again the second main theorem for three small target functions ([**11**, Theorem 2.5]), see the proof of Theorem 1.1, and recalling that f has finitely many poles, we get

$$\begin{aligned} T(r, G(z)) &\leq \overline{N}(r, G(z)) + \overline{N}\left(r, \frac{1}{G(z)}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{G(z)+S(z)}\right) + O(r^{\sigma-1+\varepsilon}) + S(r, G) \\ &\leq \overline{N}(r, f(z)) + \overline{N}(r, f(z+c)) + \overline{N}\left(r, \frac{1}{G(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + O(r^{\sigma-1+\varepsilon}) + S(r, f) \\ &\leq 2T(r, f(z)) + \overline{N}\left(r, \frac{1}{G(z)}\right) + O(r^{\sigma-1+\varepsilon}) + S(r, f). \end{aligned} \tag{4.3}$$

From (4.2) and (4.3), we get

$$\overline{N}\left(r, \frac{1}{G}\right) \geq (n-3)T(r, f) + O(r^{\sigma-1+\varepsilon}) + S(r, f).$$

We have completed the proof of Theorem 1.2.

5. Proof of Theorem 1.5. Since f is an entire function of order $1 \le \sigma(f) < \infty$ and has infinitely many zeros with the exponent of convergence of zeros $\lambda(f) = \lambda < 1$, from the Hadamard factorization theorem, we can write $f(z) = h(z)e^{P(z)}$, where $h(z)$ is the product of the zeros of f, is also an entire function, and $\lambda(f) = \lambda(h) = \sigma(h) < 1$, $P(z)$ a polynomial. Denote $P(z+c) - P(z) = R(z)$. Hence,

$$\begin{aligned} g(z) &= f(z+c) - f(z) \\ &= h(z+c)e^{P(z)+R(z)} - h(z)e^{P(z)} \\ &= (h(z+c)e^{R(z)} - h(z))e^{P(z)}. \end{aligned} \tag{5.1}$$

We will prove that $g(z)$ has infinitely many zeros.

By Lemma 2.5, there exists an ε-set E such that $h(z+c) \sim h(z)$ as $z \to \infty$ in $\mathbf{C} \setminus E$. We affirm

$$h(z+c)e^{R(z)} - h(z) \not\equiv 0.$$

Otherwise, for $z \in \mathbf{C} \setminus E$, we get $f(z+c) \equiv f(z)$, which contradicts with $\Delta_c f \not\equiv 0$.

Assume on the contrary to the assumption that $g(z)$ has finitely many zeros; then, there exists a non-zero polynomial $Q(z)$ such that

$$h(z+c)e^{R(z)} - h(z) = Q(z).$$

Writing this in the form

$$\frac{h(z+c)}{h(z)}e^{R(z)} - 1 = \frac{Q(z)}{h(z)}, \tag{5.2}$$

then the order of the right hand side of (5.2) is less than 1, and the order of the left hand side of (5.2) is at least one unless $R(z)$ is a constant. Assume that $R(z) = a$. If $a \ne 2n\pi i$, then $h(z)$ is a polynomial, which is a contradiction to the assumption that h has infinitely many zeros. Hence $a = 2n\pi i$, for some n. So $h(z+c) \equiv h(z)$, which is a contradiction. Thus, we have proved that $g(z)$ has infinitely many zeros.

Next, we will prove that $g(z)$ has infinitely many fixed points. Otherwise, from the Hadamard factorization theorem, we let

$$g^*(z) = g(z) - z = h^*(z)e^{P^*(z)}, \tag{5.3}$$

where $P^*(z)$, $h^*(z)$ are non-zero polynomials.

From (5.1) and (5.3), we get

$$h(z+c)e^{P(z+c)} - h(z)e^{P(z)} - z - h^*(z)e^{P^*(z)} \equiv 0. \tag{5.4}$$

In the following, we will prove equation (5.4) satisfies condition (ii) of Lemma 2.6.

Case 1. Suppose that $P(z+c) - P(z) \equiv A$, A is a constant. Since $P(z)$ is a polynomial, it must have the form

$$P(z) = az + d \quad \text{and} \quad ac = A, a \neq 0. \tag{5.5}$$

Combining (5.4) and (5.5), we get

$$[h(z+c)e^{ac} - h(z)]e^{az+d} - h^*(z)e^{P^*(z)} - z = 0. \tag{5.6}$$

If $P^*(z) - az - d \equiv m$, m is a given constant, then $P^*(z)$ must have the form $P^*(z) = az + b$. From (5.6), we get

$$[h(z+c)e^{ac} - h(z) - h^*(z)e^{b-d}]e^{az+d} - z = 0, \tag{5.7}$$

which is a contradiction.

If $P^*(z) - az - d \not\equiv m$, from (5.6), applying Lemma 2.6, we get $h^*(z) \equiv 0$, which is a contradiction. So we get $P(z+c) - P(z) \not\equiv A$.

