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ON LIFTING OF IDEMPOTENTS IN TOPOLOGICAL ALGEBRAS

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ABSTRACT. We extend the classical *"Lifting of Idempotents Theorem"* for unital commutative Banach algebras in the general framework of topological algebras. For this one has to give, within the same general context, new versions of the well-known *"Quasi-square Root Lemma"*, as well as of the *"Fixed Point Theorem"*, which are also presented.

0. Introduction. The *"Lifting of Idempotents Theorem"* provides an idempotent element for a given algebra E from a similar element of the quotient algebra $E/\operatorname{rad} E$, where $\operatorname{rad} E$ denotes the *topological Jacobson radical of* E. This has been proved for unital commutative Banach algebras by *Rickart* [**19**], for non-unital non-commutative Banach algebras by *Bonsall and Duncan* [**3**] and for commutative complete l.m.c. algebras by *Mallios* [**16**]. We extend the previous results to the general case of a topological algebra E, taking the *Gel'fand radical* of E, $\ker(\mathcal{G}_E)$ (the terminology is due to *Mallios*) in place of $\operatorname{rad} E$. So, we are led to examine, within the previous setting, the analogue of *"Square Root Lemma"* of *Ford* [**5**] for Banach algebras that in 1980 *Štěrbová* [**21**] generalized for complete l.m.c. algebras, as well as the *Fixed Point Theorem* of *Banach* [**4**] (see also [**20**] and/or [**13**]). We consider an algebra E topologized by the topology of its spectral radius r_E, replacing in all the preceding results the completeness of the underlying topological vector space E by the advertible completeness of the topological algebra E (Corollaries 2.7, 2.8, Theorems 3.4 and 4.1). So one has to cope with two problems: namely, in the case of an

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advertibly complete algebra, a Cauchy net converges, if it is advertibly null, while *advertible completeness is inherited to closed subalgebras.* Specifically, the first situation appears in the *"Fixed Point Theorem,"* where one classically proves that any contraction T has a fixed point; this reduces to the convergence of a Cauchy sequence of iterates, that actually amounts to finding an element in E making the previous net advertibly null. For this, one can take, for any $n \in \mathbf{N}$, $T^n(0)$ to be a spectrally zero element, i.e., $r_E(T^n(0)) = 0$, $n \in \mathbf{N}$ (Theorem 2.6, Corollaries 2.7, 2.8). We note that this is actually the case in the context of the *"Lifting of Idempotents Theorem,"* classically or not (cf. Theorem 4.1, (4.5)). Finally, we have to deal with the inheritance of advertible completeness on B, that can be arranged by remarking that B possesses already a sort of algebraic structure (cf. (3.7)).

1. Preliminaries. In all that follows by a *topological algebra* E we mean a topological $\mathbf{C}$-vector space which is also an algebra with separately continuous ring multiplication, having a non-empty *spectrum* $\mathfrak{M}(E)$ endowed with the *Gel'fand topology.* The respective *Gel'fand map* of E is given by

$$\begin{aligned}\mathcal{G} : E \longrightarrow \mathcal{C}(\mathfrak{M}(E)) : x \longmapsto \mathcal{G}(x) \equiv \widehat{x} : \mathfrak{M}(E) \longrightarrow \mathbf{C}\\ : f \longmapsto \widehat{x}(f) := f(x).\end{aligned}$$

The image of $\mathcal{G}$, denoted by $E^\wedge$, is called the *Gel'fand transform algebra* of E and is topologized as a *locally m-convex algebra* by the inclusion

$$E^\wedge \subseteq \mathcal{C}_c(\mathfrak{M}(E)),$$

where the algebra $\mathcal{C}(\mathfrak{M}(E))$ carries the topology "c" of compact convergence in $\mathfrak{M}(E)$ [**15**, page 19, Example 3.1].

Given an algebra E, an element $x \in E$ is called *quasi-invertible*, if there exists $y \in E$ such that

$$x \circ y = 0 = y \circ x, \text{ where } x \circ y = x + y - xy.$$

The last relation above defines the so-called "*circle operation*" or else "*q-operation.*" Then y is called the *quasi-inverse* of x and is unique, while the group of all quasi-invertible elements of E is denoted by E°. A subalgebra F of E is called *quasi-plane* if

$$F \cap E^\circ = F^\circ,$$

while the respective relation in the unital case defines a *plane subalgebra* of E. We denote by $Sp_E(x)$ and $r_E(x)$, the *spectrum* and *spectral radius* of $x \in E$, respectively, i.e.,

$$Sp_E(x) = \{\lambda \in \mathbf{C}\backslash\{0\} : \lambda^{-1}x \notin E^\circ\}$$

and

$$r_E(x) = \sup\{|\lambda| : \lambda \in Sp_E(x)\}.$$

If E, F are two algebras and $\phi : E \to F$ an algebra morphism, the spectra of their elements are connected by the relation

$$Sp_F(\phi(x)) \subseteq Sp_E(x), \tag{1.1}$$

for every $x \in E$ [**15**, page 49, Proposition 1.1]. An element $x \in E$ is called *spectrally zero*, if $r_E(x) = 0$.