Case 2. If $P(z+c) - P^*(z) \equiv B$, B is a constant, then from (5.4), we get

$$[h(z+c)e^{B} - h^*(z)]e^{P^*(z)} - h(z)e^{P(z)} - z \equiv 0. \tag{5.8}$$

If $P^*(z) - P(z) \equiv n$, n is a given constant, then from (5.8), we get

$$[h(z+c)e^{B} - h^*(z) - h(z)]e^{P^*(z)} - z \equiv 0. \tag{5.9}$$

From Lemma 2.6, we get $P^*(z)$ must be a constant and $P(z)$ be a constant, so $f(z) = Ch(z)$, which contradicts $\sigma(f) \geq 1$.

If $P^*(z) - P(z) \not\equiv n$, applying Lemma 2.6 to (5.8), we get $h(z) \equiv 0$, which is a contradiction. So we get $P(z+c) - P^*(z) \not\equiv B$.

Case 3. If $P(z) - P^*(z) \equiv C$, C is a constant; then, using a similar method to Case 2, we can get a contradiction to see that $P(z) - P^*(z) \not\equiv C$.

From Cases 1, 2, 3, we know that equation (5.4) satisfies condition (ii) of Lemma 2.6, so (5.4) satisfies condition (iii). Applying Lemma 2.6 to (5.4), we get $h(z) \equiv 0$, which is a contradiction. Hence $g(z)$ has infinitely many fixed points.

6. Proof of Theorem 1.3. If $f(z)$ has finitely many poles, then there exists a non-constant polynomial $S(z)$ such that $g(z) = S(z)f(z)$ is an entire function of $\sigma(g) = \sigma(f) \geq 1$, and $f(z)$ has infinitely many zeros with $\lambda(f) = \lambda(g) < 1$. Then we can write

$$f(z) = \frac{h(z)}{S(z)} e^{Q(z)},$$

where $Q(z)$ is a non-zero polynomial, $h(z)$ is an entire function of $\sigma(h) = \lambda(g) < 1$. Then

$$f(z+c) - f(z) - a = \frac{h(z+c)}{S(z+c)} e^{Q(z+c)} - \frac{h(z)}{S(z)} e^{Q(z)} - a. \tag{6.1}$$

Since $S(z)$ is a non-constant polynomial, so if

$$h(z+c)S(z)e^{Q(z+c)} - h(z)S(z+c)e^{Q(z)} - aS(z+c)S(z)$$

has infinitely many zeros, then the left hand side of (6.1) also has infinitely many zeros. Using Lemma 2.6, we get $h(z+c)S(z)e^{Q(z+c)-Q(z)} - h(z)S(z+c) \not\equiv 0$.

If $f(z+c) - f(z) - a$ has finitely many zeros, we assume that

$$h(z+c)S(z)e^{Q(z+c)} - h(z)S(z+c)e^{Q(z)} - aS(z+c)S(z) = P(z)e^{P^*(z)}, \tag{6.2}$$

where $P^*(z)$, $P(z)$ are non-zero polynomials. A similar method to the proof of Theorem 1.5, we get equation (6.2) which also satisfies Lemma 2.6; so we can get $a = 0$ or $S(z) \equiv 0$, which is a contradiction. Thus, we have completed the proof of Theorem 1.3.

7. Proof of Theorem 1.7. From Lemma 2.3, we know that $g(z) = f^n(z)\Delta_c f - a$ is a transcendental function. It is easy to get

$\sigma(\Delta_c f) \leq \sigma(f)$. Assume that $\sigma(f) < \sigma_1 < 1$, then $\sigma(g) \leq \sigma(f) < \sigma_1$. In the following, we will use Cauchy's argument principle to prove Theorem 1.7.

From Lemma 2.5, let the ε-set E contain all zeros and poles of $g(z)$, $f(z)$, $f(z+c)$, $f^n f' - a$; we define

$$\begin{matrix} E_R = \{r : z \in E, |z| = r < R\}, \\ E_\infty = \{r : z \in E, |z| = r < \infty\}, \end{matrix} \quad R \in (1, \infty).$$

Then by the property of the ε-set and $\sigma(f) < \sigma_1 < 1$, we see that E_∞ has finite linear measure and E_R has linear measure $o(1)(R/2)$.

By Lemma 2.4, we see that there exists a large R with

$$T(32R, f') < R^{\sigma_1}, \tag{7.1}$$

and there exists a set $J_R \subseteq [(R/2), R] \backslash E_R$ of linear measure $(1 - o(1))(R/2)$, such that for $r \in J_R$,

$$f(z+c) - f(z) \sim f'(z) \quad \text{on} \quad |z| = r. \tag{7.2}$$

Let

$$F_R = \left\{ r \in \left[\frac{R}{2}, R\right] : n(r, f) = n(r-1, f) \right\}; \tag{7.3}$$

then F_R has linear measure

$$m(F_R) \geq (1 - o(1)) \frac{R}{2}. \tag{7.4}$$

To see this, note that there are at most $o(R)$ points $p_k \in [(R/3), R]$ at which $n(t, f)$ is discontinuous by (7.1), and if $r \in [(R/2), R]$ is such that $n(r, f) > n(r-1, f)$, then $r \in [p_k, p_k + 1]$ for some k.