For a topological algebra E, one has

$$\widehat{x}(\mathfrak{M}(E)) \subseteq Sp_E(x), \tag{1.2}$$

for every $x \in E$ [**15**, page 74, Corollary 7.4, (7.19)]. In this concern, by a *topologically spectral algebra*, we mean a topological algebra E, whose spectrum $\mathfrak{M}(E)$ is a *spectral set*, in the sense that

$$Sp_E(x) = \widehat{x}(\mathfrak{M}(E)), \tag{1.3}$$

for every $x \in E$ [**8**, page 13, Definition 2.1]. The previous algebra is called a *topological algebra with functional spectrum* by *Abel* [**1**, page 18, (2)], while the term, *topological algebra with functional point-spectrum* is also in use (*Mallios*). In this case one has

$$r_E(x) = \sup_{f \in \mathfrak{M}(E)} |\widehat{x}(f)|, \tag{1.4}$$

for every $x \in E$. We say that E is a *quasi-inverse closed algebra*, if its spectrum $\mathfrak{M}(E)$ is a *quasi-inverting set*, in the sense that

$$x \in E^\circ \text{ if } 1 \notin \widehat{x}(\mathfrak{M}(E)), \tag{1.5}$$

[**8**, page 13, Definition 2.2]. The converse statement is always valid, in fact, quite algebraically [**15**, page 74, Lemma 7.4], while (1.3) and

(1.5) are indeed equivalent; namely, *a topological algebra is topologically spectral if and only if it is quasi-inverse closed* (see [**10**, page 52, Theorem 2.5]). On the other hand, the relations (1.3) and (1.5) referred to the set $M(E)$ of all characters of E, determine the notions of an *algebraically spectral algebra* and *algebraically quasi inverse closed algebra*, respectively, being also equivalent. It is clear that *a topologically spectral algebra is algebraically spectral*, as well.

Now, a topological algebra E is called a *Q-algebra* if E° is open, while E is called an *advertibly complete algebra*, whenever *every advertibly null Cauchy net* $(x_\delta)_{\delta\in\Delta}$ in E, that is, such that,

$$x_\delta \circ x \longrightarrow 0 \longleftarrow x \circ x_\delta, \text{ for some } x \in E, \tag{1.6}$$

converges in E; its limit is obviously the quasi-inverse of x [**15**, page 45, Definition 6.4]. The above more convenient terminology is still due to *Mallios. In any advertibly complete locally m-convex algebra* (E, p_α), *the spectral radius is* expressed by the formula (cf. *Mallios* [**15**, page 99, Theorem 6.1])

$$r_E(x) = \sup_\alpha \lim_{n\to\infty} (p_\alpha(x^n))^{1/n}, \tag{1.7}$$

so, an element $x \in E$ with $r_E(x) = 0$ is called *topologically nilpotent. In the latter case the terms spectrally zero elements and topologically nilpotent elements coincide.* Finally, E is called *t-acceptable*, if every closed maximal regular ideal is 2-sided (cf. *Najmi* [**18**]).

2. Fixed point theorem in topological algebras. In this section, we give a new version of the *"Fixed Point Theorem,"* within the general context of topological algebras, being a very useful device for the proof of a generalized *"Quasi-square Root Theorem."* In this respect, a *fixed point* of a *"self-map"* T on a set X is an element $x_0 \in X$, with $T(x_0) = x_0$. Furthermore, an endomorphism T on a (pseudo-)metric space (X, d) is called a *contraction*, if there exists a positive real number $\alpha < 1$, such that

$$d(T(x), T(y)) \le \alpha\, d(x, y), \tag{2.1}$$

for all $(x, y) \in X \times X$. Obviously, such a map is (uniformly) continuous. Based on the preceding, the "*Fixed Point Theorem*," due to *Banach* (cf. *Dugundji* [**4**, page 305, Theorem 7.2] and *Simmons* [**20**, page 338, Lemma], see also *Heuser* [**13**, page 15, Section 2 and page 372, Section 106]), says that:

(2.2) *any contraction on a complete metric space* (X, d) *has a unique fixed point.*

The crucial point of the proof is to have a *convergent sequence of "iterates"* in X. This is guaranteed by securing the sequence at issue to be Cauchy, hence, its convergence by the completeness of X. In this regard, one can actually conclude that:

(2.3) *any contraction* T *on a metric space* (X, d) *has a unique fixed point in its completion* $\widetilde{(X, d)}$.

Now, since any *Fréchet topological algebra* is a topological algebra with the underlying topological vector space Fréchet (: metrizable and complete, [**15**, page 9, Definition 1.5]), one immediately concludes by (2.3) the next theorem.

Theorem 2.1 (*Mallios* **Fixed Point Theorem**). *Any contraction on a metrizable topological algebra* E *has a unique fixed point in its completion* $\widetilde{E}$, *hence, in* E *itself if, moreover,* E *is complete, in other words, Fréchet.*

In the case of an algebra E, whose *spectral radius* r_E *is a semi-norm*, one can view the former as a (pseudo-)metric d, defining thus an r_E-*contraction*, as an endomorphism T of E, for which there exists a real number $\alpha \in (0, 1)$, such that

$$r_E\big(T(x) - T(y)\big) \leq \alpha r_E(x - y), \tag{2.4}$$

for all $(x, y) \in E \times E$. Now, in the particular case that r_E is also *submultiplicative*, one has the following generalized version of the "*Fixed Point Theorem.*"

Theorem 2.2 (Fixed Point Theorem). *Let* E *be an algebra whose spectral radius* r_E *is a submultiplicative semi-norm and let* B *be a vector* *subspace of E. Then, any r_E-contraction T on B has a unique fixed point in the r_E-completion of B, $\widetilde{(B,\, r_E)} \equiv \widetilde{B}^{r_E}$.*

Proof. Considering an element $0 \neq x_0 \in B$, and an r_E-contraction T over B, we take the following sequence of iterates in B:

$$x_1 = T(x_0), \quad x_2 = T(x_1) = T^2(x_0), \dots, x_n = T^n(x_0),$$

such that for $m < n$ one has

$$\begin{aligned} r_E(x_m - x_n) &= r_E\big(T^m(x_0) - T^n(x_0)\big) \\ &= r_E\big(T^m(x_0) - T^m(T^{n-m}(x_0))\big) \\ &\leq \alpha^m\, r_E\big(x_0 - T^{n-m}(x_0)\big) \\ &= \alpha^m\, r_E(x_0 - x_{n-m}) \\ &\leq \alpha^m\, [r_E(x_0 - x_1) + r_E(x_1 - x_2) + \cdots \\ &\quad + r_E(x_{n-m-1} - x_{n-m})] \\ &\leq \alpha^m\, r_E(x_0 - x_1)\big(1 + \alpha + \alpha^2 + \cdots + \alpha^{n-m-1}\big) \\ &< \alpha^m \frac{r_E(x_0 - x_1)}{1 + \alpha}. \end{aligned} \tag{2.5}$$

Since $\alpha < 1$, one gets, by the preceding, that $(x_n)_{n\in\mathbf{N}}$ is an r_E-Cauchy sequence in B, hence it converges in its r_E-completion $\widetilde{(B,\, r_E)}$ to an element z, such that $T(z) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty}(T(x_n)) = \lim_{n\to\infty} x_{n+1} = z$, due to the continuity of T relative to r_E (cf. (2.4)). Hence, z is a fixed point, and in fact a unique one: If $y \in \widetilde{(B,\, r_E)}$ is another fixed point, i.e., $T(y) = y$, then one gets $r_E(z - y) = r_E(T(z) - T(y)) \leq \alpha\, r_E(z - y) < r_E(z - y)$, a contradiction. □

Scholium 2.3. *A class of algebras E that have the spectral radius a submultiplicative semi-norm is*, for instance *that one* considered by *Arizmendi* and *Valov* in [**2**], *satisfying*, what we may call, (A-V) *condition*:

$$r_E(x) = \sup_{f\in M(E)} |\widehat{x}(f)| \equiv \sup |\widehat{x}|(M(E)), \quad x \in E; \tag{A–V}$$

thus, *let alone the class of algebraically spectral algebras.* Besides, in the case that B *is a subalgebra of E, the r_E-completeness of B can be* *replaced by the r_E-advertible completeness of B*, in the sense that B, endowed with the topology induced by r_E, is advertibly complete. In this respect, we note that (cf. *Mallios* [**15**, page 3, Proposition 1.4 and page 4, Proposition 1.5])

(2.6) *in any algebra E, the spectral radius r_E is a submultiplicative semi-norm iff $r_E = q_U$, with q_U the gauge function of an α-barrel U* (:absolutely convex, absorbing and multiplicative subset of E), *while if, in addition, E is topological, $r_E = q_U$ is continuous iff U is a neighborhood of zero.*

In that context, one can take now into account the characterization of a Q-algebra, given by *Tsertos* [**22**, page 550, Theorem 4.1]; namely, that

(2.7) *a topological algebra E is Q iff $r_E \leq q_U$, with q_U the gauge function of a neighborhood U of zero in E.*

Thus in conjunction with (2.6), one concludes that

(2.8) *any topological algebra E, whose spectral radius r_E is a continuous submultiplicative semi-norm, is a Q-algebra.*

More generally,

(2.9) *any algebra E with r_E a submultiplicative semi-norm is a Q-algebra, relative to the topology induced on it by r_E. That is, (E, r_E) is a semi-normed Q-algebra, hence, advertibly complete, relative to r_E; yet, in other words, E is r_E-advertibly complete. Thus, based on the comments following* (1.7), *the spectrally zero elements of (E, r_E) are exactly the topologically nilpotent elements.*

Remark 2.4. (A-V) condition implies that r_E is a submultiplicative semi-norm possibly extended-valued, but we actually work with elements $x \in E$ such that $r_E(x) < +\infty$, which they form then a subalgebra of E. Concerning the statements (2.8), (2.9) and (2.10) we remark that *an algebra E with r_E a submultiplicative semi-norm becomes either a topological Q-algebra endowed with topology induced from r_E, or if E is a topological one the continuity of r_E renders it into a Q-algebra under its own topology.*

Specifically, one obtains the following result.