From (7.2)–(7.4) and $J_R \bigcap E_R = \varnothing$, we see that there exists an $r \in F_R \bigcap J_R$ such that $g(z)$, $f(z)$, $f(z+c)$, $f^n f' - a$ have no zeros and poles on $|z| = r$. But, from the condition of Theorem 1.7, there exists an r_0, independent of R and r, such that if f has poles of multiplicities m at z_0 and $r_0 \leq |z_0| \leq r - 1$, then $f(z_0) = \infty$, $f(z_0 \pm c) \neq \infty$. Thus, from the following,

$$\begin{aligned} g(z) &= f^n(z)(f(z+c) - f(z)) - a \\ g(z-c) &= f^n(z-c)(f(z) - f(z-c)) - a, \end{aligned}$$

we know that $g(z)$ must have poles at z_0 with multiplicities $(n+1)m$ and poles at $z_0 - c$ with multiplicities m.

Hence

$$n(r,g) \geq (n+2)n(r,f) + O(1) = (n+2)n(r-1,f) + O(1). \tag{7.5}$$

By (7.2) and $g(z)$, $f^n f' - a$ have no zeros and poles on $|z| = r \in F_R \bigcap J_R$; applying Cauchy's argument principle, we obtain

$$\begin{aligned} n\left(r,\frac{1}{g}\right) &= n\left(r,\frac{1}{f^n f'-a}\right) - n(r,f^n f'-a) + n(r,g) \\ &\geq n(r,\frac{1}{f^n f'-a}) - n(r,f^n f'-a) \\ &\quad + (n+2)n(r-1,f) + O(1) \\ &\geq n\left(r,\frac{1}{f^n f'-a}\right) - n(r,f^n f') \\ &\quad + (n+2)n(r,f) + O(1) \\ &\geq n\left(r,\frac{1}{f^n f'-a}\right) + O(1). \end{aligned} \tag{7.6}$$

So, we only need to show that $f^n f' - a$ has infinitely many zeros. To see this, note that $\sigma(f) < \sigma_1 < 1$,

$$\lim_{r\to\infty} \inf \frac{T(r, f^{n+1} - (a/n+1)z)}{r} = \lim_{r\to\infty} \inf \frac{(n+1)T(r,f)}{r} = 0. \tag{7.7}$$

From Theorem B, we know that $f^n f' - a$ must have infinitely many zeros, so $g(z)$ must have infinitely many zeros. Theorem 1.7 is thus proved.

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REFERENCES

1. W. Bergweiler and A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Matem. Ibero. **11** (1995), 355–373.

2. W. Bergweiler and J.K. Langley, *Zeros of difference of meromorphic functions*, Math. Proc. Camb. Phil. Soc. **142** (2007), 133–147.

3. Z.X. Chen and K.H. Shon, *On zeros and fixed points of difference of meromorphic functions*, J. Math. Anal. Appl. **344** (2008), 373–383.

4. Y.M. Chiang and S.J. Feng, *On the Nevanlinna characteristic $f(z+\eta)$ and difference equations in the complex plane*, The Ramanujan J. **16** (2008), 105–129.

5. J. Clunie, *On integral and meromorphic functions*, J. London Math. Soc. **37** (1962), 17–27.

6. A. Eremenko, J.K. Langley and J. Rossi, *On the zeros of meromorphic functions of the form $\sum_{k=1}^{\infty} \frac{a_k}{z-z_k}$*, J. d'Analyse Math. **62** (1994), 271–286.

7. R.G. Halburd and R.J. Korhonen, *Difference analogue of lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl. **314** (2006), 477–487.

8. ———, *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 463–478.

9. W.K. Hayman, *Picard values of meromorphic functions and their derivatives*, Ann. Math. **70** (1959), 9–42.

10. ———, *Research problems in function theory*, The Athlone Press, London, 1967.

11. ———, *Meromorphic functions*, Clarendon Press, Oxford, 1964.

12. ———, *Slowly growing integral and subharmonic functions*, Comment. Math. Helv. **34** (1960), 75–84.

13. I. Laine, *Nevanlinna theory and complex differential equation*, Walter de Gruyter, Berlin, 1993.

14. I. Laine and C.C. Yang, *Value distribution of difference polynomials*, Proc. Japan Acad. **83** (2007), 148–151.

15. ———, *Clunie theorem for difference and q difference polynomials*, J. Lond. Math. Soc. **76** (2007), 556–566.

16. J. K. Langley, *Value distribution of differences of meromorphic functions*, Rocky Mountain J. Math. **41** (2011), 275–292.

17. E. Mues, *Über ein Problem von Hayman*, Math. Z. 164 (1979), 239–259.

18. C.C. Yang and H.X. Yi, *Uniqueness theory of meromorphic functions*, in *Mathematics and its application*, Kluwer Academic Publishers, Dordrecht, 2003.

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