Theorem 2.5. *In any algebra E consider the following assertions*:

1) *E is algebraically spectral.*

2) *E is algebraically quasi-inverse closed.*

3) *(E, r_E) is a semi-normed Q-algebra.*

4) *(E, r_E) is advertibly complete.*

5) *E is quasi plane in its r_E-completion.*

6) *(E, r_E) is a Mallios algebra.*

Then, one has the following relations:

$$\begin{array}{ccccccccc} 1) & \Longleftrightarrow & 2) & \Longrightarrow & 3) & \Longrightarrow & 4) & \Longleftrightarrow & 5), \\ & & & & \Downarrow & & & & \\ & & & & 6) & & & & \end{array}$$

If, moreover, (E, r_E) is t-acceptable, then 6) $\Rightarrow$ 2)*, while* 4) $\Rightarrow$ 2) *in the case $\widetilde{(E, r_E)} \equiv \widetilde{E}^{r_E}$ is a t-acceptable Mallios algebra.*

Proof. 1)$\Longleftrightarrow$2) and 4)$\Longleftrightarrow$5) follows from [**10**, page 52, Theorem 2.5], while for 3)$\Longrightarrow$4) and 3)$\Longrightarrow$6) see *Mallios* [**15**, page 45, Theorem 6.4 and page 67, Theorem 6.1].

2)$\Longrightarrow$3): By 2)$\Longleftrightarrow$1) one has that $Sp_E(x) = \widehat{x}(M(E))$, which *implies that r_E is a submultiplicative semi-norm, along with the continuity of any character of E, relative to r_E.* Besides, by (2.9), one gets that (E, r_E) is a Q-algebra.

6)$\Longrightarrow$2): Assume that $1 \notin \widehat{x}(\mathfrak{M}(E))$ and $x \notin E^{\circ}$. Then, x belongs to a maximal regular ideal M of E, being also closed in view of 6), so, by hypothesis, M is 2-sided. Thus, there exists $f \in \mathfrak{M}(E)$ such that $M = \operatorname{Ker} f$. Being x an identity of E modulo M, one has $yx - y \in M$, for every $y \in E$, hence $f(x) = 1$, a contradiction.

4)$\Longrightarrow$2): Assuming 4), let $1 \notin \widehat{x}(\mathfrak{M}(E))$, with $x \notin E^{\circ}$. Then, since 4)$\Longleftrightarrow$5), one has that $x \notin (\widetilde{E}^{r_E})^{\circ}$; thus, [**15**, page 65, (6.2)] x belongs to a maximal regular ideal M of $\widetilde{E}^{r_E}$, being also an identity of $\widetilde{E}^{r_E}$ modulo M. Since $\widetilde{E}^{r_E}$ is a Mallios algebra, M is closed, hence 2-sided, in view of the hypothesis that $\widetilde{E}^{r_E}$ is t-acceptable. Besides, the semi-normed algebra $\widetilde{E}^{r_E}$ is Gelfand-Mazur, so there exists $\phi \in \mathfrak{M}(\widetilde{E}^{r_E})$, with $M = \operatorname{Ker}\phi$, and since $yx - y \in M$, for every $y \in \widetilde{E}^{r_E}$, one has $\phi(x) = 1$. Thus, there exists $f \in \mathfrak{M}(E)$, such that $f(x) = 1$, a contradiction, therefore $x \in E^{\circ}$, implying 2). □

In toto, one concludes, by the preceding, that:

(2.10) *an algebraically spectral algebra E is made into a topological algebra, in the topology induced from the spectral radius r_E, the latter becoming then automatically a sub-multiplicative semi-norm. Moreover, any character of (E, r_E) is continuous, while the same algebra also has the Q-property.*

As a consequence of the previous discussion, one obtains the following results.

Theorem 2.6. *Let E be an algebra having the spectral radius r_E a submultiplicative semi-norm. Then, any r_E-contraction T on (E, r_E), with*

$$r_E(T^n(0)) = 0, \quad n \in \mathbf{N}, \tag{2.11}$$

(that is, $T^n(0)$, $n \in \mathbf{N}$, is a spectrally zero element), has a unique fixed point.

Proof. As in the proof of Theorem 2.2, taking $0 \neq x_0 \in (E, r_E)$, the r_E-Cauchy sequence $(x_n)_{n\in\mathbf{N}}$ of iterates in (E, r_E) is r_E-advertibly null, for $y = T^n(0)$, with $r_E(T^n(0)) = 0$, in the following sense

$$y \circ x_n \underset{r_E}{\longrightarrow} 0 \underset{r_E}{\longleftarrow} x_n \circ y.$$

Indeed, for $\varepsilon > 0$, one has

$$
\begin{aligned}
r_E(y \circ x_n) &= r_E(y + x_n - y x_n) \\
&= r_E(y + T^n(x_0) - y\, T^n(x_0)) \\
&\le r_E(y) + r_E(T^n(x_0)) + r_E(y)\, r_E(T^n(x_0)) \\
&= r_E(T^n(x_0)) = r_E(T^n(x_0) - T^n(0) + T^n(0)) \\
&\le r_E(T^n(x_0) - T^n(0)) + r_E(T^n(0)) \\
&\le \alpha^n\, r_E(x_0) < \varepsilon.
\end{aligned} \tag{2.12}
$$

and similarly $x_n \circ y \underset{r_E}{\longrightarrow} 0$, where $\alpha \in (0,\, 1)$. Since, in view of (2.9), E is r_E-advertibly complete, there exists $z \in E$, with $x_n \underset{r_E}{\longrightarrow} z$ and $y \circ z = z \circ y = 0$. The element z is the desired unique fixed point of T, according to the proof in Theorem 2.2. □

An immediate consequence of the preceding is the next.

Corollary 2.7. *Let E be an algebra with spectral radius r_E a submultiplicative semi-norm and F an advertibly complete subalgebra of E. Then, any r_E-contraction T on F, such that* (2.11) *holds true, has a unique fixed point.*

Based on Theorems 2.5, 2.6 and the fact that a closed subalgebra of an advertibly complete algebra is of the same type (cf. *Warner* [**23**, page 3, Proposition 2] and/or *Hadjigeorgiou* [**10**, page 54, Corollary 2.9]), one concludes the next result. In this context, given an element $x \in E$, we denote by

$$\overline{E(x)}^{\,r_E} \equiv \overline{(E(x),\, r_E)} \subseteq (E,\, r_E),$$

the least closed subalgebra of $(E,\, r_E)$ containing x.

Corollary 2.8. *Let E be an algebra with spectral radius r_E a submultiplicative semi-norm and $x \in E$. Then, any r_E-contraction on $\overline{E(x)}^{\,r_E}$, satisfying* (2.11)*, has a unique fixed point.*

Remark 2.9. Referring to the fixed point in all the previous theorems, we note that it is attained by the convergence of some sequence of iterates plus the inequality of the contraction T. Thus, it lies in the completion of a vector subspace of the algebra involved (cf. Theorem 2.2), or, avoiding the completion, in an advertibly complete subalgebra of it (cf. Corollaries 2.7, 2.8). In fact, in the second situation we actually need a subset of the algebra having some kind of algebraic structure; precisely, it is closed for the ring multiplication and for a scalar multiple of the addition, multiplication and the q-operation (see (3.7) in Theorem 3.4).

3. Quasi-square root lemma in topological algebras. The well-known "*Square Root Lemma*" of *Ford* (cf. [**5**] and *Bonsall-Duncan* [**3**, page 44, Proposition 13], *referred to Banach algebras, was generalized* by *Štěrbová* [**21**, Theorem 3.9] in 1980 *for complete locally m-convex algebras*, by employing the classical result of *Ford* to the Banach factors of an l.m.c. algebra. After a careful look at the proof, we remark that *we can avoid completeness and local m-convexity, by working with the spectral radius*, as a submultiplicative semi-norm, *in the completion of an appropriate subspace* of the given algebra. For the previous extension of Square Root Lemma, we shall make use of a generalized form of *"Fixed Point Theorem,"* cf. Remark 2.9. In this regard, by a *quasi square root* of an element $a \in E$, we mean an element $x \in E$, such that $x \circ x = a$.

Theorem 3.1 (Quasi-square Root Lemma). *Let E be a metrizable topological Q-algebra and $x \in E$, with $r_E(x) < 1$. Then there exists a unique quasi-square root $y \in \widetilde{E(x)}$ (*completion of $E(x) \subseteq E$*) of x, such that $r_E(y) < 1$.*

Proof. Assuming that d is the metric defining the topology of the algebra E, then, by the Q-property and (2.7), we have $r_E \leq q_{S_\varepsilon}$, with $S_\varepsilon = \{x \in E : d(x, 0) = |x| < \varepsilon\}$. So, we may suppose that $|x| < \alpha < 1$, and consider in the completion of the subalgebra $E(x)$ of E the closed subset:

$$B = \{z \in \widetilde{E(x)} : d(z, 0) = |z| \leq \alpha\}. \tag{3.1}$$

Now, setting

$$T : B \longrightarrow B : z \longmapsto T(z) := \frac{1}{2}\,(x + z^2), \tag{3.2}$$

since the elements of $\widetilde{E(x)}$ commute with each other, one obtains

$$
(3.3)\qquad \begin{aligned}
d(T(z),\,T(w)) = |\,T(z)-T(w)| &= \left|\frac{1}{2}\,(z^2+x-x-w^2)\right| \\
&\le \frac{1}{2}\,|z+w||z-w| \le \frac{1}{2}\,2\,\alpha\,|z-w| \\
&= \alpha\, d(z,\,w),
\end{aligned}
$$

for any $z,\,w \in B$, that is, T is a contraction on the complete metrizable space B. By the *"Fixed Point Theorem"* (cf. (2.3)), there exists a unique $y \in B \subseteq \widetilde{E(x)}$, such that $T(y) = y$, hence $(x+y^2)/2 = y \Leftrightarrow 2y - y^2 = x \Leftrightarrow y \circ y = x$, with $r_E(y) \le |y| < 1$, proving the assertion. □

Corollary 3.2. *In a topological Fréchet Q-algebra E, any $x \in E$, with $r_E(x) < 1$, has a unique quasi-square root $y \in \overline{E(x)}$, such that $r_E(y) < 1$.*

Since a semi-normed space is a pseudo-metric space, a direct consequence of the preceding, along with (2.9), is the next.

Corollary 3.3. *Let E be an algebra having the spectral radius r_E a complete submultiplicative semi-norm* (: the topological algebra $(E,\,r_E)$ is complete) *and $x \in E$, with $r_E(x) < 1$. Then, there exists a unique quasi-square root y of x in $\overline{E(x)}^{\,r_E}$, such that $r_E(y) < 1$.*

On the other hand, based on Corollary 2.7, *the previous result holds true, without the "completeness of r_E."* Thus, one gets

Theorem 3.4 (Quasi-Square Root Lemma). *Let E be an algebra having the spectral radius r_E a submultiplicative semi-norm and $x \in E$, with $r_E(x) < 1$. Then, there exists a unique quasi-square root y of x in $\overline{E(x)}^{\,r_E}$, with $r_E(y) < 1$, where y is the fixed point of a contraction T, provided the latter map satisfies the relation $r_E(T^n(0)) = 0$, $n \in \mathbf{N}$.*

Proof. We may suppose that $r_E(x) < \alpha < 1$, and consider the closed subset of $\overline{E(x)}^{\,r_E}$:

$$
(3.4)\qquad B = \{z \in \overline{E(x)}^{\,r_E} : r_E(z) \le \alpha\}.
$$

Now, setting

$$T : B \longrightarrow B : z \longmapsto T(z) := \frac{1}{2}(x + z^2), \tag{3.5}$$

since the elements of $\overline{E(x)}^{r_E}$ commute with each other, one obtains

$$\begin{aligned} r_E(T(z) - T(w)) &= r_E\left(\frac{1}{2}(z^2 + x - x - w^2)\right) \\ &\le \frac{1}{2} r_E(z+w)\, r_E(z-w) \\ &\le \frac{1}{2}\, 2\,\alpha\, r_E(z-w) \\ &= \alpha\, r_E(z-w), \end{aligned} \tag{3.6}$$

that is, T is an r_E-contraction on the set B. However, *the set B has the following "algebraic structure"*; namely,

(3.7)

it contains zero and is closed for the following operations:

i) *the ring multiplication,*

ii) *the multiplication by a scalar κ, with $|\kappa| \le 1$,*

iii) *any sum multiplied by λ, with $|\lambda| \le 1/2$, and*

iv) *the q-operation multiplied by μ, such that $|\mu| \le 1/3$.*

Indeed, $r_E(0) = 0 < \alpha$, while for $z, w \in B$, one has $r_E(zw) \le r_E(z) r_E(w) \le \alpha^2 < \alpha$, $r_E(\kappa z) \le |\kappa|\, r_E(z) \le \alpha$, $r_E(\lambda(z+w)) \le |\lambda|\,(r_E(z) + r_E(w)) \le 2\alpha/2 = \alpha$, and $r_E(\mu\,(z \circ w)) \le |\mu|\,(r_E(z) + r_E(w) + r_E(zw)) \le (2\alpha + \alpha^2)/3 \le 3\alpha/3 = \alpha$. Therefore, B appears to be a, so to say, "advertibly complete" subset of $\overline{E(x)}^{r_E}$, a reminder of the situation one has in [**10**, page 54, Corollary 2.9]. So one can further apply the *"Fixed Point Theorem"* (cf. Corollary 2.7 and Remark 2.9) to get a unique $y \in B$, such that $T(y) = y$; hence, $(x + y^2)/2 = y \Longleftrightarrow 2y - y^2 = x \Longleftrightarrow y \circ y = x$, proving the assertion. $\square$

4. Lifting of idempotents in topological algebras. The "*Lifting of Idempotents Theorem*" provides an idempotent element for a given algebra E from a similar element of the quotient algebra $E/\operatorname{rad} E$, where $\operatorname{rad} E$ denotes the *topological Jacobson radical of E*. This is known for E a unital commutative Banach algebra (cf. *Rickart* [**19**, page 58, Theorem 2.3.9], *Zelazko* [**24**, page 97, Lemma 20.3]), for a non-unital non-commutative Banach algebra (see *Bonsall-J. Duncan* [**3**, page 44, Theorem 14]) and for a commutative complete l.m.c. algebra by *Mallios* [**16**, page 306]. For the proof one applies the "*Quasi-square Root Lemma," along with two* basic *properties, that characterize* $\operatorname{rad} E$: the first one in terms of *the* so-called *"topologically nilpotent" elements*, the second by means of *the quasi-invertible elements*; in other words, one has

$$\begin{aligned} \operatorname{rad} E &= \{x \in E : r_E(x) = 0\} \\ &\equiv \ker(r_E) \\ &= \{x \in E : yx \in E^\circ \ \forall\ y \in E\} \\ &\equiv \{x \in E : Ex \subseteq E^\circ\} \end{aligned} \tag{4.1}$$

(cf. *Zelazko* [**24**, page 54, (12.8.1), Definition 12.8 and Theorem 12.9] as well as *Bonsall-J. Duncan* [**3**, page 125, Proposition 16 and page 126, Proposition 1], *Larsen* [**14**, page 83, Theorem 3.5.1]).

Here we extend the previous results in the framework of a topological algebra, in general, by considering in place of the *topological Jacobson radical* $\operatorname{rad} E$ *of* E the "*Gel'fand radical" of* E, $\ker(\mathcal{G}_E) = \ker(\mathfrak{M}(E)) = \bigcap_{f \in \mathfrak{M}(E)} \ker f$. (The latter terminology has been coined by *Mallios*.) Obviously, the *two radicals coincide in every commutative Banach algebra* and, *more generally, in any commutative advertibly complete l.m.c. algebra* (cf. *Fragoulopoulou* [**6**, page 51, Lemma 9.6, (i)], along with *Mallios* [**15**, page 104, Corollary 6.5, and page 201, Definition 3.1]). As already mentioned,

(4.2) (A-V) *condition renders the spectral radius* r_E, *of an algebra* E, *a submultiplicative semi-norm and the algebra* E *a semi-normed* Q*-algebra under the topology induced by* r_E *(see* (2.9)). *It also characterizes the kernel of the Gel'fand map* $\mathcal{G}_E$ (:"*Gel'fand radical*" *of* E), *as the set of the topological nilpotent elements*; *that is, one has*

$$\ker(\mathcal{G}_E) = \{x \in E : r_E(x) = 0\} \equiv \ker(r_E). \tag{4.2.1}$$

If, moreover, E is *algebraically* (equivalently topologically, under the topology of r_E) *spectral*, then, apart from the (A-V) condition and relation (4.2.1), *E shares also the following two properties*:

$$x \in E^\circ \quad \text{if} \quad 1 \notin \widehat{x}(\mathfrak{M}(E)), \tag{4.3}$$

being in fact *equivalent with the topological spectrality* of E (cf. *Hadjigeorgiou* [**10**, page 52, Theorem 2.5]), and also

$$\begin{aligned} \ker(\mathcal{G}_E) &= \{x \in E : yx \in E^\circ \ \forall\ y \in E\} \\ &\equiv \{x \in E : Ex \subseteq E^\circ\} \equiv B. \end{aligned} \tag{4.4}$$

Indeed, if $x \in \ker(\mathcal{G}_E)$, then, for every $f \in \mathfrak{M}(E)$ and $y \in E$ we have $f(yx) = f(x)f(y) = 0 \neq 1$; hence, by (4.3), $yx \in E^\circ$, that is $\ker(\mathcal{G}_E) \subseteq B$. Conversely, if $x \notin \ker(\mathcal{G}_E)$, then, there exists $g \in \mathfrak{M}(E)$, such that $g(x) \neq 0$, that is, $x \notin \ker g$, where g is now *considered as a continuous irreducible representation of E in* $\mathbf{C} \cong L(\mathbf{C})$. Thus, there exists $\lambda \in \mathbf{C}$, with $x\lambda = g(x)\lambda \neq 0$, hence (cf. *Bonsall-Duncan* [**3**, page 120, Proposition 4, (iii)], *$x\lambda$ is a cyclic vector.* Therefore, there exists $y \in E$, such that $yx\lambda = \lambda \Longleftrightarrow g(yx) = 1$, so, according to (4.3), $yx \notin E^\circ$, i.e., $x \notin B$ proving (4.4).

Theorem 4.1 (Lifting of Idempotents theorem). *Let E be an algebraically spectral algebra and $x \in E$ an idempotent, modulo the Gel'fand radical,* $\ker(\mathcal{G}_E)$. *Then, there exists a unique idempotent in E, which is equal to x, modulo* $\ker(\mathcal{G}_E)$. *In other words, if $x \in E$, with $\widehat{x^2 - x} = 0$, then, there exists a unique $y \in \overline{E(x)}^{r_E} \cap \ker(\mathcal{G}_E) \subseteq E$, such that $(x+y)^2 = x + y$.*

Proof. We consider the closed subalgebra of $\overline{E(x)}^{r_E}$,

$$\begin{aligned} F &\equiv \overline{E(x)}^{r_E} \cap \ker(\mathcal{G}_E) = \overline{E(x)}^{r_E} \cap \ker(r_E) \\ &\equiv \{z \in \overline{E(x)}^{r_E} : r_E(z) = 0\}, \end{aligned} \tag{4.5}$$

being also a closed 2-sided ideal. Since $\widehat{x^2 - x} = 0$, one also gets that $4(\widehat{x^2 - x}) = 0$; hence, by (4.2.1), $r_E(4(x^2 - x)) = 0 < 1$, therefore, $1 \notin \widehat{u}(\mathfrak{M}(E))$, where $u \equiv 4(x^2 - x)$, hence $u \in E^\circ$ (cf. [**10**, page 52, Theorem 2.5]). In fact, one has that $u \in F^\circ$: The semi-normed algebra (E, r_E) is Q, due to the continuity of r_E (cf. (2.9)), hence advertibly compete, along with its closed subalgebra F. But, F, as *a semi-normed advertibly complete algebra, is still a Q algebra* (see e.g., *Warner* [**23**, page 8, Theorem 7]), hence, by (2.6) and (2.7), one gets that $r_F \leq r_E$, which, along with the basic relation $r_E \leq r_F$, implies the coincidence of the spectral radii, that is,

$$r_E = r_F.$$

So, $r_F(u) = r_E(u) = 0 < 1$, yielding that $1 \notin \widehat{u}(\mathfrak{M}(F))$; therefore $u \in F^\circ$, since F, as a commutative semi-normed algebra, is equivalently topologically spectral (cf. [**10**, page 52, Lemma 2.2 and Theorem 2.5]). Hence, there exists $w \in F$, with $r_F(w) = r_E(w) = 0 < 1$, such that

$$4(x^2 - x) \circ w = w \circ 4(x^2 - x) = 0, \tag{4.6}$$

and by applying the *"Quasi-square Root Lemma"* for F, there exists a unique $z \in F$, with $z \circ z = w$. Since x and z are commuting elements, one has (cf. also (4.6))

$$\begin{aligned} [(2x) \circ z)] \circ [(2x) \circ z] &= (2x) \circ (2x) \circ (z \circ z) \\ &= 4(x^2 - x) \circ w = 0, \end{aligned} \tag{4.7}$$

where $(2x) \circ z = 2(x + y)$, with

$$y = \frac{1}{2}\, z - xz \in F, \tag{4.8}$$

a unique element of F, due to the uniqueness of z. By (4.7), one obtains $0 = 2(x+y) \circ 2(x+y) = 4(x + y - (x+y)^2)$, thus, $(x+y)^2 = x + y$, that is the assertion. □

Scholium 4.2. An application of the *"Lifting of Idempotents Theorem"* appears in the *"Šilov's Idempotent Theorem"*, see [**11**], [**12**, Section 10], where an *idempotent element is obtained in $E^\wedge$*. Therefore (see also [**11**, page 175, (3.5)]), the same goes to the quotient algebra $E/\ker(\mathcal{G}_E)$, in the case of a topological algebra E, with $\mathfrak{M}(E) \underset{\text{homeo}}{\cong} \mathfrak{M}(E^\wedge)$ (take, for instance, $\mathcal{G}_E$ continuous [**9**, page 136, Theorem 3.1]) *and* $E^\wedge$ *complete* (take, for instance, E local with locally equicontinuous spectrum [**17**]). On the other hand, by Theorem 4.1, one gets already an idempotent in E itself, if moreover E is algebraically spectral. In the latter case one really economizes the condition $\mathfrak{M}(E) \underset{\text{homeo}}{\cong} \mathfrak{M}(E^\wedge)$, since then E is a Q-algebra, having thus $\mathfrak{M}(E)$ equicontinuous, so $\mathcal{G}_E$ continuous, therefore, the previous identification (cf. [**9**, page 136, Theorem 3.1] along with [**15**, page 75, Proposition 7.1 and page 184, Theorem 1.2]).

Scholium 4.3. The proof of the *"Lifting of Idempotents Theorem"* is based on the *"Fixed Point Theorem"*, following its classical version for Banach algebras (cf. [**3**, page 44, Proposition 13, Theorem 14]). Now this is based on the notion of *"contraction"*, defined in terms of a (pseudo-)metric, rendering also, by the very definitions, the aforesaid map continuous, with respect to the topology of the same pseudo-metric at issue. Therefore, the motive to consider such a topology too on the algebra we work, which thus was for us the *topology of the spectral radius*, the latter being also suitably restricted, concerning the algebra structure. So the chosen in this manner framework, immediately suggests now the question (*Mallios*), whether the same context works with an arbitrary *"algebra semi-norm"*, or even, more generally, for an *l.m.c.* (topological) *algebra*: Indeed, in the case of a *semi-normed* (topological) *algebra* one gets the *"Lifting of Idempotents Theorem"* when, in particular, the said algebra is also *topologically spectral and advertibly complete*. As a corollary (see also [**15**, page 104, Corollary 6.4] and/or [**10**, page 52, Lemma 2.2]), one gets the same theorem for a *commutative advertibly complete semi-normed algebra*. In the more general case of an l.m.c. algebra, one has to suitably adjust the definition of a *"contraction in terms of a family of semi-norms", as well as*, that one of *relation* (2.11). So one defines an endomorphism T of an l.m.c. algebra $(E, \{p_\alpha\}_{\alpha\in\Gamma})$ to be a *contraction uniformly w.r.t.* Γ, if there exists a real number $\lambda \in (0, 1)$, such that

$$p_\alpha(T(x) - T(y)) \leq \lambda\, p_\alpha(x - y), \text{ for any } x,\, y \in E \text{ and } \alpha \in \Gamma,$$

while (2.11) takes the form,

$$p_\alpha(T^n(0)) = 0, \quad \alpha \in \Gamma,\; n \in \mathbf{N}.$$

Thus, one obtains the *"Lifting of Idempotents Theorem"* for a *topologically spectral advertibly complete l.m.c. algebra*; hence, a fortiori, for a *commutative advertibly complete l.m.c. algebra*, extending thus a relevant previous result of *Mallios* for such *complete* algebras [**16**, page 306].

In this context, we still remark that for the analogous adjustment of Theorem 3.4 (*"Square Root Lemma"*) that intervenes in both the above two cases, concerning, in particular, the boundedness of the square root, through that one spectral radius, one can apply [**15**, page 99, Theorem 6.1] and [**7**, page 64, Lemma 5.3 and remarks after it].

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